UE OXBRIDGE-PREP

# CAMBRIDGE SIXTH TERM EXAM PAPER

# STEP 3

## PAST PAPERS

1987-2023

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## Introduction

"STEP 3 Past Papers" is presented by UE International Education (ueie.com), which is designed as a companion to the STEP Standard Course and the STEP Question Practice. It aims to help students to prepare the Cambridge STEP mathematics exams. It is also a useful reference for teachers who are teaching STEP Exams.

All questions in this collection are reproduced from the official past papers released by the University of Cambridge, with a few typos from the source files corrected. The 2024 Edition collects a total of 511 STEP 3 questions from 1987 to 2023.

## How to Access Full Solutions

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At least one of the official solution, hand-written solution or video solution is provided for each question. Hand-written solutions are provided if official solutions are unavailable. There are video solutions for some questions.

All solutions can be accessed **ON-LINE ONLY**.

# 简介

《STEP 3 历年真题集》由优易国际教育（ueie.com）出品，是 STEP 标准课程和 STEP 刷题训练的配套资料之一。其主要用途是帮助学生提高备考剑桥 STEP 数学考试的效率，以及为教授 STEP 考试的同行老师提供参考。

真题集中的所有真题均由剑桥大学官方发布的真题重新排版制作而成，并修订了源文件中的若干印刷错误。2024 版收录了 1987 年至 2023 年共 511 道 STEP 3 真题。

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所有用户均可免费使用真题集，但所有题目的解析仅向购买以下任意优易牛剑备考系列产品之一的**付费用户开放**：

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所有解析均**只能在线查看**。

## 簡介

《STEP 3 歷年真題集》由優易國際教育（ueie.com）出品，是 STEP 標準課程和 STEP 刷題訓練的配套資料之一。其主要用途是幫助學生提高備考劍橋 STEP 數學考試的效率，以及為教授 STEP 考試的同儕老師提供參考。

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# STEP 3 1987

STEP 3 1987
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

## INFORMATION FOR CANDIDATES

There are 16 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

[STEP 3, 1987Q1]

Find the set of positive integers $n$ for which $n$ does not divide $(n-1)!$. Justify your answer.

[Note that small values of $n$ may require special consideration.]

[STEP 3, 1987Q2]

Let $I_{m,n} = \int \cos^m x \sin nx \, dx$, where $m$ and $n$ are non-negative integers. Prove that for $m, n \geq 1$,

$$(m+n)I_{m,n} = -\cos^m x \cos nx + mI_{m-1,n-1}.$$

(i) Show that $\int_0^{\pi} \cos^m x \sin nx \, dx = 0$ whenever $m, n$ are both even or both odd.

(ii) Evaluate $\int_0^{\frac{1}{2}\pi} \sin^2 x \sin 3x \, dx$.

[STEP 3, 1987Q3]

(i) If $z = x + \mathbf{i}y$, with $x, y$ real, show that

$$|x|\cos\alpha + |y|\sin\alpha \leq |z| \leq |x| + |y|$$

for all real $\alpha$.

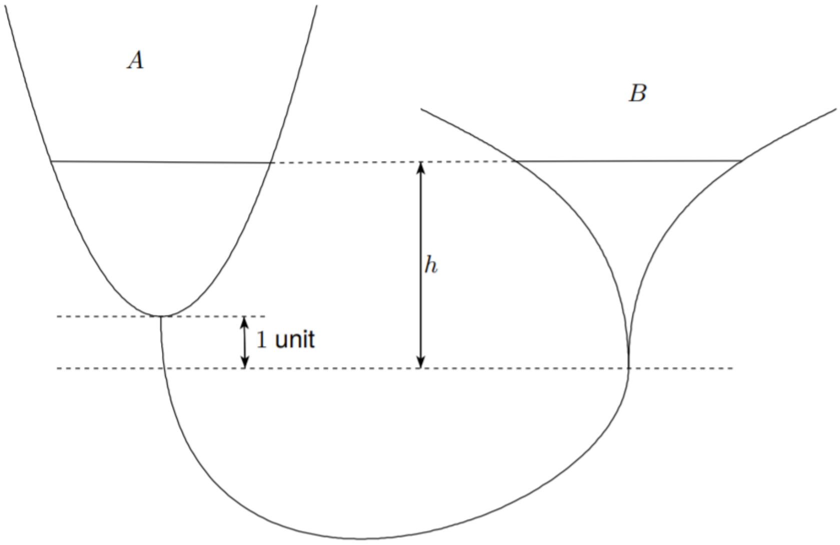
(ii) By considering $(5-\mathbf{i})^4(1+\mathbf{i})$, show that

$$\frac{1}{4}\pi = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

Prove similarly that

$$\frac{1}{4}\pi = 3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{20}\right) + \tan^{-1}\left(\frac{1}{1985}\right).$$

[STEP 3, 1987Q4]




Two funnels $A$ and $B$ have surfaces formed by rotating the curves $y = x^2$ and $y = 2\sinh^{-1} x$ $(x > 0)$ about the $y$-axis. The bottom of $B$ is one unit lower than the bottom of $A$ and they are connected by a thin rubber tube with a tap in it. The tap is closed and $A$ is filled with water to a depth of 4 units. The tap is then opened. When the water comes to rest, both surfaces are at a height $h$ above the bottom of $B$, as shown in the diagram. Show that $h$ satisfies the equation

$$h^2 - 3h + \sinh h = 15.$$



[STEP 3, 1987Q5]

A secret message consists of the numbers 1, 3, 7, 23, 24, 37, 39, 43, 43, 43, 45, 47 arranged in some order as $a_1, a_2, \dots, a_{12}$. The message is encoded as $b_1, b_2, \dots, b_{12}$, with $0 \le b_j \le 49$ and

$$b_{2j} \equiv a_{2j} + n_0 + j \pmod{50},$$
$$b_{2j+1} \equiv a_{2j+1} + n_1 + j \pmod{50},$$

for some integers $n_0$ and $n_1$. If the coded message is 35, 27, 2, 36, 15, 35, 8, 40, 40, 37, 24, 48, find the original message, explaining your method carefully.



[STEP 3, 1987Q6]

The functions $x(t)$ and $y(t)$ satisfy the simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 5y = 0,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} + ax - 2y = 2\cos t,$$

subject to $x = 0, \frac{\mathrm{d}y}{\mathrm{d}t} = 0$ at $t = 0$.

Solve these equations for $x$ and $y$ in the case when $a = 1$.

Without solving the equations explicitly, state briefly how the form of the solutions for $x$ and $y$ if $a > 1$ would differ from the form when $a = 1$.

[STEP 3, 1987Q7]

Prove that

$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots + \frac{(-1)^n t^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^t \frac{x^{2n+2}}{1+x^2} \mathrm{d}x.$$

Hence show that, if $0 \le t \le 1$, then

$$\frac{t^{2n+3}}{2(2n+3)} \le \left| \tan^{-1} t - \sum_{r=0}^{n} \frac{(-1)^r t^{2r+1}}{2r+1} \right| \le \frac{t^{2n+3}}{2n+3}.$$

Show that, as $n \to \infty$,

$$4 \sum_{r=0}^{n} \frac{(-1)^r}{2r+1} \to \pi,$$

but that the error in approximating $\pi$ by $4\sum_{r=0}^{n} \frac{(-1)^r}{2r+1}$ is at least $10^{-2}$ if $n$ is less than or equal to 98.

[STEP 3, 1987Q8]

Show that, if the lengths of the diagonals of a parallelogram are specified, then the parallelogram has maximum area when the diagonals are perpendicular. Show also that the area of a parallelogram is less than or equal to half the square of the length of its longer diagonal.

The set $A$ of points $(x, y)$ is given by

$$|a_1 x + b_1 y - c_1| \le \delta,$$
$$|a_2 x + b_2 y - c_2| \le \delta,$$

with $a_1 b_2 \ne a_2 b_1$. Sketch this set and show that it is possible to find $(x_1, y_1), (x_2, y_2) \in A$ with

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 \ge \frac{8\delta^2}{|a_1 b_2 - a_2 b_1|}.$$

[STEP 3, 1987Q9]

Let $(G, \ *)$ and $(H, \ \circ)$ be two groups and $G \times H$ the set of ordered pairs $(g, h)$ with $g \in G$ and $h \in H$. A multiplication on $G \times H$ is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1 * g_2, h_1 \circ h_2)$$

for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Show that, with this multiplication, $G \times H$ is a group.

State whether the following are true or false and prove your answers.

(i) $G \times H$ is abelian if and only if both $G$ and $H$ are abelian.

(ii) $G \times H$ contains a subgroup isomorphic to $G$.

(iii) $\mathbb{Z}_1 \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_4$.

(iv) $S_2 \times S_3$ is isomorphic to $S_6$.

[$\mathbb{Z}_n$ is the cyclic group of order $n$, and $S_n$ is the permutation group on $n$ objects.]

[STEP 3, 1987Q10]

The Bernoulli polynomials $P_n(x)$, where $n$ is a non-negative integer, are defined by $P_0(x) = 1$ and, for $n \geq 1$,

$$\frac{\mathrm{d}P_n}{\mathrm{d}x} = nP_{n-1}(x), \qquad \int_0^1 P_n(x)\,\mathrm{d}x = 0.$$

Show, by using induction or otherwise, that

$$P_n(x+1) - P_n(x) = nx^{n-1}, \qquad \text{for } n \geq 1.$$

Deduce that

$$n \sum_{m=0}^{k} m^{n-1} = P_n(k+1) - P_n(0).$$

Hence show that $\sum_{m=0}^{1000} m^3 = (500\,500)^2$.

## [STEP 3, 1987Q11]

A woman stands in a field at a distance of $a$ m from the straight bank of a river which flows with negligible speed. She sees her frightened child clinging to a tree stump standing in the river $b$ m downstream from where she stands and $c$ m from the bank. She runs at a speed of $u$ m s$^{-1}$ and swims at $v$ m s$^{-1}$ in straight lines. Find an equation to be satisfied by $x$, where $x$ m is the distance upstream from the stump at which she should enter the river if she is to reach the child in the shortest possible time.

Suppose now that the river flows with speed $v$ m s$^{-1}$ and the stump remains fixed. Show that, in this case, $x$ must satisfy the equation

$$2vx^2(b-x) = u(x^2-c^2)[a^2+(b-x)^2]^{\frac{1}{2}}.$$

For this second case, draw sketches of the woman's path for the three possibilities $b > c$, $b = c$ and $b < c$.

## [STEP 3, 1987Q12]

A firework consists of a uniform rod of mass $M$ and length $2a$, pivoted smoothly at one end so that it can rotate in a fixed horizontal plane, and a rocket attached to the other end. The rocket is a uniform rod of mass $m(t)$ and length $2l(t)$, with $m(t) = 2\alpha l(t)$ and $\alpha$ constant. It is attached to the rod by its front end and it lies at right angles to the rod in the rod's plane of rotation. The rocket burns fuel in such a way that $\frac{\mathrm{d}m}{\mathrm{d}t} = -\alpha\beta$, with $\beta$ constant. The burnt fuel is ejected from the back of the rocket, with speed $u$ and directly backwards relative to the rocket. Show that, until the fuel is exhausted, the firework's angular velocity $\omega$ at time $t$ satisfies

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3\alpha\beta a u}{2[Ma^2+2\alpha l\{3a^2+l^2\}]}.$$

## [STEP 3, 1987Q13]

A uniform rod, of mass $3m$ and length $2a$, is freely hinged at one end and held by the other end in a horizontal position. A rough particle, of mass $m$, is placed on the rod at its mid-point. If the free end is then released, prove that, until the particle begins to slide on the rod, the inclination $\theta$ of the rod to the horizontal satisfies the equation

$$5a\dot{\theta}^2 = 8g\sin\theta.$$

The coefficient of friction between the particle and the rod is $\frac{1}{2}$. Show that, when the particle begins to slide, $\tan\theta = \frac{1}{26}$.

[STEP 3, 1987Q14]

It is given that the gravitational force between a disc, of radius $a$, thickness $\delta x$ and uniform density $\rho$, and a particle of mass $m$ at a distance $b$ ($\geq 0$) from the disc on its axis is

$$2\pi mk\rho\delta x\left[1-\frac{b}{(a^2+b^2)^{\frac{1}{2}}}\right],$$

where $k$ is a constant. Show that the gravitational force on a particle of mass $m$ at the surface of a uniform sphere of mass $M$ and radius $r$ is $\frac{kmM}{r^2}$. Deduce that in a spherical cloud of particles of uniform density, which all attract one another gravitationally, the radius $r$ and inward velocity $v$ $(=-\frac{\mathrm{d}r}{\mathrm{d}t})$ of a particle at the surface satisfy the equation

$$v\frac{\mathrm{d}v}{\mathrm{d}r}=-\frac{kM}{r^2},$$

where $M$ is the mass of the cloud.

At time $t=0$, the cloud is instantaneously at rest and has radius $R$. Show that $r=R\cos^2\alpha$ after a time

$$\left[\frac{R^3}{2kM}\right]^{\frac{1}{2}}\left(\alpha+\frac{1}{2}\sin 2\alpha\right).$$

[STEP 3, 1987Q15]

A patient arrives with blue thumbs at the doctor's surgery. With probability $p$ the patient is suffering from Fenland fever and requires treatment costing £100. With probability $1-p$ he is suffering only from Steppe syndrome and will get better anyway. A test exists which infallibly gives positive results if the patient is suffering from Fenland fever but also has probability $q$ of giving positive results if the patient is not. The test costs £10. The doctor decides to proceed as follows. She will give the test repeatedly until *either* the last test is negative, in which case she dismisses the patient with kind words, *or* she has given the test $n$ times with positive results each time, in which case she gives the treatment. In the case $n=0$, she treats the patient at once. She wishes to minimise the expected cost £$E_n$, to the National Health Service.

(i) Show that

$$E_{n+1}-E_n=10p-10(1-p)q^n(9-10q),$$

and deduce that if $p=10^{-4}$, $q=10^{-2}$, she should choose $n=3$.

(ii) Show that if $q$ is larger than some fixed value $q_0$, to be determined explicitly, then, whatever the value of $p$, she should choose $n=0$.

[STEP 3, 1987Q16]

(i) $X_1, X_2, \dots, X_n$ are independent identically distributed random variables drawn from a uniform distribution on $[0,1]$ The random variables $A$ and $B$ are defined by

$$A = \min(X_1, \dots, X_n), \qquad B = \max(X_1, \dots, X_n).$$

For any fixed $k$, such that $0 < k < \frac{1}{2}$, let

$$p_n = \mathrm{P}(A \le k \text{ and } B \ge 1 - k).$$

What happens to $p_n$ as $n \to \infty$? Comment briefly on this result.

(ii) Lord Copper, the celebrated and imperious newspaper proprietor, has decided to run a lottery in which each of the 4 000 000 readers of his newspaper will have an equal probability $p$ of winning £1 000 000 and their chances of winning will be independent. He has fixed all the details leaving to you, his subordinate, only the task of choosing $p$. If nobody wins £1 000 000, you will be sacked, and if more than two readers win £1 000 000, you will also be sacked. Explaining your reasoning, show that however you choose $p$, you will have less than a 60% chance of keeping your job.

# STEP 3 1988

STEP 3 1988
On-line Exam

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**TIME ALLOWED: 180 MINUTES**

**INSTRUCTIONS TO CANDIDATES**

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**INFORMATION FOR CANDIDATES**

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Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

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You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

[STEP 3, 1988Q1]

Sketch the graph of

$$y = \frac{x^2 \mathrm{e}^{-x}}{1+x},$$

for $-\infty < x < \infty$.

Show that the value of

$$\int_0^{\infty} \frac{x^2 \mathrm{e}^{-x}}{1+x} \mathrm{d}x$$

lies between 0 and 1.

[STEP 3, 1988Q2]

The real numbers $u_0, u_1, u_2, \ldots$ satisfy the difference equation

$$a u_{n+2} + b u_{n+1} + c u_n = 0 \qquad (n = 0, 1, 2, \ldots),$$

where $a$, $b$ and $c$ are real numbers such that the quadratic equation

$$ax^2 + bx + c = 0$$

has two distinct real roots $\alpha$ and $\beta$. Show that the above difference equation is satisfied by the numbers $u_n$ defined by

$$u_n = A\alpha^n + B\beta^n,$$

where

$$A = \frac{u_1 - \beta u_0}{\alpha - \beta} \quad \text{and} \quad B = \frac{u_1 - \alpha u_0}{\beta - \alpha}.$$

Show also, by induction, that these numbers provide the only solution.

Find the numbers $v_n$ $(n = 0, 1, 2, \ldots)$ which satisfy

$$8(n+2)(n+1)v_{n+2} - 2(n+3)(n+1)v_{n+1} - (n+3)(n+2)v_n = 0$$

with $v_0 = 0$ and $v_1 = 1$.

[STEP 3, 1988Q3]

Give a parametric form for the curve in the Argand diagram determined by $|z - \mathbf{i}| = 2$.

Let $w = \frac{z+\mathbf{i}}{z-\mathbf{i}}$. Find and sketch the locus, in the Argand diagram, of the point which represents the complex number $w$ when

(i) $|z - \mathbf{i}| = 2$.

(ii) $z$ is real.

(iii) $z$ is imaginary.

[STEP 3, 1988Q4]

A kingdom consists of a vast plane with a central parabolic hill. In a vertical cross-section through the centre of the hill, with the $x$-axis horizontal and the $z$-axis vertical, the surface of the plane and hill is given by

$$z = \begin{cases} \dfrac{1}{2a}(a^2 - x^2) & \text{for } |x| \leq a, \\ 0 & \text{for } |x| > a. \end{cases}$$

The whole surface is formed by rotating this cross-section about the $z$-axis. In the $(x, z)$ plane through the centre of the hill, the king has a summer residence at $(-R, 0)$ and a winter residence at $(R, 0)$, where $R > a$. He wishes to connect them by a road, consisting of the following segments:

(i) a path in the $(x, z)$ plane joining $(-R, 0)$ to $\left(-b, \frac{a^2-b^2}{2a}\right)$, where $0 \leq b \leq a$.

(ii) a horizontal semicircular path joining the two points $\left(\pm b, \frac{a^2-b^2}{2a}\right)$, if $b \neq 0$.

(iii) a path in the $(x, z)$ plane joining $\left(b, \frac{a^2-b^2}{2a}\right)$ to $(R, 0)$.

The king wants the road to be as short as possible. Advise him on his choice of $b$.

[STEP 3, 1988Q5]

A firm of engineers obtains the right to dig and exploit an undersea tunnel. Each day the firm borrows enough money to pay for the day's digging, which costs £$c$, and to pay the daily interest of $100k\%$ on the sum already borrowed. The tunnel takes $T$ days to build, and, once finished, earns £$d$ a day, all of which goes to pay the daily interest and repay the debt until it is fully paid. The financial transactions take place at the end of each day's work. Show that $S_n$, the total amount borrowed by the end of day $n$, is given by

$$S_n = \frac{c[(1+k)^n - 1]}{k}$$

for $n \leq T$.

Given that $S_{T+m} > 0$, where $m > 0$, express $S_{T+m}$ in terms of $c$, $d$, $k$, $T$ and $m$.

Show that, if $\frac{d}{c} > (1+k)^T - 1$, the firm will eventually pay off the debt.

[STEP 3, 1988Q6]

Let $f(x) = \sin 2x \cos x$. Find the 1988th derivative of $f(x)$.

Show that the smallest positive value of $x$ for which this derivative is zero is $\frac{\pi}{3} + \epsilon$, where $\epsilon$ is approximately equal to

$$\frac{3^{-1989}\sqrt{3}}{2}.$$

[STEP 3, 1988Q7]

For $n = 0, 1, 2, \ldots$, the functions $y_n(x)$ satisfy the differential equation

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - \omega^2 x^2 y_n = -(2n+1)\omega y_n \,,$$

where $\omega$ is a positive constant, and $y_n \to 0$ and $\frac{\mathrm{d}y_n}{\mathrm{d}x} \to 0$ as $x \to +\infty$ and as $x \to -\infty$. Verify that these conditions are satisfied, for $n = 0$ and $n = 1$, by

$$y_0(x) = \mathrm{e}^{-\lambda x^2} \qquad \text{and} \qquad y_1(x) = x\mathrm{e}^{-\lambda x^2}$$

for some constant $\lambda$, to be determined.

Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y_m \frac{\mathrm{d}y_n}{\mathrm{d}x} - y_n \frac{\mathrm{d}y_m}{\mathrm{d}x}\right] = 2(m-n)\omega y_m y_n \,,$$

and deduce that, if $m \neq n$,

$$\int_{-\infty}^{\infty} y_m(x) y_n(x) \, \mathrm{d}x = 0.$$

[STEP 3, 1988Q8]

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

For $i = 1, 2$, and 3, let $P_i$ be the point $\left(at_i^2, 2at_i\right)$, where $t_1$, $t_2$ and $t_3$ are all distinct. Let $A_1$ be the area of the triangle formed by the tangents at $P_1$, $P_2$ and $P_3$, and let $A_2$ be the area of the triangle formed by the normals at $P_1$, $P_2$ and $P_3$. Using the fact that the area of the triangle with vertices at $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ is the absolute value of

$$\frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix},$$

show that $A_2 = (t_1 + t_2 + t_3)^2 A_1$.

Deduce a necessary and sufficient condition in terms of $t_1$, $t_2$ and $t_3$ for the normal at $P_1$, $P_2$ and $P_3$ to be concurrent.

[STEP 3, 1988Q9]

Let $G$ be a finite group with identity $e$. For each element $g$ in $G$, the order of $g$, $o(g)$, is defined to be the smallest positive integer $n$ for which $g^n = e$.

(i) Show that, if $o(g) = n$ and $g^N = e$, then $n$ divides $N$.

(ii) Let $g$ and $h$ be elements of $G$. Prove that, for any integer $m$,

$$gh^m g^{-1} = (ghg^{-1})^m.$$

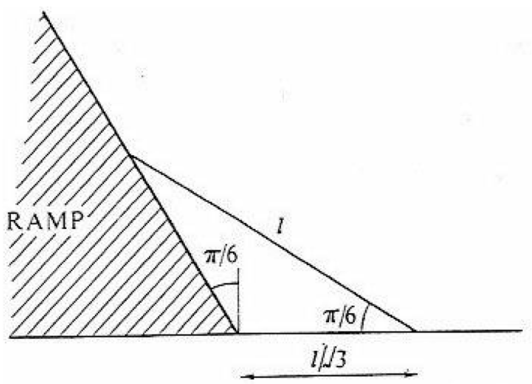
(iii) Let $g$ and $h$ be elements of $G$, such that $g^5 = e$, $h \neq e$ and $ghg^{-1} = h^2$. Prove that $g^2 h g^{-2} = h^4$ and find $o(h)$.

## [STEP 3, 1988Q10]

Four greyhounds, $A$, $B$, $C$ and $D$, are held at positions such that $ABCD$ is a large square. At a given instant, the dogs are released and $A$ runs directly towards $B$ at constant speed $v$, $B$ runs directly towards $C$ at constant speed $v$, and so on. Show that $A$'s path is given in polar coordinates (referred to an origin at the centre of the field and a suitable initial line) by $r = \lambda e^{-\theta}$, where $\lambda$ is a constant.

Generalise this result to the case of $n$ dogs held at the vertices of a regular $n$-gon ($n \geq 3$).

## [STEP 3, 1988Q11]




A uniform ladder of length $l$ and mass $m$ rests with one end in contact with a smooth ramp inclined at an angle of $\frac{\pi}{6}$ to the vertical. The foot of the ladder rests, on horizontal ground, at a distance $\frac{l}{\sqrt{3}}$ from the foot of the ramp, and the coefficient of friction between the ladder and the ground is $\mu$. The ladder is inclined at an angle $\frac{\pi}{6}$ to the horizontal, in the vertical plane containing a line of greatest slope of the ramp. A labourer of mass $m$ intends to climb slowly to the top of the ladder.

(i) Find the value of $\mu$ if the ladder slips as soon as the labourer reaches the midpoint.

(ii) Find the minimum value of $\mu$ which will ensure that the labourer can reach the top of the ladder.



## [STEP 3, 1988Q12]

A smooth billiard ball moving on a smooth horizontal table strikes another identical ball which is at rest. The coefficient of restitution between the balls is $e$ ($< 1$). Show that after the collision the angle between the velocities of the balls is less than $\frac{1}{2}\pi$.

Show also that the maximum angle of deflection of the first ball is

$$\sin^{-1}\left(\frac{1+e}{3-e}\right).$$

## [STEP 3, 1988Q13]

A goalkeeper stands on the goal-line and kicks the football directly into the wind, at an angle $\alpha$ to the horizontal. The ball has mass $m$ and is kicked with velocity $v_0$. The wind blows horizontally with constant velocity $w$ and the air resistance on the ball is $mk$ times its velocity relative to the wind velocity, where $k$ is a positive constant. Show that the equation of motion of the ball can be written in the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} + kv = g + kw,$$

where $v$ is the ball's velocity relative to the ground, and $g$ is the acceleration due to gravity.

By writing down horizontal and vertical equations of motion for the ball, or otherwise, find its position at time $t$ after it was kicked.

On the assumption that the goalkeeper moves out of the way, show that if $\tan\alpha = \frac{|g|}{k|w|}$, then the goalkeeper scores an own goal.

## [STEP 3, 1988Q14]

A small heavy bead can slide smoothly in a vertical plane on a fixed wire with equation

$$y = x - \frac{x^2}{4a},$$

where the $y$-axis points vertically upwards and $a$ is a positive constant. The bead is projected from the origin with initial speed $V$ along the wire.

(i) Show that for a suitable value of $V$, to be determined, a motion is possible throughout which the bead exerts no pressure on the wire.

(ii) Show that $\theta$, the angle between the particle's velocity at time $t$ and the $x$-axis, satisfies

$$\frac{4a^2\dot{\theta}^2}{\cos^6\theta} + 2ga(1 - \tan^2\theta) = V^2.$$

[STEP 3, 1988Q15]

Each day, books returned to a library are placed on a shelf in order of arrival, and left there. When a book arrives for which there is no room on the shelf, that book and all books subsequently returned are put on a trolley. At the end of each day, the shelf and trolley are cleared. There are just two-sizes of book: thick, requiring two units of shelf space; and thin, requiring one unit. The probability that a returned book is thick is $p$, and the probability that it is thin is $q = 1 - p$. Let $M(n)$ be the expected number of books that will be put on the shelf, when the length of the shelf is $n$ units and $n$ is an integer, on the assumption that more books will be returned each day than can be placed on the shelf. Show, giving reasoning, that:

(i) $M(0) = 0$.

(ii) $M(1) = q$.

(iii) $M(n) - qM(n-1) - pM(n-2) = 1$, for $n \geq 2$.

Verify that a possible solution to these equations is

$$M(n) = A(-p)^n + B + Cn,$$

where $A$, $B$ and $C$ are numbers independent of $n$ which you should express in terms of $p$.

[STEP 3, 1988Q16]

Balls are chosen at random without replacement from an urn originally containing $m$ red balls and $M - m$ green balls. Find the probability that exactly $k$ red balls will be chosen in $n$ choices $(0 \leq k \leq m, 0 \leq n \leq M)$.

The random variables $X_i$ $(i = 1, 2, \dots, n)$ are defined for $n \leq M$ by

$$X_i = \begin{cases} 0 & \text{if the } i\text{th ball chosen is green} \\ 1 & \text{if the } i\text{th ball chosen is red} \end{cases}$$

Show that

(i) $\mathrm{P}\{X_i = 1\} = \frac{m}{M}$,

(ii) $\mathrm{P}\{X_i = 1 \text{ and } X_j = 1\} = \frac{m(m-1)}{M(M-1)}$, for $i \neq j$.

Find the mean and variance of the random variable $X$ defined by

$$X = \sum_{i=1}^{n} X_i.$$

# STEP 3 1989

STEP 3 1989

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## [STEP 3, 1989Q1]

Prove that the area of the zone of the surface of a sphere between two parallel planes cutting the sphere is given by

$$2\pi \times (\text{radius of sphere}) \times (\text{perpendicular distance between the planes}).$$

A tangent from the origin $O$ to the curve with cartesian equation

$$(x-c)^2 + y^2 = a^2,$$

where $a$ and $c$ are positive constants with $c > a$, touches the curve at $P$. The $x$-axis cuts the curve at $Q$ and $R$, the points lying in the order $OQR$ on the axis. The line $OP$ and the arc $PR$ are rotated through $2\pi$ radians about the line $OQR$ to form a surface. Find the area of this surface.

## [STEP 3, 1989Q2]

The points $A$, $B$ and $C$ lie on the surface of the ground, which is an inclined plane. The point $B$ is 100m due north of $A$, and $C$ is 60m due east of $B$. The vertical displacements from $A$ to $B$, and from $B$ to $C$, are each 5m downwards. A plane coal seam lies below the surface and is to be located by making vertical bore-holes at $A$, $B$, and $C$. The bore-holes strike the coal seam at 95m, 45m and 76m below $A$, $B$ and $C$ repectively. Show that the coal seam is inclined at $\cos^{-1}\left(\frac{4}{5}\right)$ to the horizontal.

The coal seam comes to the surface along a line. Find the bearing of this line.



## [STEP 3, 1989Q3]

The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} \cos\dfrac{2\pi}{m} & -\sin\dfrac{2\pi}{m} \\ \sin\dfrac{2\pi}{m} & \cos\dfrac{2\pi}{m} \end{pmatrix},$$

where $m$ is an integer greater than 1. Prove that

$$\mathbf{M}^{m-1} + \mathbf{M}^{m-2} + \cdots + \mathbf{M}^2 + \mathbf{M} + \mathbf{I} = \mathbf{O},$$

where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The sequence $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2$ ... is defined by

$$\mathbf{X}_{k+1} = \mathbf{P}\mathbf{X}_k + \mathbf{Q},$$

where $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{X}_0$ are given $2 \times 2$ matrices. Suggest a suitable expression for $\mathbf{X}_k$ in terms of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{X}_0$, and justify it by induction.

The binary operation $*$ is defined as follows:

$$\mathbf{X}_i * \mathbf{X}_i \quad \text{is the result of substituting } \mathbf{X}_j \text{ for } \mathbf{X}_0 \text{ in the expression for } \mathbf{X}_i.$$

Show that if $\mathbf{P} = \mathbf{M}$, the set $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots\}$ forms a finite group under the operation $*$.

[STEP 3, 1989Q4]

Sketch the curve whose cartesian equation is

$$y = \frac{2x(x^2-5)}{x^2-4},$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence, or otherwise, determine (giving reasons) the number of real roots of the following equations:

(i) $4x(x^2-5) = (5x-2)(x^2-4)$.

(ii) $4x^4(x^2-5)^2 = (x^2-4)^2(x^2+1)$.

(iii) $4z^2(z-5)^2 = (z-4)^2(z+1)$.

[STEP 3, 1989Q5]

Given that $y = \cosh(n\cosh^{-1}x)$, for $x \geq 1$, prove that

$$y = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right].$$

Explain why, when $n = 2k+1$ and $k \in \mathbb{Z}^+$, $y$ can also be expressed as the polynomial

$$a_0x + a_1x^3 + a_2x^5 + \cdots + a_kx^{2k+1}.$$

Find $a_0$, and show that:

(i) $a_1 = (-1)^{k-1}\frac{2k(k+1)(2k+1)}{3}$.

(ii) $a_2 = (-1)^k\frac{2(k-1)k(k+1)(k+2)(2k+1)}{15}$.

Find also the value of $\sum_{r=0}^{k} a_r$.

[STEP 3, 1989Q6]

Show that, for a given constant $\gamma$ ($\sin\gamma \neq 0$) and with suitable choice of the constants $A$ and $B$, the line with cartesian equation $lx + my = 1$ has polar equation

$$\frac{1}{r} = A\cos\theta + B\cos(\theta-\gamma).$$

The distinct points $P$ and $Q$ on the conic with polar equation

$$\frac{a}{r} = 1 + e\cos\theta$$

correspond to $\theta = \gamma - \delta$ and $\theta = \gamma + \delta$ respectively, and $\cos\delta \neq 0$. Obtain the polar equation of the chord $PQ$. Hence, or otherwise, obtain the equation of the tangent at the point where $\theta = \gamma$.

The tangents at $L$ and $M$ to a conic with focus $S$ meet at $T$. Show that $ST$ bisects the angle $LSM$ and find the position of the intersection of $ST$ and $LM$ in terms of your chosen parameters for $L$ and $M$.

## [STEP 3, 1989Q7]

The linear transformation $T$ is a shear which transforms a point $P$ to the point $P'$ defined by:

(i) $\overrightarrow{PP'}$ makes an acute non-zero angle $\alpha$ (anticlockwise) with the $x$-axis,

(ii) $\angle POP'$ is clockwise (i.e. the rotation from $OP$ to $OP'$ clockwise is less than $\pi$),

(iii) $PP' = k \times PN$, where $PN$ is the perpendicular onto the line $y = x\tan\alpha$, where $k$ is a given non-zero constant.

If $T$ is represented in matrix form by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}$, show that

$$\mathbf{M} = \begin{pmatrix} 1 - k\sin\alpha\cos\alpha & k\cos^2\alpha \\ -k\sin^2\alpha & 1 + k\sin\alpha\cos\alpha \end{pmatrix}.$$

Show that the necessary and sufficient condition for $\begin{pmatrix} p & q \\ r & t \end{pmatrix}$ to commute with $\mathbf{M}$ is

$$t - p = 2q\tan\alpha = -2r\cot\alpha.$$

## [STEP 3, 1989Q8]

Given that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4(x - y) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x - 12(\mathrm{e}^{2t} + \mathrm{e}^{-2t}),$$

obtain a differential equation for $x$ which does not contain $y$. Hence, or otherwise, find $x$ and $y$ in terms of $t$ given that $x = y = 0$ when $t = 0$.



## [STEP 3, 1989Q9]

Obtain the sum to infinity of each of the following series.

(i) $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots + \frac{r}{2^{r-1}} + \cdots.$

(ii) $1 + \frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2^2} + \cdots + \frac{1}{r} \times \frac{1}{2^{r-1}} + \cdots.$

(iii) $\frac{1\times3}{2!} \times \frac{1}{3} + \frac{1\times3\times5}{3!} \times \frac{1}{3^2} + \cdots + \frac{1\times3\times\cdots\times(2k-1)}{k!} \times \frac{1}{3^{k-1}} + \cdots.$

[Questions of convergence need not be considered.]

[STEP 3, 1989Q10]

(i) Prove that

$$\sum_{r=1}^{n} r(r+1)(r+2)(r+3)(r+4) = \frac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5)$$

and deduce that

$$\sum_{r=1}^{n} r^5 < \frac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5).$$

(ii) Prove that, if $n > 1$,

$$\sum_{r=0}^{n-1} r^5 > \frac{1}{6}(n-5)(n-4)(n-3)(n-2)(n-1)\,n.$$

(iii) Let $f$ be an increasing function. If the limits

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{a}{n}f\left(\frac{ra}{n}\right) \quad \text{and} \quad \lim_{n\to\infty}\sum_{r=1}^{n}\frac{a}{n}f\left(\frac{ra}{n}\right)$$

both exist and are equal, the definite integral $\int_0^a f(x)\,\mathrm{d}x$ is defined to be their common value. Using this definition, prove that

$$\int_0^a x^5\,\mathrm{d}x = \frac{a^6}{6}.$$



[STEP 3, 1989Q11]

A smooth uniform sphere, with centre $A$, radius $2a$ and mass $3m$, is suspended from a fixed point $O$ by means of a light inextensible string, of length $3a$, attached to its surface at $C$. A second smooth uniform sphere, with centre $B$, radius $3a$ and mass $25m$, is held with its surface touching $O$ and with $OB$ horizontal. The second sphere is released from rest, falls and strikes the first sphere. The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speed $U$ of $A$ immediately after the impact in terms of the speed $V$ of $B$ immediately before impact.

The same system is now set up with a light rigid rod replacing the string and rigidly attached to the sphere so that $OCA$ is a straight line. The rod can turn freely about $O$. The sphere with centre $B$ is dropped as before. Show that the speed of $A$ immediately after impact is $\frac{125U}{127}$.

[STEP 3, 1989Q12]

A smooth horizontal plane rotates with constant angular velocity $\Omega$ about a fixed vertical axis through a fixed point $O$ of the plane. The point $A$ is fixed in the plane and $OA = a$. A particle $P$ lies on the plane and is joined to $A$ by a light rod of length $b$ $(b < a)$ freely pivoted at $A$. Initially $OAP$ is a straight line and $P$ is moving with speed $\left(a + 2\sqrt{ab}\right)\Omega$ perpendicular to $OP$ in the same sense as $\Omega$. At time $t$, $AP$ makes an angle $\phi$ with $OA$ produced. Obtain an expression for the component of the acceleration of $P$ perpendicular to $AP$ in terms of $\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}$, $\phi$, $a$, $b$ and $\Omega$.

Hence find $\frac{\mathrm{d}\phi}{\mathrm{d}t}$, in terms of $\phi$, $a$, $b$ and $\Omega$, and show that $P$ never crosses $OA$.

[STEP 3, 1989Q13]

The points $A, B, C, D$ and $E$ lie on a thin smooth horizontal table and are equally spaced on a circle with centre $O$ and radius $a$. At each of these points there is a small smooth hole in the table. Five elastic strings are threaded through the holes, one end of each being attached at $O$ under the table and the other end of each being attached to a particle $P$ of mass $m$ on top of the table. Each of the strings has natural length $a$ and modulus of elasticity $\lambda$. If $P$ is displaced from $O$ to any point $F$ on the table and released from rest, show that $P$ moves with simple harmonic motion of period $T$, where

$$T = 2\pi\sqrt{\frac{am}{5\lambda}}.$$

The string $PAO$ is replaced by one of natural length $a$ and modulus $k\lambda$. $P$ is displaced along $OA$ from its equilibrium position and released. Show that $P$ still moves in a straight line with simple harmonic motion, and, given that the period is $\frac{T}{2}$, find $k$.

[STEP 3, 1989Q14]

(i) A solid circular disc has radius $a$ and mass $m$. The density is proportional to the distance from the centre $O$. Show that the moment of inertia about an axis through $O$ perpendicular to the plane of the disc is $\frac{3}{5}ma^2$.

(ii) A light inelastic string has one end fixed at $A$. It passes under and supports a smooth pulley $B$ of mass $m$. It then passe over a rough pulley $C$ which is a disc of the type described in (i), free to turn about its axis which is fixed and horizontal. The string carries a particle $D$ of mass $M$ at its other end. The sections of the string which are not in contact with the pulleys are vertical. The system is released from rest and moves under gravity for $t$ seconds. At the end of this interval the pulley $B$ is suddenly stopped. Given that $m < 2M$, find the resulting impulse on $D$ in terms of $m$, $M$, $g$ and $t$.

[You may assume that the string is long enough for there to be no collisions between the elements of the system, and that the pulley $C$ is rough enough to prevent slipping throughout.]

[STEP 3, 1989Q15]

The continuous random variable $X$ is uniformly distributed over the interval $[-c, c]$. Write down expressions for the probabilities that:

(i) $n$ independently selected values of $X$ are all greater than $k$,

(ii) $n$ independently selected values of $X$ are all less than $k$,

where $k$ lies in $[-c, c]$.

A sample of $2n+1$ values of $X$ is selected at random and $Z$ is the median of the sample. Show that $Z$ is distributed over $[-c, c]$ with probability density function

$$\frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}(c^2 - z^2)^n.$$

Deduce the value of $\int_{-c}^{c}(c^2 - z^2)^n \, \mathrm{d}z$.

Evaluate $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.

[STEP 3, 1989Q16]

It is believed that the population of Ruritania can be described as follows:

(i) 25% are fair-haired and the rest are dark-haired.

(ii) 20% are green-eyed and the rest hazel-eyed.

(iii) the population can also be divided into narrow-headed and broad-headed.

(iv) no narrow-headed person has green eyes and fair hair.

(v) those who are green-eyed are as likely to be narrow-headed as broad-headed.

(vi) those who are green-eyed and broad-headed are as likely to be fair-haired as dark-haired.

(vii) half the population is broad-headed and dark-haired.

(viii) a hazel-eyed person is as likely to be fair-haired and broad-headed as dark-haired and narrow-headed.

Find the proportion believed to be narrow-headed.

I am acquainted with only six Ruritanians, all of whom are broad-headed. Comment on this observation as evidence for or against the given model.

A random sample of 200 Ruritanians is taken and is found to contain 50 narrow-heads. On the basis of the given model, calculate (to a reasonable approximation) the probability of getting 50 or fewer narrow-heads. Comment on the result.

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## [STEP 3, 1990Q1]

Show, using de Moivre's Theorem, or otherwise, that

$$\tan 9\theta = \frac{t(t^2-3)(t^6-33t^4+27t^2-3)}{(3t^2-1)(3t^6-27t^4+33t^2-1)},$$

where $t = \tan\theta$.

By considering the equation $\tan 9\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{9}\right), \qquad \tan^2\left(\frac{2\pi}{9}\right) \qquad \text{and} \qquad \tan^2\left(\frac{4\pi}{9}\right).$$

Deduce the value of

$$\tan\left(\frac{\pi}{9}\right)\tan\left(\frac{2\pi}{9}\right)\tan\left(\frac{4\pi}{9}\right).$$

Show that

$$\tan^6\left(\frac{\pi}{9}\right) + \tan^6\left(\frac{2\pi}{9}\right) + \tan^6\left(\frac{4\pi}{9}\right) = 33273.$$

## [STEP 3, 1990Q2]

The distinct points $O(0,0,0)$, $A(a^3, a^2, a)$, $B(b^3, b^2, b)$ and $C(c^3, c^2, c)$ lie in three-dimensional space.

(i) Prove that the lines $OA$ and $BC$ do not intersect.

(ii) Given that $a$ and $b$ can vary with $ab = 1$, show that $\widehat{AOB}$ can take any value in the interval $0 < \widehat{AOB} < \frac{\pi}{2}$, but no others.

## [STEP 3, 1990Q3]

The elements $a, b, c, d$ belong to the group $\mathcal{G}$ with binary operation $*$. Show that

(i) if $a$, $b$ and $a * b$ are of order 2, then $a$ and $b$ commute.

(ii) $c * d$ and $d * c$ have the same order.

(iii) if $c^{-1} * b * c = b^r$, then $c^{-1} * b^s * c = b^{sr}$ and $c^{-n} * b^s * c^n = b^{sr^n}$.

[STEP 3, 1990Q4]

Given that $\sin\beta \neq 0$, sum the series

$$\cos\alpha + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + 2r\beta) + \cdots + \cos(\alpha + 2n\beta)$$

and

$$\cos\alpha + \binom{n}{1}\cos(\alpha + 2\beta) + \cdots + \binom{n}{r}\cos(\alpha + 2r\beta) + \cdots + \cos(\alpha + 2n\beta).$$

Given that $\sin\theta \neq 0$. Prove that

$$1 + \cos\theta\sec\theta + \cos 2\theta\sec^2\theta + \cdots + \cos r\theta\sec^r\theta + \cdots + \cos n\theta\sec^n\theta = \frac{\sin(n+1)\theta\sec^n\theta}{\sin\theta}.$$

[STEP 3, 1990Q5]

Prove that, for any integers $n$ and $r$, with $1 \leq r \leq n$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

Hence or otherwise, prove that

$$(uv)^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \cdots + uv^{(n)},$$

where $u$ and $v$ are functions of $x$ and $z^{(r)}$ means $\frac{\mathrm{d}^r z}{\mathrm{d}x^r}$.

Prove that, if $y = \sin^{-1} x$, then $(1 - x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2 y^{(n)} = 0$.



[STEP 3, 1990Q6]

The transformation $T$ from $\binom{x}{y}$ to $\binom{X}{Y}$ is given by

$$\binom{X}{Y} = \frac{2}{5}\begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}\binom{x}{y}.$$

Show that $T$ leaves the vector $\binom{1}{2}$ unchanged in direction but multiplied by a scalar, and that $\binom{2}{-1}$ is similarly transformed.

The circle $C$ whose equation is $x^2 + y^2 = 1$ transforms under $T$ to a curve $E$. Show that $E$ has equation

$$8X^2 + 12XY + 17Y^2 = 80,$$

and state the area of the region bounded by $E$. Show also that the greatest value of $X$ on $E$ is $2\sqrt{\frac{17}{5}}$.

Find the equation of the tangent to $E$ at the point which corresponds to the point $\left(\frac{3}{5}, \frac{4}{5}\right)$ on $C$.

[STEP 3, 1990Q7]

The points $P(0,a)$, $Q(a,0)$ and $R(a,-a)$ lie on the curve $C$ with cartesian equation

$$xy^2 + x^3 + a^2y - a^3 = 0, \qquad \text{where } a > 0.$$

At each of $P$, $Q$ and $R$, express $y$ as a Taylor series in $h$, where $h$ is a small increment in $x$, as far as the term in $h^2$. Hence, or otherwise, sketch the shape of $C$ near each of these points.

Show that, if $(x,y)$ lies on $C$, then

$$4x^4 - 4a^3x - a^4 \leq 0.$$

Sketch the graph of $y = 4x^4 - 4a^3x - a^4$.

Given that the $y$-axis is an asymptote to $C$, sketch the curve $C$.

[STEP 3, 1990Q8]

Let $P$, $Q$ and $R$ be functios of $x$. Prove that, for any function $y$ of $x$, the function $Py'' + Qy' + Ry$ can be written in the form $\frac{\mathrm{d}}{\mathrm{d}x}(py' + qy)$, where $p$ and $q$ are finctions of $x$, if and only if $P'' - Q' + R \equiv 0$.

Solve the differential equation

$$(x - x^4)y'' + (1 - 7x^3)y' - 9x^2y = (x^3 + 3x^2)\mathrm{e}^x,$$

given that when $x = 2$, $y = 2\mathrm{e}^2$ and $y' = 0$.



[STEP 3, 1990Q9]

The real variables $\theta$ and $u$ are related by the equation $\tan\theta = \sinh u$ and $0 \leq \theta < \frac{\pi}{2}$ Let $v = \operatorname{sech} u$. Prove that

(i) $v = \cos\theta$.

(ii) $\frac{\mathrm{d}\theta}{\mathrm{d}u} = v$.

(iii) $\sin 2\theta = -2\frac{\mathrm{d}v}{\mathrm{d}u}$ and $\cos 2\theta = -\cosh u \frac{\mathrm{d}^2v}{\mathrm{d}u^2}$.

(iv) $\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2} + \frac{\mathrm{d}v}{\mathrm{d}\theta}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = 0$.

[STEP 3, 1990Q10]

By considering the graphs of $y = kx$ and $y = \sin x$, show that the equation $kx = \sin x$, where $k > 0$, may have 0, 1, 2 or 3 roots in the interval $(4n+1)\frac{\pi}{2} < x < (4n+5)\frac{\pi}{2}$, where $n$ is a positive integer.

For a certain given value of $n$, the equation has exactly one root in this interval. Show that $k$ lies in an interval which may be written $\sin\delta < k < \frac{2}{(4n+1)\pi}$, where $0 < \delta < \frac{\pi}{2}$ and

$$\cos\delta = \left((4n+5)\frac{\pi}{2} - \delta\right)\sin\delta.$$

Show that, if $n$ is large, then $\delta \approx \frac{2}{(4n+5)\pi}$ and obtain a second, improved, approximation.

[STEP 3, 1990Q11]

The points $O$, $A$, $B$ and $C$ are the vertices of a uniform square lamina of mass $M$. The lamina can turn freely under gravity about a horizontal axis perpendicular to the plane of the lamina through $O$. The sides of the lamina are of length $2a$. When the lamina is hanging at rest with the diagonal $OB$ vertically downwards it is struck at the mid-point of $OC$ by a particle of mass $6M$ moving horizontally in the plane of the lamina with speed $V$. The particle adheres to the lamina. Find, in terms of $a$, $M$ and $g$, the value which $V^2$ must exceed for the lamina and particle to make complete revolutions about the axis.



[STEP 3, 1990Q12]

A uniform smooth wedge of mass $m$ has congruent triangular end faces $A_1B_1C_1$ and $A_2B_2C_2$, and $A_1A_2$, $B_1B_2$ and $C_1C_2$ are perpendicular to these faces. The points $A$, $B$ and $C$ are the mid-points of $A_1A_2$, $B_1B_2$ and $C_1C_2$ respectively. The sides of the triangle $ABC$ have lengths $AB = AC = 5a$ and $BC = 6a$. The wedge is placed with $BC$ on a smooth horizontal table, a particle of mass $2m$ is placed at $A$ on $AC$, and the system is released from rest. The particle slides down $AC$, strikes the table, bounces perfectly elastically and lands again on the table at $D$. At this time the point $C$ of the wedge has reached the point $E$; show that $DE = \frac{192a}{19}$.

[STEP 3, 1990Q13]

A particle $P$ is projected, from the lowest point, along the smooth inside surface of a fixed sphere with centre $O$. It leaves the surface when $OP$ makes an angle $\theta$ with the upward vertical. Find the smallest angle that must be exceeded by $\theta$ to ensure that $P$ will next strike the surface below the level of $O$.

[You may find it helpful to find the time at which the particle strikes the sphere.]

## [STEP 3, 1990Q14]

The edges $OA$, $OB$, $OC$ of a rigid cube are taken as coordinate axes and $O'$, $A'$, $B'$, $C'$are the vertices diagonally opposite $O, A, B, C$, respectively. The four forces acting on the cube are $\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$ at $O(0,0,0)$, $\begin{pmatrix}\lambda\\0\\1\end{pmatrix}$ at $O'(a,a,a)$, $\begin{pmatrix}-1\\0\\2\end{pmatrix}$ at $B(0,a,0)$ and $\begin{pmatrix}1\\\mu\\\nu\end{pmatrix}$ at $B'(a,0,a)$. The moment of the system about $O$ is zero: find $\lambda$, $\mu$ and $\nu$.

(i) Given that $\alpha = \beta = \gamma = 0$, find the system consisting of a single force at $B$ together with a couple which is equivalent to the given system.

(ii) Given that $\alpha = 2, \beta = 3$ and $\gamma = 2$, find the equation of the locus about each point of which the moment of the system is zero. Find the number of units of work done on the cube when it moves (without rotation) a unit distance in the direction of this line under the action of the given forces only.

## [STEP 3, 1990Q15]

An unbiased twelve-sided die has its faces marked $A, A, A, B, B, B, B, B, B, B, B, B$. In a series of throws of the die the first $M$ throws show $A$, the next $N$ throws show $B$ and the $(M + N + 1)$th throw shows $A$. Write down the probability that $M = m$ and $N = n$, where $m \geq 0$ and $n \geq 1$.

Find (i) the marginal distributions of $M$ and $N$,

(ii) the mean values of $M$ and $N$.

Investigate whether $M$ and $N$ are independent.

Find the probability that $N$ is greater than a given integer $k$, where $k \geq 1$, and find $\mathrm{P}(N > M)$. Find also $\mathrm{P}(N = M)$ and show that $\mathrm{P}(N < M) = \frac{1}{52}$.



## [STEP 3, 1990Q16]

(i) A rod of unit length is cut into pieces of length $X$ and $1 - X$; the latter is then cut in half. The random variable $X$ is uniformly distributed over $[0, 1]$. For some values of $X$ a triangle can be formed from the three pieces of the rod. Show that the conditional probability that, if a triangle can be formed, it will be obtuse-angled is $3 - 2\sqrt{2}$.

(ii) The bivariate distribution of the random variables $X$ and $Y$ is uniform over the triangle with vertices $(1, 0)$, $(1, 1)$ and $(0, 1)$. A pair of values $x, y$ is chosen at random from this distribution and a (perhaps degenerate) triangle $ABC$ is constructed with $BC = x$ and $CA = y$ and $AB = 2 - x - y$. Show that the construction is always possible and that $\widehat{ABC}$ is obtuse if and only if

$$y > \frac{x^2 - 2x + 2}{2 - x}.$$

Deduce that the probability that $\widehat{ABC}$ is obtuse is $3 - 4\ln 2$.

# STEP 3 1991

STEP 3 1991
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**Calculators are not permitted.**

[STEP 3, 1991Q1]

(i) Evaluate

$$\sum_{r=1}^{n} \frac{6}{r(r+1)(r+3)}.$$

(ii) Expand $\ln(1+x+x^2+x^3)$ as a series in powers of $x$, where $|x|<1$, giving the first five non-zero terms and the general term.

(iii) Expand $e^{x\ln(1+x)}$ as a series in powers of $x$, where $-1<x\le 1$, as far as the term in $x^4$.

[STEP 3, 1991Q2]

The distinct points $P_1, P_2, P_3, Q_1, Q_2$ and $Q_3$ in the Argand diagram are represented by the complex numbers $z_1, z_2, z_3, w_1, w_2$ and $w_3$ respectively. Show that the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are similar, with $P_i$ corresponding to $Q_i$ $(i=1,2,3)$ and the rotation from 1 to 2 to 3 being in the same sense for both triangles, if and only if

$$\frac{z_1-z_2}{z_1-z_3}=\frac{w_1-w_2}{w_1-w_3}.$$

Verify that this condition may be written

$$\det\begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{pmatrix}=0.$$

(i) Show that $w_i=z_i^2$ $(i=1,2,3)$ then triangle $P_1P_2P_3$ is not similar to triangle $Q_1Q_2Q_3$.

(ii) Show that if $w_i=z_i^3$ $(i=1,2,3)$ then triangle $P_1P_2P_3$ is similar to triangle $Q_1Q_2Q_3$ if and only if the centroid of triangle $P_1P_2P_3$ is the origin. [The *centroid* of triangle $P_1P_2P_3$ is represented by the complex number $\frac{1}{3}(z_1+z_2+z_3)$.]

(iii) Show that the triangle $P_1P_2P_3$ is equilateral if and only if

$$z_2z_3+z_3z_1+z_1z_2=z_1^2+z_2^2+z_3^2.$$

[STEP 3, 1991Q3]

The function $f$ is defined for $x<2$ by

$$f(x)=2|x^2-x|+|x^2-1|-2|x^2+x|.$$

Find the maximum and minimum points and the points of inflection of the graph of $f$ and sketch this graph. Is $f$ continuous everywhere? Is $f$ differentiable everywhere?

Find the inverse of the function $f$, i.e. expressions for $f^{-1}(x)$, defined in the various appropriate intervals.

[STEP 3, 1991Q4]

The point $P$ moves on a straight line in three-dimensional space. The position of $P$ is observed from the points $O_1(0,0,0)$ and $O_2(8a,0,0)$. At times $t = t_1$ and $t = t_1'$, the lines of sight from $O_1$ are along the lines $\frac{x}{2} = \frac{z}{3}$, $y = 0$ and $x = 0$, $\frac{y}{3} = \frac{z}{4}$ respectively. At time $t = t_2$ and $t = t_2'$, the lines of sight from $O_2$ are $\frac{x-8a}{-3} = \frac{y}{1} = \frac{z}{3}$ and $\frac{x-8a}{-4} = \frac{y}{2} = \frac{z}{5}$ respectively. Find an equation or equations for the path of $P$.

[STEP 3, 1991Q5]

The curve $C$ has the differential equation in polar coordinates

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} + 4r = 5\sin 3\theta\,, \qquad \text{for} \quad \frac{\pi}{5} \le \theta \le \frac{3\pi}{5},$$

and, when $\theta = \frac{\pi}{2}$, $r = 1$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2$. Show that $C$ forms a closed loop and that the area of the region enclosed by $C$ is

$$\frac{\pi}{5} + \frac{25}{48}\left[\sin\left(\frac{\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right)\right].$$

[STEP 3, 1991Q6]

The transformation $T$ from $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x' \\ y' \end{pmatrix}$ in two-dimensional space is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $u$ is a positive real constant. Show that the curve with equation $x^2 - y^2 = 1$ is transformed into itself. Find the equations of two straight lines through the origin which transform into themselves.

A line, not necessarily through the origin, which has gradient $\tanh v$ transforms under $T$ into a line with gradient $\tanh v'$. Show that $v' = v + u$.

The lines $l_1$ and $l_2$ with gradients $\tanh v_1$ and $\tanh v_2$ transform under $T$ into lines with gradients $\tanh v_1'$ and $\tanh v_2'$ respectively. Find the relation satisfied by $v_1$ and $v_2$ that is the necessary and sufficient for $l_1$ and $l_2$ to intersect at the same angle as their transforms.

In the case when $l_1$ and $l_2$ meet at the origin, illustrate in a diagram the relation between $l_1$, $l_2$ and their transforms.

[STEP 3, 1991Q7]

(i) Prove that

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, dx = \int_0^{\frac{\pi}{2}} \ln \cos x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln \sin 2x \, dx - \frac{1}{4}\pi \ln 2.$$

and

$$\int_0^{\frac{\pi}{2}} \ln \sin 2x \, dx = \frac{1}{2}\int_0^{\pi} \ln \sin x \, dx = \int_0^{\frac{\pi}{2}} \ln \sin x \, dx.$$

Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \ln \sin x \, dx$.

[You may assume that all the integrals converge.]

(ii) Given that $\ln u < u$ for $u \geq 1$ deduce that

$$\frac{1}{2}\ln x < \sqrt{x} \quad \text{for} \quad x \geq 1.$$

Deduce that $\frac{\ln x}{x} \to 0$ as $x \to \infty$ and that $x \ln x \to 0$ as $x \to 0$ through positive values.

(iii) Using the results of parts (i) and (ii), or otherwise, evaluate $\int_0^{\frac{\pi}{2}} x \cot x \, dx$.

[STEP 3, 1991Q8]

(i) The integral $I_k$ is defined by

$$I_k = \int_0^{\theta} \cos^k x \cos kx \, dx.$$

Prove that $2kI_k = kI_{k-1} + \cos^k \theta \sin k\theta$.

(ii) Prove that

$$1 + m\cos 2\theta + \binom{m}{2}\cos 4\theta + \cdots + \binom{m}{r}\cos 2r\theta + \cdots + \cos 2m\theta = 2^m \cos^m \theta \cos m\theta.$$

(iii) Using the results of parts (i) and (ii), show that

$$m\frac{\sin 2\theta}{2} + \binom{m}{2}\frac{\sin 4\theta}{4} + \cdots + \binom{m}{r}\frac{\sin 2r\theta}{2r} + \cdots + \frac{\sin 2m\theta}{2m}$$
$$= \cos\theta \sin\theta + \cos^2\theta \sin 2\theta + \cdots + \frac{1}{r}2^{r-1}\cos^r\theta \sin r\theta + \cdots + \frac{1}{m}2^{m-1}\cos^m\theta \sin m\theta.$$

[STEP 3, 1991Q9]

The parametric equations $E_1$ and $E_2$ define the same ellipse, in terms of the parameters $\theta_1$ and $\theta_2$, (though not referred to the same coordinate axes).

$$E_1\colon\ x = a\cos\theta_1\,, \qquad y = b\sin\theta_1\,,$$
$$E_2\colon\ x = \frac{k\cos\theta_2}{1+e\cos\theta_2}, \qquad y = \frac{k\sin\theta_2}{1+e\cos\theta_2},$$

where $0 < b < a$, $0 < e < 1$ and $0 < k$. Find the position of the axes for $E_2$ relative to the axes for $E_1$ and show that $k = a(1-e^2)$ and $b^2 = a^2(1-e^2)$.

[The standard polar equation of an ellipse is $r = \frac{l}{1+e\cos\theta}$.]

By considering expressions for the length of the perimeter of the ellipse, or otherwise, prove that

$$\int_0^{\pi} \sqrt{1-e^2\cos^2\theta}\,\mathrm{d}\theta = \int_0^{\pi} \frac{1-e^2}{(1+e\cos\theta)^2}\sqrt{1+e^2+2e\cos\theta}\,\mathrm{d}\theta.$$

Given that $e$ is so small that $e^6$ may be neglected, show that the value of either integral is

$$\frac{1}{64}\pi(64-16e^2-3e^4).$$

[STEP 3, 1991Q10]

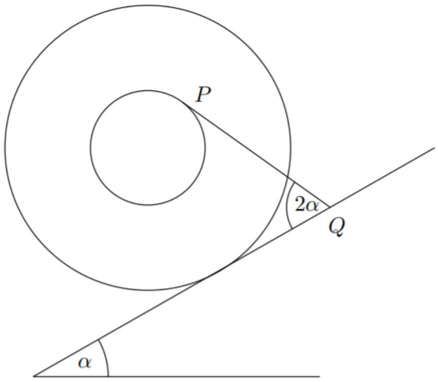
The equation $x^n - qx^{n-1} + r = 0$, where $n \geq 5$ and $q$ and $r$ are real constants, has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. The sum of the products of $m$ distinct roots is denoted by $\sum_m$ (so that, for example, $\sum_3 = \sum \alpha_i\alpha_j\alpha_k$ where the sum runs over all values of $i, j$ and $k$ with $n \geq i > j > k \geq 1$). The sum of the $m$th powers of the roots is denoted by $S_m$ (so that, for example, $S_3 = \sum_{i=1}^{n} \alpha_i^3$). Prove that $S_p = q^p$ for $1 \leq p \leq n-1$.

[You may assume that for any $n$th degree equation and $1 \leq p \leq n$

$S_p - S_{p-1}\sum_1 + S_{p-2}\sum_2 - \cdots + (-1)^{p-1}S_1\sum_{p-1} + (-1)^p p\sum_p = 0$.].

Find expressions for $S_n$, $S_{n+1}$ and $S_{n+2}$ in terms of $q$, $r$ and $n$. Suggest an expression for $S_{n+m}$, where $m < n$, and prove its validity by induction.

[STEP 3, 1991Q11]




A uniform circular cylinder of radius $2a$ with a groove of radius $a$ cut in its central cross-section has mass $M$. It rests, as shown in the diagram, on a rough plane inclined at an acute angle $\alpha$ to the horizontal. It is supported by a light inextensible string wound round the groove and attached to the cylinder at one end. The other end of the string is attached to the plane at $Q$, the free part of the string, $PQ$, making an angle $2\alpha$ with the inclined plane. The coefficient of friction at the contact between the cylinder and the plane is $\mu$. Show that $\mu \geq \frac{1}{3}\tan\alpha$.

The string $PQ$ is now detached from the plane and the end $Q$ is fastened to a particle of mass $3M$ which is placed on the plane, the position of the string remaining unchanged. Given that $\tan\alpha = \frac{1}{2}$ and that the system remains in equilibrium, find the least value of the coefficient of friction between the particle and the plane.



[STEP 3, 1991Q12]

A smooth tube whose axis is horizontal has an elliptic cross-section in the form of the curve with parametric equations

$$x = a\cos\theta \qquad y = b\sin\theta$$

where the $x$-axis is horizontal and the $y$-axis is vertically upwards. A particle moves freely under gravity on the inside of the tube in the plane of this cross-section. By first finding $\ddot{x}$ and $\ddot{y}$, or otherwise, show that the acceleration along the inward normal at the point with parameter $\theta$ is

$$\frac{ab\dot{\theta}^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.$$

The particle is projected along the surface in the vertical cross-section plane, with speed $2\sqrt{bg}$, from the lowest point. Given that $2a = 3b$, show that it will leave the surface at the point with parameter $\theta$ where

$$5\sin^3\theta + 12\sin\theta - 8 = 0.$$

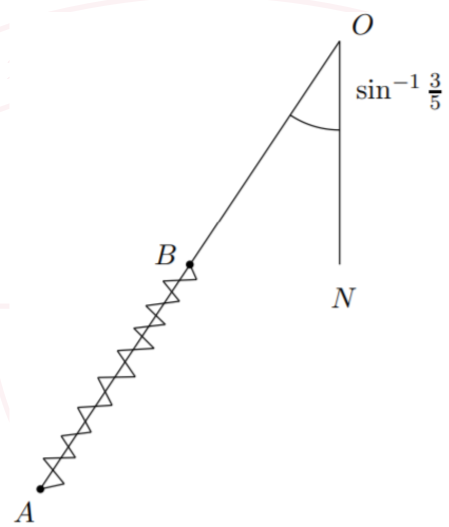
[STEP 3, 1991Q13]

A smooth particle $P_1$ is projected from a point $O$ on the horizontal floor of a room which has a horizontal ceiling at a height $h$ above the floor. The speed of projection is $\sqrt{8gh}$ and the direction of projection makes an acute angle $\alpha$ with the horizontal. The particle strikes the ceiling and rebounds, the impact being perfectly elastic. Show that for this to happen $\alpha$ must be at least $\frac{1}{6}\pi$ and that the range on the floor is then

$$8h\cos\alpha\left(2\sin\alpha - \sqrt{4\sin^2\alpha - 1}\right).$$

Another particle $P_2$ is projected from $O$ with the same velocity as $P_1$ but its impact with the ceiling is perfectly inelastic. Find the difference $D$ between the ranges of $P_1$ and $P_2$ on the floor and show that, as $\alpha$ varies, $D$ has a maximum value when $\alpha = \frac{1}{4}\pi$.

[STEP 3, 1991Q14]






The end $O$ of a smooth light rod $OA$ of length $2a$ is a fixed point. The rod $OA$ makes a fixed angle $\sin^{-1}\left(\frac{3}{5}\right)$ with the downward vertical $ON$, but is free to rotate about $ON$. A particle of mass $m$ is attached to the rod at $A$ and a small ring $B$ of mass $m$ is free to slide on the rod but is joined to $A$ by a spring of natural length $a$ and modulus of elasticity $kmg$. The vertical plane containing the rod $OA$ rotates about $ON$ with constant angular velocity $\sqrt{\frac{5g}{2a}}$ and $B$ is at rest relative to the rod. Show that the length of $OB$ is

$$\frac{(10k+8)a}{10k-9}.$$

Given that the reaction of the rod on the particle at $A$ makes an angle $\tan^{-1}\left(\frac{13}{21}\right)$ with the horizontal, find the value of $k$. Find also the magnitude of the reaction between the rod and the ring $B$.

[STEP 3, 1991Q15]

A pack of $2n$ (where $n \geq 4$) cards consists of two each of $n$ different sorts. If four cards are drawn from the pack without replacement show that the probability that no pairs of identical cards have been drawn is

$$\frac{4(n-2)(n-3)}{(2n-1)(2n-3)}.$$

Find the probability that exactly one pair of identical cards is included in the four.

If $k$ cards are drawn without replacement and $2 < k < 2n$, find an expression for the probability that there are exactly $r$ pairs of identical cards included when $r < \frac{1}{2}k$.

For even values of $k$ show that the probability that the drawn cards consist of $\frac{1}{2}k$ pairs is

$$\frac{1 \times 3 \times 5 \times \cdots \times (k-1)}{(2n-1)(2n-3)\cdots(2n-k+1)}.$$

[STEP 3, 1991Q16]

The random variables $X$ and $Y$ take integer values $x$ and $y$ respectively which are restricted by $x \geq 1$, $y \geq 1$ and $2x + y \leq 2a$ where $a$ is an integer greater than 1. The joint probability is given by

$$\mathrm{P}(X = x, Y = y) = c(2x + y),$$

where $c$ is a positive constant, within this region and zero elsewhere. Obtain, in terms of $x$, $c$ and $a$, the marginal probability $\mathrm{P}(X = x)$ and show that

$$c = \frac{6}{a(a-1)(8a+5)}.$$

Show that when $y$ is an even number the marginal probability $\mathrm{P}(Y = y)$ is

$$\frac{3(2a-y)(2a+2+y)}{2a(a-1)(8a+5)}$$

and find the corresponding expression when $y$ is odd.

Evaluate $\mathrm{E}(Y)$ in terms of $a$.

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[STEP 3, 1992Q1]

(i) Given that $f(x) = \ln(1 + e^x)$, prove that $\ln(f'(x)) = x - f(x)$ and that $f''(x) = f'(x) - [f'(x)]^2$. Hence, or otherwise, expand $f(x)$ as a series in power of $x$ up to the term in $x^4$.

(ii) Given that

$$g(x) = \frac{1}{\sinh x \cosh 2x},$$

explain why $g(x)$ can not be expanded as a series of non-negative powers of $x$ but that $xg(x)$ can be so expanded. Explain also why this latter expansion will consist of even powers of $x$ only. Expand $xg(x)$ as a series as far as the term in $x^4$.

[STEP 3, 1992Q2]

The matrices $\mathbf{I}$ and $\mathbf{J}$ are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ respectively and $\mathbf{A} = \mathbf{I} + a\mathbf{J}$, where $a$ is a non-zero real constant. Prove that $\mathbf{A}^2 = \mathbf{I} + \frac{1}{2}((1+2a)^2 - 1)\mathbf{J}$ and $\mathbf{A}^3 = \mathbf{I} + \frac{1}{2}((1+2a)^3 - 1)\mathbf{J}$ and obtain a similar form for $\mathbf{A}^4$.

If $\mathbf{A}^k = \mathbf{I} + p_k\mathbf{J}$, suggest a suitable form for $p_k$ and prove that it is correct by induction, or otherwise.

[STEP 3, 1992Q3]



Sketch the curve $C_1$ whose parametric equations are $x = t^2$, $y = t^3$.

The circle $C_2$ passes through the origin $O$. The points $R$ and $S$ with real non-zero parameters $r$ and $s$ respectively are other intersections of $C_1$ and $C_2$. Show that $r$ and $s$ are roots of an equation of the form

$$t^4 + t^2 + at + b = 0,$$

where $a$ and $b$ are real constants.

By obtaining a quadratic equation, with coefficients expressed in terms of $r$ and $s$, whose roots would be the parameters of any further intersections of $C_1$ and $C_2$, or otherwise, show that $O$, $R$ and $S$ are the only real intersections of $C_1$ and $C_2$.

[STEP 3, 1992Q4]

A set of curves $S_1$ is defined by the equation $y = \frac{x}{x-a}$, where $a$ is a constant which is different for different members of $S_1$. Sketch on the same axes the curves for which $a = -2, -1, 1$ and 2.

A second set of curve $S_2$ is such that at each intersection between a member of $S_2$ and a member of $S_1$ the tangents of the intersecting curves are perpendicular. On the same axes as the already sketched members of $S_1$, sketch the member of $S_2$ that passes through the point $(1, -1)$.

Obtain the first order differential equation for $y$ satisfied at all points on all members of $S_1$ (i.e. an equation connecting $x$, $y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ which does not involve $a$).

State the relationship between the values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ on two intersecting curves, one from $S_1$ and one from $S_2$, at their intersection. Hence show that the differential equation for the curves of $S_2$ is

$$x = y(y-1)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Find an equation for the member of $S_2$ that you have sketched.

[STEP 3, 1992Q5]

The tetrahedron $ABCD$ has $A$ at the point $(0, 4, -2)$. It is symmetrical about the plane $y + z = 2$, which passes through $A$ and $D$. The mid-point of $BC$ is $N$. The centre, $Y$, of the sphere $ABCD$ is at the point $(3, -2, 4)$ and lies on $AN$ such that $\overrightarrow{AY} = 3\overrightarrow{YN}$. Show that $BN = 6\sqrt{2}$ and find the coordinates of $B$ and $C$.

The angle $AYD$ is $\cos^{-1}\left(\frac{1}{3}\right)$. Find the coordinates of $D$.

[There are two alternative answers for each point.]

[STEP 3, 1992Q6]

Given that $I_n = \int_0^{\pi} \frac{x\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, where $n$ is a positive integer, show that $I_n - I_{n-1} = J_n$, where

$$J_n = \int_0^{\pi} \frac{x\sin(2n-1)x}{\sin x}\,\mathrm{d}x.$$

Obtain also a reduction formula for $J_n$.

The curve $C$ is given by the cartesian equation $y = \frac{x\sin^2(nx)}{\sin^2 x}$, where $n$ is a positive integer and $0 \le x \le \pi$. Show that the area under the curve $C$ is $\frac{n\pi^2}{2}$.

[STEP 3, 1992Q7]

The points $P$ and $R$ lie on the sides $AB$ and $AD$, respectively, of the parallelogram $ABCD$. The point $Q$ is the fourth vertex of the parallelogram $APQR$. Prove that $BR$, $CQ$ and $DP$ meet in a point.

[STEP 3, 1992Q8]

Show that

$$\sin(2n+1)\,\theta = \sin^{2n+1}\theta \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{2r} \cot^{2r}\theta,$$

where $n$ is a positive integer. Deduce that the equation

$$\sum_{r=0}^{n} (-1)^{r} \binom{2n+1}{2r} x^{r} = 0$$

has roots $\cot^2\left(\frac{k\pi}{2n+1}\right)$ for $k = 1, 2, \dots, n$.

Show that

(i) $\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right) = \frac{n(2n-1)}{3}$,

(ii) $\sum_{k=1}^{n} \tan^2\left(\frac{k\pi}{2n+1}\right) = n(2n+1)$,

(iii) $\sum_{k=1}^{n} \operatorname{cosec}^2\left(\frac{k\pi}{2n+1}\right) = \frac{2n(n+1)}{3}$.

[STEP 3, 1992Q9]

The straight line $OSA$, where $O$ is the origin, bisects the angle between the positive $x$ and $y$ axes. The ellipse $E$ has $S$ as focus. In polar coordinates with $S$ as pole and $SA$ as the initial line, $E$ has equation $l = r(1 + \mathrm{e}\cos\theta)$. Show that, at the point on $E$ given by $\theta = \alpha$, the gradient of the tangent to the ellipse is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\alpha - \cos\alpha - \mathrm{e}}{\sin\alpha + \cos\alpha + \mathrm{e}}.$$

The points on $E$ given by $\theta = \alpha$ and $\theta = \beta$ are the ends of a diameter of $E$. Show that

$$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = -\frac{1+\mathrm{e}}{1-\mathrm{e}}.$$

[A diameter of an ellipse is a chord through its centre.]

[STEP 3, 1992Q10]

Sketch the curve $C$ whose polar equation is

$$r = 4a\cos 2\theta \quad \text{for} \quad -\frac{\pi}{4} < \theta < \frac{\pi}{4}.$$

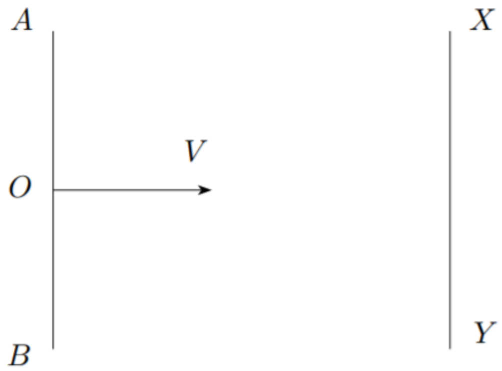
The ellipse $E$ has parametric equations

$$x = 2a\cos\phi\,, \qquad y = a\sin\phi.$$

Show, without evaluating the integrals, that the perimeters of $C$ and $E$ are equal.

Show also that the areas of the regions enclosed by $C$ and $E$ are equal.

[STEP 3, 1992Q11]

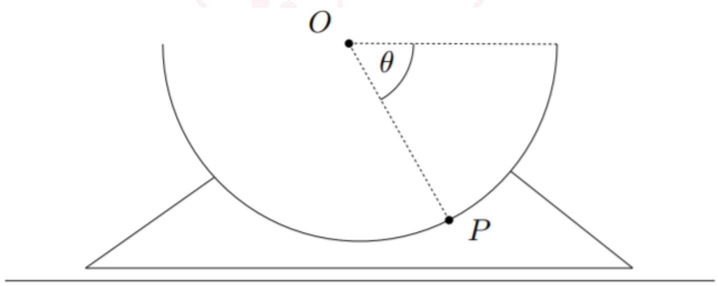



$AOB$ represents a smooth vertical wall and $XY$ represents a parallel smooth vertical barrier, both standing on a smooth horizontal table. A particle $P$ is projected along the table from $O$ with speed $V$ in a direction perpendicular to the wall. At the time of projection, the distance between the wall and the barrier is $\left(\frac{75}{32}\right)VT$, where $T$ is a constant. The barrier moves directly towards the wall, remaining parallel to the wall, with initial speed $4V$ and with constant acceleration $\frac{4V}{T}$ directly away from the wall. The particle strikes the barrier $XY$ and rebounds. Show that this impact take place at time $\frac{5T}{8}$.

The barrier is sufficiently massive for its motion to be unaffected by the impact. Given that the coefficient of restitution is $\frac{1}{2}$, find the speed of $P$ immediately after impact.

$P$ strikes $AB$ and rebounds. Given that the coefficient of restitution for this collision is also $\frac{1}{2}$, show that the next collision of $P$ with the barrier is at time $\frac{9T}{8}$ from the start of the motion.

[STEP 3, 1992Q12]



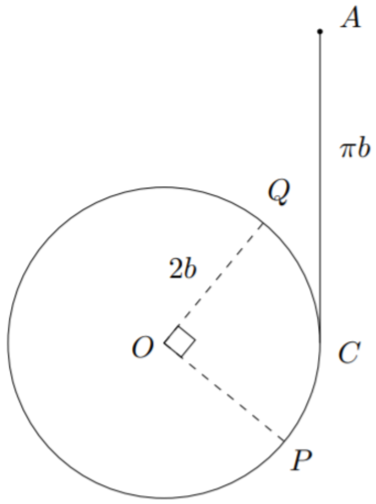

A smooth hemispherical bowl of mass $2m$ is rigidly mounted on a light carriage which slides freely on a horizontal table as shown in the diagram. The rim of the bowl is horizontal and has centre $O$. A particle $P$ of mass $m$ is free to slide on the inner surface of the bowl. Initially, $P$ is in contact with the rim of the bowl and the system is at rest. The system is released and when $OP$ makes an angle $\theta$ with the horizontal the velocity of the bowl is $v$. Show that $3v = a\dot{\theta}\sin\theta$ and that

$$v^2 = \frac{2ga\sin^3\theta}{3(3-\sin^2\theta)},$$

where $a$ is the interior radius of the bowl.

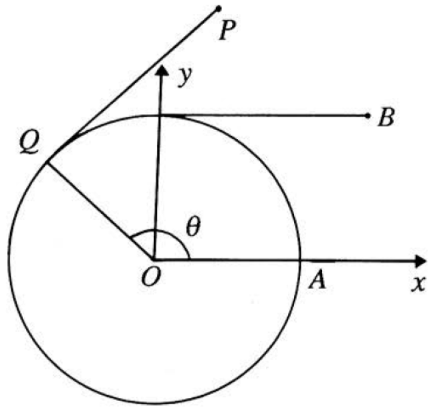
Find, in terms of $m$, $g$ and $\theta$, the reaction between the bowl and the particle.

[STEP 3, 1992Q13]




A uniform circular disc of radius $2b$, mass $m$ and centre $O$ is free to turn about a fixed horizontal axis through $O$ perpendicular to the plane of the disc. A light elastic string of modulus $kmg$, where $k > \frac{4}{\pi}$, has one end attached to a fixed point $A$ and the other end to the rim of the disc at $P$. The string is in contact with the rim of the disc along the arc $PC$, and $OC$ is horizontal. The natural length of the string and the length of the line $AC$ are each $\pi b$ and $AC$ is vertical. A particle $Q$ of mass $m$ is attached to the rim of the disc and $\widehat{POQ} = 90°$ as shown in the diagram. The system is released from rest with $OP$ vertical and $P$ below $O$. Show that $P$ reaches $C$ and that then the upward vertical component of the reaction on the axis is $mg\,\frac{10-\pi k}{3}$.

[STEP 3, 1992Q14]




A horizontal circular disc of radius $a$ and centre $O$ lies on a horizontal table and is fixed to it so that it cannot rotate. A light inextensible string of negligible thickness is wrapped round the disc and attached at its free end to a particle $P$ of mass $m$. When the string is all in contact with the disc, $P$ is at $A$. The string is unwound so that the part not in contact with the disc is taut and parallel to $OA$. $P$ is then at $B$. The particle is projected along the table from $B$ with speed $V$ perpendicular to and away from $OA$. In the general position, the string is tangential to the disc at $Q$ and $\widehat{AOQ} = \theta$. Show that, in the general position, the $x$-coordinate of $P$ with respect to the axes shown in the figure is $a\cos\theta + a\theta\sin\theta$, and find $y$-coordinate of $P$. Hence, or otherwise, show that the acceleration of $P$ has components $a\theta\dot{\theta}^2$ and $a\dot{\theta}^2 + a\theta\ddot{\theta}$ along and perpendicular to $PQ$, respectively.

The friction force between $P$ and the table is $\frac{2\lambda m v^2}{a}$, where $v$ is the speed of $P$ and $\lambda$ is a constant. Show that

$$\frac{\ddot{\theta}}{\dot{\theta}} = -\left(\frac{1}{\theta} + 2\lambda\theta\right)\dot{\theta}$$

and find $\dot{\theta}$ in terms of $\theta$, $\lambda$ and $a$. Find also the tension in the string when $\theta = \pi$.



[STEP 3, 1992Q15]

A goat $G$ lives in a square field $OABC$ of side $a$. It wanders randomly round its field, so that at any time the probability of its being in any given region is proportional to the area of this region. Write down the probability that its distance, $R$, from $O$ is less than $r$ if $0 < r \leq a$, and show that if $r \geq a$ the probability is

$$\left(\frac{r^2}{a^2} - 1\right)^{\frac{1}{2}} + \frac{\pi r^2}{4a^2} - \frac{r^2}{a^2}\cos^{-1}\left(\frac{a}{r}\right).$$

Find the median of $R$ and probability density function of $R$.

The goat is then tethered to the corner $O$ by a chain of length $a$. Find the conditional probability that its distance from the fence $OC$ is more than $\frac{a}{2}$.

## [STEP 3, 1992Q16]

The probability that there are exactly $n$ misprints in an issue of a newspaper is $\frac{e^{-\lambda}\lambda^n}{n!}$ where $\lambda$ is a positive constant. The probability that I spot a particular misprint is $p$, independent of what happens for other misprints, and $0 < p < 1$.

(i) If there are exactly $m + n$ misprints, what is the probability that I spot exactly $m$ of them?

(ii) show that, if I spot exactly $m$ misprints, the probability that I have failed to spot exactly $n$ misprints is

$$\frac{(1-p)^n\lambda^n}{n!}e^{-(1-p)\lambda}.$$

# STEP 3 1993

STEP 3 1993
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

**INSTRUCTIONS TO CANDIDATES**

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

**INFORMATION FOR CANDIDATES**

There are 16 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

[STEP 3, 1993Q1]

The curve $P$ has the parametric equations

$$x = \sin\theta\,, \qquad y = \cos 2\theta \quad \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

Show that $P$ is part of the parabola $y = 1 - 2x^2$ and sketch $P$.

Show that the length of $P$ is $\sqrt{17} + \frac{1}{4}\sinh^{-1} 4$.

Obtain the volume of the solid enclosed when $P$ is rotated through $2\pi$ radians about the line $y = -1$.

[STEP 3, 1993Q2]

The curve $C$ has the equation $x^3 + y^3 = 3xy$.

(i) Show that there is no point of inflection on $C$. You may assume that the origin is not a point of inflection.

(ii) The part of $C$ which lies in the first quadrant is a closed loop touching the axes at the origin. By converting to polar coordinates, or otherwise, evaluate the area of this loop.

[STEP 3, 1993Q3]

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{M}$ are given by

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 13 & 5 \\ 3 & 8 & 7 \end{pmatrix},$$

where $a, b, \dots, r$ are real numbers. Given that $\mathbf{M} = \mathbf{AB}$, show that $a = 1, b = 4, c = 1, d = 3$, $e = -1, f = -2, p = 3, q = 2, r = -3$ gives the *unique* solution for $\mathbf{A}$ and $\mathbf{B}$. Evaluate $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

Hence, or otherwise, solve the simultaneous equations

$$\begin{aligned} x + 3y + 2z &= 7 \\ 4x + 13y + 5z &= 18 \\ 3x + 8y + 7z &= 25. \end{aligned}$$

[STEP 3, 1993Q4]

Sum the following infinite series.

(i) $1 + \frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{5}\left(\frac{1}{2}\right)^4 + \cdots + \frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n} + \cdots$.

(ii) $2 - x - x^3 + 2x^4 - \cdots + 2x^{4k} - x^{4k+1} - x^{4k+3} + \cdots$ where $|x| < 1$.

(iii) $\sum_{r=2}^{\infty} \frac{r2^{r-2}}{3^{r-1}}$.

(iv) $\sum_{r=2}^{\infty} \frac{2}{r(r^2-1)}$.

[STEP 3, 1993Q5]

The set $S$ consists of ordered pairs of complex numbers $(z_1, z_2)$ and a binary operation $\circ$ on $S$ is defined by

$$(z_1, z_2) \circ (w_1, w_2) = (z_1w_1 - z_2w_2^*, z_1w_2 + z_2w_1^*).$$

Show that the operation $\circ$ is associative and determine whether it is commutative. Evaluate $(z, 0) \circ (w, 0)$, $(z, 0) \circ (0, w)$, $(0, z) \circ (w, 0)$ and $(0, z) \circ (0, w)$.

The set $S_1$ is the subset of $S$ consisting of $A, B, \dots, H$, where $A = (1, 0)$, $B = (0, 1)$, $C = (\mathbf{i}, 0)$, $D = (0, \mathbf{i})$, $E = (-1, 0)$, $F = (0, -1)$, $G = (-\mathbf{i}, 0)$ and $H = (0, -\mathbf{i})$. Show that $S_1$ is closed under $\circ$ and that it has an identity element. Determine the inverse and order of each element of $S_1$. Show that $S_1$ is a group under $\circ$.

[You are not required to compute the multiplication table in full.]

Show that $\{A, B, E, F\}$ is a subgroup of $S_1$ and determine whether it is isomorphic to the group generated by the $2 \times 2$ matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under matrix multiplication.

[STEP 3, 1993Q6]

The point in the Argand diagram representing the complex number $z$ lies on the circle with centre $K$ and radius $r$, where $K$ represents the complex number $k$. Show that

$$zz^* - kz^* - k^*z + kk^* - r^2 = 0.$$

The point $P$, $Q_1$ and $Q_2$ represent the complex numbers $z$, $w_1$ and $w_2$ respectively. The point $P$ lies on the circle with $OA$ as diameter, where $O$ and $A$ represent $0$ and $2\mathbf{i}$ respectively. Given that $w_1 = \frac{z}{z-1}$, find the equation of the locus $L$ of $Q_1$ in the terms of $w_1$ and describe the geometrical form of $L$.

Given that $w_2 = z^*$, show that the locus of $Q_2$ is also $L$. Determine the positions of $P$ for which $Q_1$ coincides with $Q_2$.



[STEP 3, 1993Q7]

The real numbers $x$ and $y$ satisfy the simultaneous equations

$$\sinh(2x) = \cosh y \quad \text{and} \quad \sinh(2y) = 2\cosh x.$$

Show that $\sinh^2 y$ is a root of the equation

$$4t^3 + 4t^2 - 4t - 1 = 0$$

and demonstrate that this gives at most one valid solution for $y$. Show that the relevant value of $t$ lies between 0.7 and 0.8, and use an iterative process to find $t$ to 6 decimal places.

Find $y$ and hence find $x$, checking your answers and stating the final answers to four decimal places.

## [STEP 3, 1993Q8]

A square pyramid has its base vertices at the points $A(a,0,0)$, $B(0,a,0)$, $C(-a,0,0)$ and $D(0,-a,0)$, and its vertex at $E(0,0,a)$. The point $P$ lies on $AE$ with $x$-coordinate $\lambda a$, where $0 < \lambda < 1$, and the point $Q$ lies on $CE$ with $x$-coordinate $-\mu a$, where $0 < \mu < 1$. The plane $BPQ$ cuts $DE$ at $R$ and the $y$-coordinate of $R$ is $-\gamma a$. Prove that

$$\gamma = \frac{\lambda\mu}{\lambda + \mu - \lambda\mu}.$$

Show that the quadrilateral $BPRQ$ cannot be a parallelogram.

## [STEP 3, 1993Q9]

For the real numbers $a_1, a_2, a_3, \ldots$,

(i) prove that $a_1^2 + a_2^2 \geq 2a_1a_2$,

(ii) prove that $a_1^2 + a_2^2 + a_3^2 \geq a_2a_3 + a_3a_1 + a_1a_2$,

(iii) prove that $3(a_1^2 + a_2^2 + a_3^2 + a_4^2) \geq 2(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)$,

(iv) state and prove a generalisation of (iii) to the case of $n$ real numbers,

(v) prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geq \frac{2n}{n-1}\sum a_i a_j\,,$$

where the latter sum is taken over all pairs $(i,j)$ with $1 \leq i < j \leq n$.



## [STEP 3, 1993Q10]

The transformation $T$ of the point $P$ in the $x, y$ plane to the point $P'$ is constructed as follows: Lines are drawn through $P$ parallel to the lines $y = mx$ and $y = -mx$ to cut the line $y = kx$ at $Q$ and $R$ respectively, $m$ and $k$ being given constant. $P'$ is the fourth vertex of the parallelogram $PQP'R$.

Show that if $P$ is $(x_1, y_1)$ then $Q$ is

$$\left(\frac{mx_1 - y_1}{m-k}, \frac{k(mx_1 - y_1)}{m-k}\right).$$

Obtain the coordinates of $P'$ in terms of $x_1$, $y_1$, $m$ and $k$, and express $T$ as a matrix transformation. Show that areas are transformed under $T$ into areas of the same magnitude.

[STEP 3, 1993Q11]

[In this question, all gravitational forces are to be neglected.] A rigid frame is constructed from 12 equal uniform rods, each of length $a$ and mass $m$, forming the edges of a cube. Three of the edges are $OA$, $OB$ and $OC$, and the vertices opposite $O$, $A$, $B$ and $C$ are $O'$, $A'$, $B'$ and $C'$ respectively. Forces act along the lines as follows, in the directions indicated by the order of the letters:

$$\begin{array}{ccc} 2mg \text{ along } OA, & mg \text{ along } AC', & \sqrt{2}mg \text{ along } O'A, \\ \sqrt{2}mg \text{ along } OA', & 2mg \text{ along } C'B, & mg \text{ along } A'C. \end{array}$$

Initially the frame is at rest and there are no other forces acting on it.

(i) The frame is freely pivoted at $O$. Show that the direction of the line about which it will start to rotate is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ with respect to axes along $OA$, $OB$ and $OC$ respectively.

(ii) Show that the moment of inertia of the rod $OA$ about the axis $OO'$ is $\frac{2ma^2}{9}$ and about a parallel axis through its mid-point is $\frac{ma^2}{18}$. Hence find the moment of inertia of $B'C$ about $OO'$ and show that the moment of inertia of the frame about $OO'$ is $\frac{14ma^2}{3}$. If the frame is freely pivoted about the line $OO'$ and the forces continue to act along the specified lines, find the initial angular acceleration of the frame.

[STEP 3, 1993Q12]

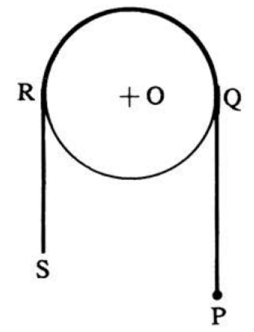


$ABCD$ is a horizontal line with $AB = CD = a$ and $BC = 6a$. There are fixed smooth pegs at $B$ and $C$. A uniform string of natural length $2a$ and modulus of elasticity $kmg$ is stretched from $A$ to $D$, passing over the pegs at $B$ and $C$. A particle of mass $m$ is attached to the midpoint $P$ of the string. When the system is in equilibrium, $P$ is a distance $\frac{a}{4}$ below $BC$. Evaluate $k$.

The particle is pulled down to a point $Q$, which is at a distance $pa$ below the mid-point of $BC$, and is released form rest. $P$ rises to a point $R$, which is at a distance $3a$ above $BC$. Show that $2p^2 - p - 17 = 0$.

Show also that the tension in the strings is less when the particle is at $R$ than when the particle is at $Q$.

[STEP 3, 1993Q13]




A uniform circular disc with radius $a$, mass $4m$ and centre $O$ is freely mounted on a fixed horizontal axis which is perpendicular to its plane and passes through $O$. A uniform heavy chain $PS$ of length $(4+\pi)a$, mass $(4+\pi)m$ and negligible thickness is hung over the rim of the disc as shown in the diagram: $Q$ and $R$ are the points of the chain at the same level as $O$. The contact between the chain and the rim of the disc is sufficiently rough to prevent slipping. Initially, the system is at rest with $PQ = RS = 2a$. A particle of mass $m$ is attached to the chain at $P$ and the system is released. By considering the energy of the system, show that when $P$ has descended a distance $x$, its speed $v$ is given by

$$(\pi+7)av^2 = 2g(x^2+ax).$$

By considering the part $PQ$ of the chain as a body of variable mass, show that when $S$ reaches $R$ the tension in the chain at $Q$ is

$$\frac{5\pi-2}{\pi+7}mg.$$



[STEP 3, 1993Q14]

A particle rests at a point $A$ on a horizontal table and is joined to a point $O$ on the table by a taut inextensible string of length $c$. The particle is projected vertically upwards at a speed $64\sqrt{6gc}$. It next strikes the table at a point $B$ and rebounds. The coefficient of restitution for any impact between the particle and the table is $\frac{1}{2}$. After rebounding at $B$, the particle will rebound alternately at $A$ and $B$ until the string becomes slack. Show that when the string becomes slack the particle is at height $\frac{c}{2}$ above the table.

Determine whether the first rebound *between* $A$ and $B$ is nearer to $A$ or to $B$.

[STEP 3, 1993Q15]

The probability of throwing a head with a certain coin is $p$ and the probability of throwing a tail is $q = 1-p$. The coin is thrown until at least two heads and at least two tails have been thrown; this happens when the coin has been thrown $N$ times. Write down an expression for the probability that $N = n$.

Show that the expectation of $N$ is

$$2\left(\frac{1}{pq}-1-pq\right).$$

[STEP 3, 1993Q16]

The time taken for me to set an acceptable examination question it $T$ hours. The distribution of $T$ is a truncated normal distribution with probability density $f$ where

$$f(t) = \begin{cases} \dfrac{1}{k\sigma\sqrt{2\pi}} \exp\left(-\dfrac{1}{2}\left(\dfrac{t-\sigma}{\sigma}\right)^2\right) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Sketch the graph of $f(t)$. Show that $k$ is approximately 0.841 and obtain the mean of $T$ as a multiple of $\sigma$.

Over a period of years, I find that the mean setting time is 3 hours.

(i) Find the approximate probability that none of the 16 questions on next year's paper will take more than 4 hours to set.

(ii) Given that a particular question is unsatisfactory after 2 hours work, find the probability that it will still be unacceptable after a further 2 hours work.

# STEP 3 1994

STEP 3 1994
On-line Exam

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**TIME ALLOWED: 180 MINUTES**

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## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

| | |
|---|---|
| **Section A** | Pure Mathematics |
| **Section B** | Mechanics |
| **Section C** | Probability and Statistics |

There are 14 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

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Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 1994Q1]

Calculate

$$\int_0^x \operatorname{sech} t \, \mathrm{d}t.$$

Find the reduction formula involving $I_n$ and $I_{n-2}$, where

$$I_n = \int_0^x \operatorname{sech}^n t \, \mathrm{d}t$$

and, hence or otherwise, find $I_5$ and $I_6$.

## [STEP 3, 1994Q2]

(i) By setting $y = x + x^{-1}$, find the solutions of

$$x^4 + 10x^3 + 26x^2 + 10x + 1 = 0.$$

(ii) Solve

$$x^4 + x^3 - 10x^2 - 4x + 16 = 0.$$

## [STEP 3, 1994Q3]

Describe geometrically the possible intersections of a plane with a sphere.

Let $P_1$ and $P_2$ be the planes with equations

$$3x - y - 1 = 0,$$
$$x - y + 1 = 0,$$

respectively, and let $S_1$ and $S_2$ be the spheres with equations

$$x^2 + y^2 + z^2 = 7,$$
$$x^2 + y^2 + z^2 - 6y - 4z + 10 = 0,$$

respectively. Let $C_1$ be the intersection of $P_1$ and $S_1$, let $C_2$ be the intersection of $P_2$ and $S_2$ and let $L$ be the intersection of $P_1$ and $P_2$. Find the points where $L$ meets each of $S_1$ and $S_2$. Determine, giving your reasons, whether the circles $C_1$ and $C_2$ are linked.

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[STEP 3, 1994Q4]

Find the two solutions of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y$$

which pass through the point $(a, b^2)$, where $b \neq 0$.

Find two distinct points $(a_1, 1)$ and $(a_2, 1)$ such that one of the solutions through each of them also passes through the origin. Show that the graphs of these two solutions coincide and sketch their common graph, together with the other solutions through $(a_1, 1)$ and $(a_2, 1)$.

Now sketch sufficient members of the family of solutions (for varying $a$ and $b$) to indicate the general behaviour. Use your sketch to identify a common tangent, and comment briefly on its relevance to the differential equation.

[STEP 3, 1994Q5]

The function $f$ is given by $f(x) = \sin^{-1} x$ for $-1 < x < 1$. Prove that

$$(1 - x^2)f''(x) - xf'(x) = 0.$$

Prove also that

$$(1 - x^2)f^{n+2}(x) - (2n + 1)xf^{n+1}(x) - n^2 f^n(x) = 0,$$

for all $n > 0$, where $f^n$ denotes the $n$th derivative of $f$. Hence express $f(x)$ as a Maclaurin series.



The function $g$ is given by

$$g(x) = \ln\sqrt{\frac{1 + x}{1 - x}},$$

for $-1 < x < 1$. Write down a power series expansion for $g(x)$, and show that the coefficient of $x^{2n+1}$ is greater than that in the expansion of $f$, for each $n > 0$.



[STEP 3, 1994Q6]

The four points $A, B, C, D$ in the Argand diagram (complex plane) correspond to the complex numbers $a, b, c, d$ respectively. The point $P_1$ is mapped to $P_2$ by rotating about $A$ through $\frac{\pi}{2}$ radians. Then $P_2$ is mapped to $P_3$ by rotating about $B$ through $\frac{\pi}{2}$ radians, $P_3$ is mapped to $P_4$ by rotating about $C$ through $\frac{\pi}{2}$ radians and $P_4$ is mapped to $P_5$ by rotating about $D$ through $\frac{\pi}{2}$ radians, each rotation being in the positive sense. If $z_i$ is the complex number corresponding to $P_i$, find $z_5$ in terms of $a, b, c, d$ and $z_1$.

Show that $P_5$ will coincide with $P_1$, irrespective of the choice of the latter if, and only if, $a - c = \mathbf{i}(b - d)$ and interpret this condition geometrically.

The points $A, B$ and $C$ are now chosen to be distinct points on the unit circle and the angle of rotation is changed to $\theta$, where $\theta \neq 0$, on each occasion. Find the necessary and sufficient condition on $\theta$ and the points $A, B$ and $C$ for $P_4$ always to coincide with $P_1$.

[STEP 3, 1994Q7]

Let $S_3$ be the group of permutations of three objects and $Z_6$ be the group of integers under addition modulo 6. List all the elements of each group, stating the order of each element. State, with reasons, whether $S_3$ is isomorphic with $Z_6$.

Let $C_6$ be the group of 6th roots of unity. That is, $C_6 = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$ where $\alpha = e^{\frac{\pi i}{3}}$ and the group operation is complex multiplicaion. Prove that $C_6$ is isomorphic with $Z_6$. Is there any (multiplicative or additive) subgroup of the complex numbers which is isomorphic with $S_3$? Give a reason for your answer.

[STEP 3, 1994Q8]

Let $a$, $b$, $c$, $d$, $p$, $q$, $r$ and $s$ be real numbers. By considering the determinant of the matrix product

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}\begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix},$$

where $z_1$, $z_2$, $z_3$ and $z_4$ are suitably chosen complex numbers, find expressions $L_1$, $L_2$, $L_3$ and $L_4$, each of which is linear in $a$, $b$, $c$ and $d$ and also linear in $p$, $q$, $r$ and $s$, such that

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = L_1^2 + L_2^2 + L_3^2 + L_4^2.$$

## Section B: Mechanics

### [STEP 3, 1994Q9]

A smooth, axially symmetric bowl has its vertical cross-sections determined by $s = 2\sqrt{ky}$, where $s$ is the arc-length measured from its lowest point $V$, and $y$ is the height above $V$. A particle is released from rest at a point on the surface at a height $h$ above $V$. Explain why

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + 2gy$$

is constant.

Show that the time for the particle to reach $V$ is

$$\pi\sqrt{\frac{k}{2g}}.$$

Two elastic particles of mass $m$ and $\alpha m$, where $\alpha < 1$, are released simultaneously from opposite sides of the bowl at heights $\alpha^2 h$ and $h$ respectively. If the coefficient of restitution between the particles is $\alpha$, describe the subsequent motion.

### [STEP 3, 1994Q10]

The island of Gammaland is totally flat and subject to a constant wind of $w$ km h$^{-1}$, blowing from the West. Its southernmost shore stretches almost indefinitely, due east and west, from the coastal city of Alphaport. A novice pilot is making her first solo flight from Alphaport to the town of Betaville which lies north-east of Alphaport. Her instructor has given her the correct heading to reach Betaville, flying at the plane's recommended airspeed of $v$ km h$^{-1}$, where $v > w$.

On reaching Betaport the pilot returns with the opposite heading to that of the outward flight and, so featureless is Gammaland, that she only realises her error as she crosses the coast with Alphaport nowhere in sight. Assuming that she then turns West along the coast, and that her outward flight took $t$ hours, show that her return flight takes

$$\left(\frac{v+w}{v-w}\right) t \text{ hours.}$$

If Betaville is $d$ kilometres from Alphaport, show that, with the correct heading, the return flight should have taken

$$t + \frac{\sqrt{2}wd}{v^2 - w^2} \text{ hours.}$$

[STEP 3, 1994Q11]

A step-ladder has two sections $AB$ and $AC$, each of length $4a$, smoothly hinged at $A$ and connected by a light elastic rope $DE$, of natural length $\frac{a}{4}$ and modulus $W$, where $D$ is on $AB$, $E$ is on $AC$ and $AD = AE = a$. The section $AB$, which contains the steps, is uniform and of weight $W$ and the weight of $AC$ is negligible.

The step-ladder rests on a smooth horizontal floor and a man of weight $4W$ carefully ascends it to stand on a rung distant $\beta a$ from the end of the ladder resting on the floor. Find the height above the floor of the rung on which the man is standing when $\beta$ is the maximum value at which equilibrium is possible.

## Section C: Probability and Statistics

### [STEP 3, 1994Q12]

In certain forms of Tennis two players $A$ and $B$ serve alternate games. Player $A$ has probability $p_A$ of winning a game in which she serves and probability $p_B$ of winning a game in which player $B$ serves. Player $B$ has probability $q_B = 1 - p_B$ of winning a game in which she serves and probability $q_A = 1 - p_A$ of winning a game in which player $A$ serves. In Shortened Tennis the first player to lead by 2 games wins the match. Find the probability $P_{\text{short}}$ that $A$ wins a Shortened Tennis match in which she serves first and show that it is the same as if $B$ serves first.

In Standard Tennis the first player to lead by 2 or more games after 4 or more games have been played wins the match. Show that the probability that the match is decided in 4 games is

$$p_A^2 p_B^2 + q_A^2 q_B^2 + 2(p_A p_B + q_A q_B)(p_A q_B + q_A p_B).$$

If $p_A = p_B = p$ and $q_A = q_B = q$, find the probability $P_{\text{stan}}$ that $A$ wins a Standard Tennis match in which she serves first. Show that

$$P_{\text{stan}} - P_{\text{short}} = \frac{p^2 q^2 (p - q)}{p^2 + q^2}.$$

### [STEP 3, 1994Q13]

During his performance a trapeze artist is supported by two identical ropes, either of which can bear his weight. Each rope is such that the time, in hours of performance, before it fails is exponentially distributed, independently of the other, with probability density function $\lambda \exp(-\lambda t)$ for $t \geq 0$ (and 0 for $t < 0$), for some $\lambda > 0$. A particular rope has already been in use for $t_0$ hours of performance. Find the distribution for the length of time the artist can continue to use it before it fails. Interpret and comment upon your result.

Before going on tour the artist insists that the management purchase two new ropes of the above type. Show that the probability density function of the time until both ropes fail is

$$f(t) = \begin{cases} 2\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right) & \text{if } t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If each performance lasts for $h$ hours, find the probability that both ropes fail during the $n$th performance. Show that the probability that both ropes will fail during the same performance is $\tanh\left(\frac{\lambda h}{2}\right)$.

### [STEP 3, 1994Q14]

Three points, $P$, $Q$ and $R$, are independently randomly chosen on the perimeter of a circle. Prove that the probability that at least one of the angles of the triangle $PQR$ will exceed $k\pi$ is $3(1-k)^2$ if $\frac{1}{2} \leq k \leq 1$. Find the probability if $\frac{1}{3} \leq k \leq \frac{1}{2}$.

# STEP 3 1995

STEP 3 1995
[On-line Exam]

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Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

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Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

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**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 1995Q1]

Find the simultaneous solutions of the three linear equations

$$a^2x + ay + z = a^2$$
$$ax + y + bz = 1$$
$$a^2bx + y + bz = b$$

for all possible real values of $a$ and $b$.

## [STEP 3, 1995Q2]

If

$$I_n = \int_0^a x^{n+\frac{1}{2}}(a-x)^{\frac{1}{2}}\,\mathrm{d}x,$$

show that $I_0 = \frac{\pi a^2}{8}$.

Show that $(2n+4)I_n = (2n+1)aI_{n-1}$ and hence evaluate $I_n$.

## [STEP 3, 1995Q3]

What is the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

for each of the cases: (i) $k > 1$; (ii) $k = 1$; (iii) $0 < k < 1$?

In case(iii) the equation represents damped simple harmonic motion with damping factor $k$. Let $x(0) = 0$ and let $x_1, x_2, \ldots, x_n, \ldots$ be the sequence of successive maxima and minima, so that if $x_n$ is a maximum then $x_{n+1}$ is the next minimum. Show that $\left|\frac{x_{n+1}}{x_n}\right|$ takes a value $\alpha$ which is independent of $n$, and that

$$k^2 = \frac{(\ln\alpha)^2}{\pi^2 + (\ln\alpha)^2}.$$

[STEP 3, 1995Q4]

Let

$$C_n(\theta) = \sum_{k=0}^{n} \cos k\theta$$

and

$$S_n(\theta) = \sum_{k=0}^{n} \sin k\theta,$$

where $n$ is a positive integer and $0 < \theta < 2\pi$. Show that

$$C_n(\theta) = \frac{\cos\left(\frac{1}{2}n\theta\right)\sin\left(\frac{1}{2}(n+1)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)},$$

and obtain the corresponding expression for $S_n(\theta)$.

Hence, or otherwise, show that for $0 < \theta < 2\pi$,

$$\left|C_n(\theta) - \frac{1}{2}\right| \leq \frac{1}{2\sin\left(\frac{1}{2}\theta\right)}.$$

[STEP 3, 1995Q5]

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3

Show that $y = \sin^2(m\sin^{-1}x)$ satisfies the differential equation

$$(1-x^2)y^2 = xy^1 + 2m^2(1-2y),$$

and deduce that, for all $n \geq 1$,

$$(1-x^2)y^{n+2} = (2n+1)xy^{n+1} + (n^2 - 4m^2)y^n,$$

where $y^n$ denotes the $n$th derivative of $y$.

Derive the Maclaurin series for $y$, making it clear what the general term is.



[STEP 3, 1995Q6]

The variable non-zero complex number $z$ is such that

$$|z - \mathbf{i}| = 1.$$

Find the modulus of $z$ when its argument is $\theta$. Find also the modulus and argument of $\frac{1}{z}$ in terms of $\theta$ and show in an Argand diagram the loci of points which represent $z$ and $\frac{1}{z}$.

Find the locus $C$ in the Argand diagram such that $w \in C$ if, and only if, the real part of $\frac{1}{w}$ is $-1$.

[STEP 3, 1995Q7]

Consider the following sets with the usual definition of multiplication appropriate to each. In each case you may assume that the multiplication is associative. In each case state, giving adequate reasons, whether or not the set is a group:

(i) the complex numbers of unit modulus.

(ii) the integers modulo 4.

(iii) the matrices

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $0 \leq \theta < 2\pi$.

(iv) the integers 1, 3, 5, 7 modulo 8.

(v) the $2 \times 2$ matrices all of whose entries are integers.

(vi) the integers 1, 2, 3, 4 modulo 5.

In the case of each pair of groups above state, with reasons, whether or not they are isomorphic.

[STEP 3, 1995Q8]

A plane $\pi$ in 3-dimensional space is given by the vector equation $\mathbf{r} \cdot \hat{\mathbf{n}} = p$, where $\hat{\mathbf{n}}$ is a unit vector and $p$ is a non-negative real number. If $\mathbf{x}$ is the position vector of a general point $X$, find the equation of the normal to $\pi$ through $X$ and the perpendicular distance of $X$ from $\pi$.



The unit circles $C_i$, $i = 1,2$, with centres $\mathbf{r}_i$, lie in the planes $\pi_i$ given by $\mathbf{r} \cdot \hat{\mathbf{n}}_i = p_i$, where the $\hat{\mathbf{n}}_i$ are unit vectors, and $p_i$ are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number $\lambda$ such that

$$\mathbf{r}_1 + \lambda\widehat{\mathbf{n}_1} = \mathbf{r}_2 \pm \lambda\widehat{\mathbf{n}_2}.$$

Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\widehat{\mathbf{n}_1} \times \widehat{\mathbf{n}_2}) = 0$$

and that

$$(p_1 - \widehat{\mathbf{n}_1} \cdot \mathbf{r}_2)^2 = (p_2 - \widehat{\mathbf{n}_2} \cdot \mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.

## Section B: Mechanics

### [STEP 3, 1995Q9]

A thin circular disc of mass $m$, radius $r$ and with its centre of mass at its centre $C$ can rotate freely in a vertical plane about a fixed horizontal axis through a point $O$ of its circumference. A particle $P$, also of mass $m$, is attached to the circumference of the disc so that the angle $OCP$ is $2\alpha$, where $\alpha \leq \frac{\pi}{2}$.

(i) In the position of stable equilibrium $OC$ makes an angle $\beta$ with the vertical. Prove that

$$\tan\beta = \frac{\sin 2\alpha}{2 - \cos 2\alpha}.$$

(ii) The density of the disc at a point distant $x$ from $C$ is $\frac{\rho x}{r}$. Show that its moment of inertia about the horizontal axis through $O$ is $\frac{8mr^2}{5}$.

(iii) The mid-point of $CP$ is $Q$. The disc is held at rest with $OQ$ horizontal and $C$ lower than $P$ and it is then released. Show that the speed $v$ with which $C$ is moving when $P$ passes vertically below $O$ is given by

$$v^2 = \frac{15gr\sin\alpha}{2(2 + 5\sin^2\alpha)}.$$

Find the maximum value of $v^2$ as $\alpha$ is varied.



### [STEP 3, 1995Q10]

A cannon is situated at the bottom of a plane inclined at angle $\beta$ to the horizontal. A (small) cannon ball is fired from the cannon at an initial speed $u$. Ignoring air resistance, find the angle of firing which will maximize the distance up the plane travelled by the cannon ball and show that in this case the ball will land at a distance

$$\frac{u^2}{g(1 + \sin\beta)}$$

from the cannon.



### [STEP 3, 1995Q11]

A ship is sailing due west at $V$ knots while a plane, with an airspeed of $kV$ knots, where $k > \sqrt{2}$, patrols so that it is always to the north west of the ship. If the wind in the area is blowing from north to south at $V$ knots and the pilot is instructed to return to the ship every thirty minutes, how long will her outward flight last?

Assume that the maximum distance of the plane from the ship during the above patrol was $d_w$ miles. If the air now becomes dead calm, and the pilot's orders are maintained, show that the ratio $\frac{d_w}{d_c}$ of $d_w$ to the new maximum distance, $d_c$ miles, of the plane from the ship is

$$\frac{k^2 - 2}{2k(k^2 - 1)}\sqrt{4k^2 - 2}.$$

## Section C: Probability and Statistics

### [STEP 3, 1995Q12]

The random variables $X$ and $Y$ are independently normally distributed with means 0 and variances 1. Show that the joint probability density function for $(X, Y)$ is

$$f(x, y) = \frac{1}{2\pi} \mathrm{e}^{-\frac{1}{2}(x^2+y^2)} \qquad -\infty < x < \infty, \qquad -\infty < y < \infty.$$

If $(x, y)$ are the coordinates, referred to rectangular axes, of a point in the plane, explain what is meant by saying that this density is radially symmetrical.

The random variables $U$ and $V$ have a joint probability density function which is radially symmetrical (in the above sense). By considering the straight line with equation $U = kV$, or otherwise, show that

$$\mathrm{P}\left(\frac{U}{V} < k\right) = 2\mathrm{P}(U < kV, V > 0).$$

Hence, or otherwise, show that the probability density function of $\frac{U}{V}$ is

$$g(k) = \frac{1}{\pi(1+k^2)} \qquad -\infty < k < \infty.$$

### [STEP 3, 1995Q13]

A message of $10^k$ binary digits is sent along a fibre optic cable with high probabilities $p_0$ and $p_1$ that the digits 0 and 1, respectively, are received correctly. If the probability of a digit in the original message being a 1 is $\alpha$, find the probability that the entire message is received correctly.

Find the probability $\beta$ that a randomly chosen digit in the message is received as a 1 and show that $\beta = \alpha$ if, and only if,

$$\alpha = \frac{q_0}{q_1 + q_0},$$

where $q_0 = 1 - p_0$ and $q_1 = 1 - p_1$. If this condition is satisfied and the received message consists entirely of zeros, what is the probability that it is correct?

If now $q_0 = q_1 = q$ and $\alpha = \frac{1}{2}$, find the approximate value of $q$ which will ensure that a message of one million binary digits has a fifty-fifty chance of being received entirely correctly.

The probability of error $q$ is proportional to the square of the length of the cable. Initially the length is such that the probability of a message of one million binary bits, among which 0 and 1 are equally likely, being received correctly is $\frac{1}{2}$. What would this probability become if a booster station were installed at its mid-point, assuming that the booster station re-transmits the received version of the message, and assuming that terms of order $q^2$ may be ignored?

[STEP 3, 1995Q14]

A candidate finishes examination questions in time $T$, where $T$ has probability density function

$$f(t) = te^{-t} \qquad t \geq 0,$$

the probabilities for the various questions being independent. Find the moment generating function of $T$ and hence find the moment generating function for the total time $U$ taken to finish two such questions. Show that the probability density function for $U$ is

$$g(u) = \frac{1}{6}u^3 e^{-u} \quad u \geq 0.$$

Find the probability density function for the total time taken to answer $n$ such questions.

# STEP 3 1996

STEP 3 1996
On-line Exam

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# Section A: Pure Mathematics

## [STEP 3, 1996Q1]

Define $\cosh x$ and $\sinh x$ in terms of exponentials and prove, from your definitions, that

$$\cosh^4 x - \sinh^4 x = \cosh 2x$$

and

$$\cosh^4 x + \sinh^4 x = \frac{1}{4}\cosh 4x + \frac{3}{4}.$$

Find $a_0, a_1, \dots, a_n$ in terms of $n$ such that

$$\cosh^n x = a_0 + a_1 \cosh x + a_2 \cosh 2x + \cdots + a_n \cosh nx.$$

Hence, or otherwise, find expressions for $\cosh^{2m} x - \sinh^{2m} x$ and $\cosh^{2m} x + \sinh^{2m} x$, in terms of $\cosh kx$, where $k = 0, \dots, 2m$.

## [STEP 3, 1996Q2]

For all values of $a$ and $b$, either solve the simultaneous equations

$$\begin{aligned} x + y + az &= 2 \\ x + ay + z &= 2 \\ 2x + y + z &= 2b \end{aligned}$$

or prove that they have no solution.



## [STEP 3, 1996Q3]

Find

$$\int_0^\theta \frac{1}{1 - a\cos x}\,\mathrm{d}x,$$

where $0 < \theta < \pi$ and $-1 < a < 1$.

Hence show that

$$\int_0^{\frac{\pi}{2}} \frac{1}{2 - a\cos x}\,\mathrm{d}x = \frac{2}{\sqrt{4 - a^2}}\tan^{-1}\sqrt{\frac{2+a}{2-a}},$$

and also that

$$\int_0^{\frac{3\pi}{4}} \frac{1}{\sqrt{2} + \cos x}\,\mathrm{d}x = \frac{\pi}{2}.$$

[STEP 3, 1996Q4]

Find the integers $k$ satisfying the inequality $k \leq 2(k-2)$.

Given that $N$ is a strictly positive integer consider the problem of finding strictly positive integers whose sum is $N$ and whose product is as large as possible. Call this largest possible product $P(N)$. Show that $P(5) = 2 \times 3$, $P(6) = 3^2$, $P(7) = 2^2 \times 3$, $P(8) = 2 \times 3^2$ and $P(9) = 3^3$.

Find $P(1000)$ explaining your reasoning carefully.

[STEP 3, 1996Q5]

Show, using de Moivre's Theorem, or otherwise, that

$$\tan 7\theta = \frac{t(t^6 - 21t^4 + 35t^2 - 7)}{7t^6 - 35t^4 + 21t^2 - 1},$$

where $t = \tan\theta$.

(i) By considering the equation $\tan 7\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{7}\right), \tan^2\left(\frac{2\pi}{7}\right) \text{ and } \tan^2\left(\frac{3\pi}{7}\right)$$

and deduce the value of

$$\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right).$$

(ii) Find, without using a calculator, the value of

$$\tan^2\left(\frac{\pi}{14}\right) + \tan^2\left(\frac{3\pi}{14}\right) + \tan^2\left(\frac{5\pi}{14}\right).$$

[STEP 3, 1996Q6]

(i) Let $S$ be the set of matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix},$$

where $a$ is any real non-zero number. Show that $S$ is closed under matrix multiplication and, further, that $S$ is a group under matrix multiplication.

(ii) Let $G$ be a set of $n \times n$ matrices which is a group under matrix multiplication, with identity element $\mathbf{E}$. By considering equations of the form $\mathbf{BC} = \mathbf{D}$ for suitable elements $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of $G$, show that if a given element $\mathbf{A}$ of $G$ is a singular matrix (i.e. $\det \mathbf{A} = 0$), then all elements of $G$ are singular. Give, with justification, an example of such a group of singular matrices in the case $n = 3$.

[STEP 3, 1996Q7]

(i) If $x + y + z = \alpha$, $xy + yz + zx = \beta$ and $xyz = \gamma$, find numbers $A$, $B$ and $C$ such that

$$x^3 + y^3 + z^3 = A\alpha^3 + B\alpha\beta + C\gamma.$$

Solve the equations

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 4. \end{aligned}$$

(ii) The area of a triangle whose sides are $a$, $b$ and $c$ is given by the formula

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s$ is the semi-perimeter $\frac{1}{2}(a+b+c)$. If $a$, $b$ and $c$ are the roots of the equation

$$x^3 - 16x^2 + 81x - 128 = 0,$$

find the area of the triangle.

[STEP 3, 1996Q8]

A transformation $T$ of the real numbers is defined by

$$y = T(x) = \frac{ax - b}{cx - d},$$

where $a, b, c, d$ are real numbers such that $ad \neq bc$. Find all numbers $x$ such that $T(x) = x$. Show that the inverse operation, $x = T^{-1}(y)$ expressing $x$ in terms of $y$ is of the same form as $T$ and find corresponding numbers $a', b', c', d'$.

Let $S_r$ denote the set of all real numbers excluding $r$. Show that, if $c \neq 0$, there is a value of $r$ such that $T$ is defined for all $x \in S_r$ and find the image $T(S_r)$. What is the corresponding result if $c = 0$?

If $T_1$, given by numbers $a_1, b_1, c_1, d_1$, and $T_2$, given by numbers $a_2, b_2, c_2, d_2$, are two such transformations, show that their composition $T_3$, defined by $T_3(x) = T_2(T_1(x))$, is of the same form.

Find necessary and sufficient conditions on the numbers $a, b, c, d$ for $T^2$, the composition of $T$ with itself, to be the identity. Hence, or otherwise, find transformations $T_1$, $T_2$ and their composition $T_3$ such that $T_1^2$ and $T_2^2$ are each the identity but $T_3^2$ is not.

## Section B: Mechanics

### [STEP 3, 1996Q9]

A particle of mass $m$ is at rest on top of a smooth fixed sphere of radius $a$. Show that, if the particle is given a small displacement, it reaches the horizontal plane through the centre of the sphere at a distance

$$\frac{a(5\sqrt{5}+4\sqrt{23})}{27}$$

from the centre of the sphere.

[Air resistance should be neglected.]

### [STEP 3, 1996Q10]

Two rough solid circular cylinders, of equal radius and length and of uniform density, lie side by side on a rough plane inclined at an angle $\alpha$ to the horizontal, where $0 < \alpha < \frac{\pi}{2}$. Their axes are horizontal and they touch along their entire length. The weight of the upper cylinder is $W_1$ and the coefficient of friction between it and the plane is $\mu_1$. The corresponding quantities for the lower cylinder are $W_2$ and $\mu_2$ respectively and the coefficient of friction between the two cylinders is $\mu$. Show that for equilibrium to be possible:

(i) $W_1 > W_2$.

(ii) $\mu \geq \frac{W_1+W_2}{W_1-W_2}$.

(iii) $\mu_1 \geq \left(\frac{2W_1 \cot\alpha}{W_1+W_2} - 1\right)^{-1}$.

Find the similar inequality to (iii) for $\mu_2$.



### [STEP 3, 1996Q11]

A smooth circular wire of radius $a$ is held fixed in a vertical plane with light elastic strings of natural length $a$ and modulus $\lambda$ attached to the upper and lower extremities, $A$ and $C$ respectively, of the vertical diameter. The other ends of the two strings are attached to a small ring $B$ of mass $m$ which is free to slide on the wire. Show that, while both strings remain taut, the equation for the motion of the ring is

$$2ma\ddot{\theta} = \lambda(\cos\theta - \sin\theta) - mg\sin 2\theta,$$

where $\theta$ is the angle $\widehat{CAB}$.

Initially the system is at rest in equilibrium with $\sin\theta = \frac{3}{5}$. Deduce that $5\lambda = 24mg$.

The ring is now displaced slightly. Show that in the ensuing motion it will oscillate with period approximately

$$10\pi\sqrt{\frac{a}{91g}}.$$

# Section C: Probability and Statistics

## [STEP 3, 1996Q12]

It has been observed that Professor Ecks proves three types of theorems: 1, those that are correct and new; 2, those that are correct, but already known; 3, those that are false. It has also been observed that, if a certain of her theorems is of type $i$, then her next theorem is of type $j$ with probability $p_{ij}$, where $p_{ij}$ is the entry in the $i$th row and $j$th column of the following array:

$$\begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

Let $a_i$, $i = 1,2,3$, be the probability that a given theorem is of type $i$, and let $b_j$ be the consequent probability that the next theorem is of type $j$.

(i) Explain why $b_j = a_1 p_{1j} + a_2 p_{2j} + a_3 P_{3j}$.

(ii) Find values of $a_1$, $a_2$ and $a_3$ such that $b_i = a_i$ for $i = 1, 2, 3$.

(iii) For these values of the $a_i$ find the probabilities $q_{ij}$ that, if a particular theorem is of type $j$, then the *preceding* theorem was of type $i$.

## [STEP 3, 1996Q13]

Let $X$ be a random variable which takes only the finite number of different possible real values $x_1, x_2, \dots, x_n$. Define the expectation $\mathrm{E}(X)$ and the variance $\mathrm{var}(X)$ of $X$. Show that, if $a$ and $b$ are real numbers, then $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and express $\mathrm{var}(aX + b)$ similarly in terms of $\mathrm{var}(X)$.

Let $\lambda$ be a positive real number. By considering the contribution to $\mathrm{var}(X)$ of those $x_i$ for which $|x_i - \mathrm{E}(X)| \geq \lambda$, or otherwise, show that

$$\mathrm{P}[|X - \mathrm{E}(X)| \geq \lambda] \leq \frac{\mathrm{var}(X)}{\lambda^2}.$$

Let $k$ be a real number satisfying $k \geq \lambda$. If $|x_i - \mathrm{E}(X)| \leq k$ for all $i$, show that

$$\mathrm{P}[|X - \mathrm{E}(X)| \geq \lambda] \geq \frac{\mathrm{var}(X) - \lambda^2}{k^2 - \lambda^2}.$$

[STEP 3, 1996Q14]

Whenever I go cycling I start with my bike in good working order. However if all is well at time $t$, the probability that I get a puncture in the small interval $(t, t+\delta t)$ is $\alpha\delta t$. How many punctures can I expect to get on a journey during which my total cycling time is $T$?

When I get a puncture I stop immediately to repair it and the probability that, if I am repairing it at time $t$, the repair will be completed in time $(t, t+\delta t)$ is $\beta\delta t$. If $p(t)$ is the probability that I am repairing a puncture at time $t$, write down an equation relating $p(t)$ to $p(t+\delta t)$, and derive from this a differential equation relating $p'(t)$ and $p(t)$. Show that

$$p(t) = \frac{\alpha}{\alpha+\beta}\big(1 - \exp(-(\alpha+\beta)t)\big),$$

satisfies this differential equation with the appropriate initial condition.

Find an expression, involving $\alpha$, $\beta$ and $T$, for the time expected to be spent mending punctures during a journey of total time $T$. Hence, or otherwise, show that, the fraction of the journey expected to be spent mending punctures is given approximately by

$$\frac{\alpha T}{2} \quad \text{if} \quad (\alpha+\beta)T \quad \text{is small,}$$

and by

$$\frac{\alpha}{\alpha+\beta} \quad \text{if} \quad (\alpha+\beta)T \quad \text{is large.}$$

# STEP 3 1997

STEP 3 1997
[On-line Exam]

Scan the QR code or click on the link to take an on-line exam.
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**TIME ALLOWED: 180 MINUTES**

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**Calculators are not permitted.**

## Section A: Pure Mathematics

### [STEP 3, 1997Q1]

(i) By considering the series expansion of $(x^2+5x+4)\mathrm{e}^x$ or otherwise, show that

$$10\mathrm{e} = 4 + \frac{3^2}{1!} + \frac{4^2}{2!} + \frac{5^2}{3!} + \cdots.$$

(ii) Show that

$$5\mathrm{e} = 1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots.$$

(iii) Evaluate

$$1 + \frac{2^3}{1!} + \frac{3^3}{2!} + \frac{4^3}{3!} + \cdots.$$

### [STEP 3, 1997Q2]

Let

$$f(t) = \frac{\ln t}{t} \quad \text{for } t > 0.$$

Sketch the graph of $f(t)$ and find its maximum value. How many values of $t$ correspond to a given positive value of $f(t)$?

Find how many positive values of $y$ satisfy $x^y = y^x$ for a given positive value of $x$. Sketch the set of points $(x, y)$ which satisfy $x^y = y^x$ with $x, y > 0$.

### [STEP 3, 1997Q3]

By considering the solutions of the equation $z^n - 1 = 0$, or otherwise, show that

$$(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}) = 1 + z + z^2 + \cdots + z^{n-1},$$

where $z$ is any complex number and $\omega = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}$.

Let $A_1, A_2, A_3, \ldots, A_n$ be points equally spaced around a circle of radius $r$ centred at $O$ (so that they are the vertices of a regular $n$-sided polygon).

Show that

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \cdots + \overrightarrow{OA_n} = \mathbf{0}.$$

Deduce, or prove otherwise, that

$$\sum_{k=1}^{n} |A_1A_k|^2 = 2r^2n.$$

[STEP 3, 1997Q4]

In this question, you may assume that if $k_1, \dots, k_n$ are distinct positive real numbers, then

$$\frac{1}{n}\sum_{r=1}^{n} k_r > \left(\prod_{r=1}^{n} k_r\right)^{\frac{1}{n}},$$

i.e. their arithmetic mean is greater than their geometric mean.

Suppose that $a$, $b$, $c$ and $d$ are positive real numbers such that the polynomial

$$f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$$

has four distinct positive roots $p$, $q$, $r$ and $s$.

(i) Show that $pqr$, $qrs$, $rsp$, and $spq$ are distinct.

(ii) By considering the relationship between the coefficients of $f$ and its roots, show that $c > d$.

(iii) Explain why the polynomial $f'(x)$ must have three distinct roots.

(iv) By differentiating $f$, show that $b > c$.

(v) Show that $a > b$.

[STEP 3, 1997Q5]

Find the ratio, over one revolution, of the distance moved by a wheel rolling on a flat surface to the distance traced out by a point on its circumference.



[STEP 3, 1997Q6]

Suppose that $y_n$ satisfies the equations

$$(1-x^2)\frac{\mathrm{d}^2y_n}{\mathrm{d}x^2} - x\frac{\mathrm{d}y_n}{\mathrm{d}x} + n^2y_n = 0,$$

$$y_n(1) = 1, \qquad y_n(x) = (-1)^n y_n(-x).$$



If $x = \cos\theta$, show that

$$\frac{\mathrm{d}^2y_n}{\mathrm{d}\theta^2} + n^2y_n = 0,$$

and hence obtain $y_n$ as a function of $\theta$. Deduce that for $|x| \le 1$

$$y_0 = 1, \qquad y_1 = x,$$

$$y_{n+1} - 2xy_n + y_{n-1} = 0.$$

[STEP 3, 1997Q7]

For each positive integer $n$, let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots;$$
$$b_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots.$$

(i) Evaluate $b_n$.

(ii) Show that $0 < a_n < \frac{1}{n}$.

(iii) Deduce that $a_n = n!\,\mathrm{e} - [n!\,\mathrm{e}]$ (where $[x]$ is the integer part of $x$).

(iv) Hence show that e is irrational.

[STEP 3, 1997Q8]

Let $\mathbf{R}_\alpha$ be the $2 \times 2$ matrix that represents a rotation through the angle $\alpha$ and let

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(i) Find in terms of $a$, $b$ and $c$ an angle $\alpha$ such that the matrix $\mathbf{R}_{-\alpha}\mathbf{A}\mathbf{R}_\alpha$ is diagonal (i.e. has the value zero in top-right and bottom-left positions).

(ii) Find values of $a$, $b$ and $c$ such that the equation of the ellipse

$$x^2 + (y + 2x\cot 2\theta)^2 = 1 \quad \left(0 < \theta < \frac{\pi}{4}\right)$$

can be expressed in the form

$$(x \quad y)\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

Show that, for this $\mathbf{A}$, the matrix $\mathbf{R}_{-\alpha}\mathbf{A}\mathbf{R}_\alpha$ is diagonal if $\alpha = \theta$. Express the non-zero elements of this diagonal matrix in terms of $\theta$.

(iii) Deduce, or show otherwise, that the minimum and maximum distances from the centre to the circumference of this ellipse are $\tan\theta$ and $\cot\theta$.

## Section B: Mechanics

### [STEP 3, 1997Q9]

A uniform rigid rod $BC$ is suspended from a fixed point $A$ by light stretched springs $AB$, $AC$. The springs are of different natural lengths but the ratio of tension to extension is the same constant $\kappa$ for each. The rod is *not* hanging vertically. Show that the ratio of the lengths of the stretched springs is equal to the ratio of the natural lengths of the unstretched springs.

### [STEP 3, 1997Q10]

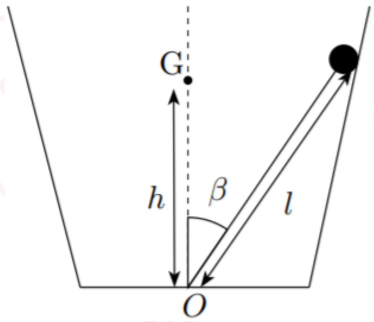
By pressing a finger down on it, a uniform spherical marble of radius $a$ is made to slide along a horizontal table top with an initial linear speed $v_0$ and an initial *backward* angular speed $\omega_0$ about the horizontal axis perpendicular to $v_0$. The frictional force between the marble and the table is constant (independent of speed).

Find the value of $\frac{v_0}{a\omega_0}$ for which the marble

(i) slides to a complete stop,

(ii) comes to a stop and then rolls back towards its initial position with linear speed $\frac{v_0}{7}$.

### [STEP 3, 1997Q11]




A heavy symmetrical bell and clapper can both swing freely in a vertical plane about a point $O$ on a horizontal beam at the apex of the bell. The mass of the bell is $M$ and its moment of inertia about the beam is $Mk^2$. Its centre of mass, $G$, is a distance $h$ from $O$. The clapper may be regarded as a small heavy ball on a light rod of length $l$. Initially the bell is held with its axis vertical and its mouth above the beam. The clapper ball rests against the side of the bell, with the rod making an angle $\beta$ with the axis. The bell is then released. Show that, at the moment when the clapper and bell separate, the clapper rod makes an angle $\alpha$ with the upward vertical, where

$$\cot\alpha = \cot\beta - \frac{k^2}{hl}\operatorname{cosec}\beta.$$

## Section C: Probability and Statistics

### [STEP 3, 1997Q12]

(i) I toss a biased coin which has a probability $p$ of landing heads and a probability $q = 1 - p$ of landing tails. Let $K$ be the number of tosses required to obtain the first head and let

$$G(s) = \sum_{k=1}^{\infty} \mathrm{P}(K = k)s^k.$$

Show that

$$G(s) = \frac{ps}{1 - qs}$$

and hence find the expectation and variance of $K$.

(ii) I sample cards at random with replacement from a normal pack of 52. Let $N$ be the total number of draws I make in order to sample every card at least once. By expressing $N$ as a sum $N = N_1 + N_2 + \cdots + N_{52}$ of random variables, or otherwise, find the expectation of $N$. Estimate the numerical value of this expectation, using the approximations $\mathrm{e} \approx 2.7$ and $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx 0.5 + \ln n$ if $n$ is large.

### [STEP 3, 1997Q13]

Let $X$ and $Y$ be independent standard normal random variables. The probability density function, $f$, of each is therefore given by

$$f(x) = (2\pi)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}x^2}.$$

(i) Find the moment generating function $E\left(\mathrm{e}^{\theta X}\right)$ of $X$.

(ii) Find the moment generating function of $aX + bY$ and hence obtain the condition on $a$ and $b$ which ensures that $aX + bY$ has the same distribution as $X$ and $Y$.

(iii) Let $Z = \mathrm{e}^{\mu + \sigma X}$. Show that

$$E\left(Z^\theta\right) = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2},$$

and hence find the expectation and variance of $Z$.

### [STEP 3, 1997Q14]

An industrial process produces rectangular plates of mean length $\mu_1$ and mean breadth $\mu_2$. The length and breadth vary independently with non-zero standard deviations $\sigma_1$ and $\sigma_2$ respectively. Find the means and standard deviations of the perimeter and of the area of the plates. Show that the perimeter and area are not independent.

# STEP 3 1998

STEP 3 1998
On-line Exam

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# Section A: Pure Mathematics

## [STEP 3, 1998Q1]

Let

$$f(x) = \sin^2 x + 2\cos x + 1$$

for $0 \le x \le 2\pi$. Sketch the curve $y = f(x)$, giving the coordinates of the stationary points. Now let

$$g(x) = \frac{af(x)+b}{cf(x)+d} \qquad ad \neq bc,\ \ d \neq -3c,\ \ d \neq c.$$

Show that the stationary points of $y = g(x)$ occur at the same values of $x$ as those of $y = f(x)$, and find the corresponding values of $g(x)$.

Explain why, if $\frac{d}{c} < -3$ or $\frac{d}{c} > 1$, $|g(x)|$ cannot be arbitrarily large.

## [STEP 3, 1998Q2]

Let

$$I(a,b) = \int_0^1 t^a(1-t)^b \, \mathrm{d}t \qquad (a \ge 0, b \ge 0).$$

(i) Show that $I(a,b) = I(b,a)$.

(ii) Show that $I(a,b) = I(a+1,b) + I(a,b+1)$.

(iii) Show that $(a+1)I(a,b) = bI(a+1,b-1)$ when $a$ and $b$ are positive and hence calculate $I(a,b)$ when $a$ and $b$ are positive integers.

## [STEP 3, 1998Q3]

The value $V_N$ of a bond after $N$ days is determined by the equation

$$V_{N+1} = (1+c)V_N - d \qquad (c > 0, d > 0),$$

where $c$ and $d$ are given constants. By looking for solutions of the form $V_T = Ak^T + B$ for some constants $A$, $B$ and $k$, or otherwise, find $V_N$ in terms of $V_0$.

What is the solution for $c = 0$? Show that this is the limit (for fixed $N$) as $c \to 0$ of your solution for $c > 0$.

[STEP 3, 1998Q4]

Show that the equation (in plane polar coordinates) $r = \cos\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, represents a circle.

Sketch the curve $r = \cos 2\theta$ for $0 \le \theta \le 2\pi$, and describe the curves $r = \cos 2n\theta$, where $n$ is an integer. Show that the area enclosed by such a curve is independent of $n$.

Sketch also the curve $r = \cos 3\theta$ for $0 \le \theta \le 2\pi$.

[STEP 3, 1998Q5]

The exponential of a square matrix $\mathbf{A}$ is defined to be

$$\exp(\mathbf{A}) = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r,$$

where $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{I}$ is the identity matrix.

Let

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that $\mathbf{M}^2 = -\mathbf{I}$ and hence express $\exp(\theta\mathbf{M})$ as a single $2 \times 2$ matrix, where $\theta$ is a real number. Explain the geometrical significance of $\exp(\theta\mathbf{M})$.

Let

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Express similarly $\exp(s\mathbf{N})$, where $s$ is a real number, and explain the geometrical significance of $\exp(s\mathbf{N})$.

For which values of $\theta$ does

$$\exp(s\mathbf{N})\exp(\theta\mathbf{M}) = \exp(\theta\mathbf{M})\exp(s\mathbf{N})$$

for all $s$? Interpret this fact geometrically.



[STEP 3, 1998Q6]

(i) Show that four vertices of a cube, no two of which are adjacent, form the vertices of a regular tetrahedron. Hence, or otherwise, find the volume of a regular tetrahedron whose edges are of unit length.

(ii) Find the volume of a regular octahedron whose edges are of unit length.

(iii) Show that the centres of the faces of a cube form the vertices of a regular octahedron. Show that its volume is half that of the tetrahedron whose vertices are the vertices of the cube.

[A regular tetrahedron (octahedron) has four (eight) faces, all equilateral triangles.]

[STEP 3, 1998Q7]

Sketch the graph of $f(s) = e^s(s-3)+3$ for $0 \le s < \infty$. Taking $e \approx 2.7$, find the smallest positive integer, $m$, such that $f(m) > 0$.

Now let

$$b(x) = \frac{x^3}{e^{\frac{x}{T}} - 1}$$

where $T$ is a positive constant. Show that $b(x)$ has a single turning point in $0 < x < \infty$. By considering the behaviour for small $x$ and for large $x$, sketch $b(x)$ for $0 \le x < \infty$.

Let

$$\int_0^\infty b(x)\,dx = B,$$

which may be assumed to be finite. Show that $B = KT^n$ where $K$ is a constant, and $n$ is an integer which you should determine.

Given that $B \approx 2\int_0^{Tm} b(x)\,dx$, use your graph of $b(x)$ to find a rough estimate for $K$.

[STEP 3, 1998Q8]

(i) Show that the line $\mathbf{r} = \mathbf{b} + \lambda\hat{\mathbf{m}}$, where $\hat{\mathbf{m}}$ is a unit vector, intersects the sphere $\mathbf{r}.\mathbf{r} = a^2$ at two points if

$$a^2 > \mathbf{b}.\mathbf{b} - (\mathbf{b}.\hat{\mathbf{m}})^2.$$

Write down the corresponding condition for there to be precisely one point of intersection. If this point has position vector $\mathbf{p}$, show that $\hat{\mathbf{m}}.\mathbf{p} = 0$.

(ii) Now consider a second sphere of radius $a$ and a plane perpendicular to a unit vector $\hat{\mathbf{n}}$. The centre of the sphere has position vector $\mathbf{d}$ and the minimum distance from the origin to the plane is $l$. What is the condition for the plane to be tangential to this second sphere?

(iii) Show that the first and second spheres intersect at right angles (i.e. the two radii to each point of intersection are perpendicular) if

$$\mathbf{d}.\mathbf{d} = 2a^2.$$

# Section B: Mechanics

## [STEP 3, 1998Q9]

A uniform right circular cone of mass $m$ has base of radius $a$ and perpendicular height $h$ from base to apex. Show that its moment of inertia about its axis is $\frac{3}{10}ma^2$, and calculate its moment of inertia about an axis through its apex parallel to its base.

[Any theorems used should be stated clearly.]

The cone is now suspended from its apex and allowed to perform small oscillations. Show that their period is

$$2\pi\sqrt{\frac{4h^2+a^2}{5gh}}.$$

[You may assume that the centre of mass of the cone is a distance $\frac{3}{4}h$ from its apex.]

## [STEP 3, 1998Q10]

Two identical spherical balls, moving on a horizontal, smooth table, collide in such a way that both momentum and kinetic energy are conserved. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities of the balls before the collision and let $\mathbf{v}'_1$ and $\mathbf{v}'_2$ be the velocities of the balls after the collision, where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}'_1$ and $\mathbf{v}'_2$ are two-dimensional vectors. Write down the equations for conservation of momentum and kinetic energy in terms of these vectors. Hence show that their relative speed is also conserved.

Show that, if one ball is initially at rest but after the collision both balls are moving, their final velocities are perpendicular.

Now suppose that one ball is initially at rest, and the second is moving with speed $V$. After a collision in which they lose a proportion $k$ of their original kinetic energy ($0 \leq k \leq 1$), the direction of motion of the second ball has changed by an angle $\theta$. Find a quadratic equation satisfied by the final speed of the second ball, with coefficients depending on $k$, $V$ and $\theta$. Hence show that $k \leq \frac{1}{2}$.

[STEP 3, 1998Q11]

Consider a simple pendulum of length $l$ and angular displacement $\theta$, which is **not** assumed to be small. Show that

$$\frac{1}{2}l\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = g(\cos\theta - \cos\gamma),$$

where $\gamma$ is the maximum value of $\theta$. Show also that the period $P$ is given by

$$P = 2\sqrt{\frac{l}{g}}\int_0^{\gamma}\left(\sin^2\left(\frac{\gamma}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)^{-\frac{1}{2}}\mathrm{d}\theta.$$

By using the substitution $\sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{\gamma}{2}\right)\sin\phi$, and then finding an approximate expression for the integrand using the binomial expansion, show that for small values of $\gamma$ the period is approximately

$$2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\gamma^2}{16}\right).$$

## Section C: Probability and Statistics

### [STEP 3, 1998Q12]

The mountain villages $A$, $B$, $C$ and $D$ lie at the vertices of a tetrahedron, and each pair of villages is joined by a road. After a snowfall the probability that any road is blocked is $p$, and is independent of the conditions of any other road. The probability that, after a snowfall, it is possible to travel from any village to any other village by some route is $P$. Show that

$$P = 1 - p^2(6p^3 - 12p^2 + 3p + 4).$$

### [STEP 3, 1998Q13]

Write down the probability of obtaining $k$ heads in $n$ tosses of a fair coin. Now suppose that $k$ is known but $n$ is unknown. A *maximum likelihood estimator* (MLE) of $n$ is defined to be a value (which must be an integer) of $n$ which maximizes the probability of $k$ heads.

A friend has thrown a fair coin a number of times. She tells you that she has observed one head. Show that in this case there are *two* MLEs of the number of tosses she has made.

She now tells you that in a repeat of the exercise she has observed $k$ heads. Find the two MLEs of the number of tosses she has made.

She next uses a coin biased with probability $p$ (known) of showing a head, and again tells you that she has observed $k$ heads. Find the MLEs of the number of tosses made. What is the condition for the MLE to be unique?



### [STEP 3, 1998Q14]

A hostile naval power possesses a large, unknown number $N$ of submarines. Interception of radio signals yields a small number $n$ of their identification numbers $X_i$ $(i = 1, 2, \dots, n)$, which are taken to be independent and uniformly distributed over the continuous range from 0 to $N$. Show that $Z_1$ and $Z_2$, defined by

$$Z_1 = \frac{n+1}{n}\text{Max}\{X_1, X_2, \dots, X_n\} \quad \text{and} \quad Z_2 = \frac{2}{n}\sum_{i=1}^{n} X_i\,,$$

both have means equal to $N$.

Calculate the variance of $Z_1$ and of $Z_2$. Which estimator do you prefer, and why?

# STEP 3 1999

STEP 3 1999
On-line Exam

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Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

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## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

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| **Section A** | Pure Mathematics |
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There are 14 questions in this paper.

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**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 1999Q1]

Consider the cubic equation

$$x^3 - px^2 + qx - r = 0,$$

where $p \neq 0$ and $r \neq 0$.

(i) If the three roots can be written in the form $ak^{-1}$, $a$ and $ak$ for some constants $a$ and $k$, show that one root is $\frac{q}{p}$ and that $q^3 - rp^3 = 0$.

(ii) If $r = \frac{q^3}{p^3}$, show that $\frac{q}{p}$ is a root and that the product of the other two roots is $\left(\frac{q}{p}\right)^2$. Deduce that the roots are in geometric progression.

(iii) Find a necessary and sufficient condition involving $p$, $q$ and $r$ for the roots to be in arithmetic progression.

## [STEP 3, 1999Q2]

(i) Let $f(x) = (1 + x^2)\mathrm{e}^x$. Show that $f'(x) \geq 0$ and sketch the graph of $f(x)$. Hence, or otherwise, show that the equation

$$(1 + x^2)\mathrm{e}^x = k,$$

where $k$ is a constant, has exactly one real root if $k > 0$ and no real roots if $k \leq 0$.



(ii) Determine the number of real roots of the equation

$$(\mathrm{e}^x - 1) - k\tan^{-1} x = 0$$

in the cases (a) $0 < k \leq \frac{2}{\pi}$ and (b) $\frac{2}{\pi} < k < 1$.

## [STEP 3, 1999Q3]

Justify, by means of a sketch, the formula

$$\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n} f\left(1 + \frac{m}{n}\right)\right\} = \int_1^2 f(x)\,\mathrm{d}x.$$

Show that

$$\lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right\} = \ln 2.$$

Evaluate

$$\lim_{n\to\infty}\left\{\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2}\right\}.$$

## [STEP 3, 1999Q4]

A polyhedron is a solid bounded by $F$ plane faces, which meet in $E$ edges and $V$ vertices. You may assume *Euler's formula*, that $V - E + F = 2$.

In a regular polyhedron the faces are equal regular $m$-sided polygons, $n$ of which meet at each vertex. Show that

$$F = \frac{4n}{h},$$

where $h = 4 - (n-2)(m-2)$.

By considering the possible values of $h$, or otherwise, prove that there are only five regular polyhedra, and find $V$, $E$ and $F$ for each.

## [STEP 3, 1999Q5]

The sequence $u_0, u_1, u_2, \dots$ is defined by

$$u_0 = 1, \qquad u_1 = 1, \qquad u_{n+1} = u_n + u_{n-1} \quad \text{for} \quad n \geq 1.$$

Prove that

$$u_{n+2}^2 + u_{n-1}^2 = 2(u_{n+1}^2 + u_n^2).$$

Using induction, or otherwise, prove the following result:

$$u_{2n} = u_n^2 + u_{n-1}^2 \quad \text{and} \quad u_{2n+1} = u_{n+1}^2 - u_{n-1}^2$$

for any positive integer $n$.



## [STEP 3, 1999Q6]

A closed curve is given by the equation

$$x^{\frac{2}{n}} + y^{\frac{2}{n}} = a^{\frac{2}{n}} \qquad (*)$$

where $n$ is an odd integer and $a$ is a positive constant. Find a parametrization $x = x(t)$, $y = y(t)$ which describes the curve anticlockwise as $t$ ranges from 0 to $2\pi$.

Sketch the curve in the case $n = 3$, justifying the main features of your sketch.

The area $A$ enclosed by such a curve is given by the formula

$$A = \frac{1}{2}\int_0^{2\pi}\left[x(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} - y(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]\mathrm{d}t.$$

Use this result to find the area enclosed by $(*)$ for $n = 3$.

[STEP 3, 1999Q7]

Let $a$ be a non-zero real number and define a binary operation on the set of real numbers by

$$x * y = x + y + axy.$$

Show that the operation $*$ is associative.

Show that $(G, *)$ is a group, where $G$ is the set of all real numbers except for one number which you should identify.

Find a subgroup of $(G, *)$ which has exactly 2 elements.

[STEP 3, 1999Q8]

The function $y(x)$ is defined for $x \geq 0$ and satisfies the conditions

$$y = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad \text{at} \quad x = 0.$$

When $x$ is in the range $2(n-1)\pi < x < 2n\pi$, where $n$ is a positive integer, $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + n^2 y = 0.$$

Both $y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ are continuous at $x = 2n\pi$ for $n = 0, 1, 2, \ldots$.

(i) Find $y(x)$ for $0 \leq x \leq 2\pi$.

(ii) Show that $y(x) = \frac{1}{2}\sin 2x$ for $2\pi \leq x \leq 4\pi$, and find $y(x)$ for all $x \geq 0$.

(iii) Show that

$$\int_0^\infty y^2 \,\mathrm{d}x = \pi \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

## Section B: Mechanics

### [STEP 3, 1999Q9]

The gravitational force between two point particles of masses $m$ and $m'$ is mutually attractive and has magnitude

$$\frac{Gmm'}{r^2}$$

where $G$ is a constant and $r$ is the distance between them.

A particle of unit mass lies on the axis of a thin uniform circular ring of radius $r$ and mass $m$, at a distance $x$ from its centre. Explain why the net force on the particle is directed towards the centre of the ring and show that its magnitude is

$$\frac{Gmx}{(x^2+r^2)^{\frac{3}{2}}}.$$

The particle now lies inside a thin hollow spherical shell of uniform density, mass $M$ and radius $a$, at a distance $b$ from its centre. Show that the particle experiences no gravitational force due to the shell.

### [STEP 3, 1999Q10]

A chain of mass $m$ and length $l$ is composed of $n$ small smooth links. It is suspended vertically over a horizontal table with its end just touching the table, and released so that it collapses inelastically onto the table. Calculate the change in momentum of the $(k+1)$th link from the bottom of the chain as it falls onto the table.

Write down an expression for the total impulse sustained by the table in this way from the whole chain. By approximating the sum by an integral, show that this total impulse is approximately

$$\frac{2}{3}m\sqrt{2gl}$$

when $n$ is large.



### [STEP 3, 1999Q11]

Calculate the moment of inertia of a uniform thin circular hoop of mass $m$ and radius $a$ about an axis perpendicular to the plane of the hoop through a point on its circumference.

The hoop, which is rough, rolls with speed $v$ on a rough horizontal table straight towards the edge and rolls over the edge without initially losing contact with the edge. Show that the hoop will lose contact with the edge when it has rotated about the edge of the table through an angle $\theta$, where

$$\cos\theta = \frac{1}{2} + \frac{v^2}{2ag}.$$

# Section C: Probability and Statistics

## [STEP 3, 1999Q12]

In the game of endless cricket the scores $X$ and $Y$ of the two sides are such that

$$P(X = j, Y = k) = e^{-1}\frac{(j+k)\lambda^{j+k}}{j!\,k!},$$

for some positive constant $\lambda$, where $j, k = 0, 1, 2, \ldots$.

(i) Find $P(X + Y = n)$ for each $n > 0$.

(ii) Show that $2\lambda e^{2\lambda - 1} = 1$.

(iii) Show that $2xe^{2x-1}$ is an increasing function of $x$ for $x > 0$ and deduce that the equation in (ii) has at most one solution and hence determine $\lambda$.

(iv) Calculate the expectation $E(2^{X+Y})$.

## [STEP 3, 1999Q13]

The cakes in our canteen each contain exactly four currants, each currant being randomly placed in the cake. I take a proportion $X$ of a cake where $X$ is a random variable with density function

$$f(x) = Ax$$

for $0 \le x \le 1$ where $A$ is a constant.

(i) What is the expected number of currants in my portion?

(ii) If I find all four currants in my portion, what is the probability that I took more than half the cake?

## [STEP 3, 1999Q14]

In the basic version of Horizons (H1) the player has a maximum of $n$ turns, where $n \ge 1$. At each turn, she has a probability $p$ of success, where $0 < p < 1$. If her first success is at the $r$th turn, where $1 \le r \le n$, she collects $r$ pounds and then withdraws from the game. Otherwise, her winnings are nil. Show that in H1, her expected winnings are

$$p^{-1}[1 + nq^{n+1} - (n+1)q^n] \text{ pounds},$$

where $q = 1 - p$.

The rules of H2 are the same as those of H1, except that $n$ is randomly selected from a Poisson distribution with parameter $\lambda$. If $n = 0$ her winnings are nil. Otherwise she plays H1 with the selected $n$. Show that in H2, her expected winnings are

$$\frac{1}{p}\left(1 - e^{-\lambda p}\right) - \lambda q e^{-\lambda p} \text{ pounds}.$$

# STEP 3 2000

STEP 3 2000
On-line Exam

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# Section A: Pure Mathematics

## [STEP 3, 2000Q1]

Sketch on the same axes the two curves $C_1$ and $C_2$, given by

$$\begin{aligned} C_1 &: \quad xy = 1, \\ C_2 &: \quad x^2 - y^2 = 2. \end{aligned}$$

The curves intersect at $P$ and $Q$. Given that the coordinates of $P$ are $(a, b)$ (which you need not evaluate), write down the coordinates of $Q$ in terms of $a$ and $b$.

The tangent to $C_1$ through $P$ meets the tangent to $C_2$ through $Q$ at the point $M$, and the tangent to $C_2$ through $P$ meets the tangent to $C_1$ through $Q$ at $N$. Show that the coordinates of $M$ are $(-b, a)$ and write down the coordinates of $N$.

Show that $PMQN$ is a square.

## [STEP 3, 2000Q2]

Use the substitution $x = 2 - \cos\theta$ to evaluate the integral

$$\int_{\frac{3}{2}}^{2} \left(\frac{x-1}{3-x}\right)^{\frac{1}{2}} \mathrm{d}x.$$

Show that, for $a < b$,

$$\int_{p}^{q} \left(\frac{x-a}{b-x}\right)^{\frac{1}{2}} \mathrm{d}x = \frac{(b-a)\left(\pi + 3\sqrt{3} - 6\right)}{12},$$

where $p = \frac{3a+b}{4}$ and $q = \frac{a+b}{2}$.



## [STEP 3, 2000Q3]

Given that $\alpha = \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$, prove that $1 + \alpha^2 = \alpha$.

A triangle in the Argand plane has vertices $A$, $B$, and $C$ represented by the complex numbers $p$, $q\alpha^2$ and $-r\alpha$ respectively, where $p$, $q$ and $r$ are positive real numbers. Sketch the triangle $ABC$.

Three equilateral triangles $ABL$, $BCM$ and $CAN$ (each lettered clockwise) are erected on sides $AB$, $BC$ and $CA$ respectively. Show that the complex number representing $N$ is $(1-\alpha)p - \alpha^2 r$ and find similar expressions for the complex numbers representing $L$ and $M$.

Show that lines $LC$, $MA$ and $NB$ all meet at the origin, and that these three line segments have the common length $p + q + r$.

[STEP 3, 2000Q4]

The function $f(x)$ is defined by

$$f(x)=\frac{x(x-2)(x-a)}{x^2-1}.$$

Prove algebraically that the line $y=x+c$ intersects the curve $y=f(x)$ if $|a|\geq 1$, but there are values of $c$ for which there are no points of intersection if $|a|<1$.

Find the equation of the oblique asymptote of the curve $y=f(x)$. Sketch the graph in the two cases (i) $a<-1$; and (ii) $-1<a<-\frac{1}{2}$. (You need not calculate the turning points.)

[STEP 3, 2000Q5]

Given two non-zero vectors $\mathbf{a}=\binom{a_1}{a_2}$ and $\mathbf{b}=\binom{b_1}{b_2}$ we define $\Delta(\mathbf{a},\mathbf{b})$ by $\Delta(\mathbf{a},\mathbf{b})=a_1b_2-a_2b_1$.

Let $A$, $B$ and $C$ be points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, no two of which are parallel. Let $P$, $Q$ and $R$ be points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, none of which are parallel.

(i) Show that there exists a $2\times 2$ matrix $\mathbf{M}$ such that $P$ and $Q$ are the images of $A$ and $B$ under the transformation represented by $\mathbf{M}$.

(ii) Show that $\Delta(\mathbf{a},\mathbf{b})\mathbf{c}+\Delta(\mathbf{c},\mathbf{a})\mathbf{b}+\Delta(\mathbf{b},\mathbf{c})\mathbf{a}=\mathbf{0}$.

Hence, or otherwise, prove that a necessary and sufficient condition for the points $P$, $Q$, and $R$ to be the images of points $A$, $B$ and $C$ under the transformation represented by some $2\times 2$ matrix $\mathbf{M}$ is that

$$\Delta(\mathbf{a},\mathbf{b}):\Delta(\mathbf{b},\mathbf{c}):\Delta(\mathbf{c},\mathbf{a})=\Delta(\mathbf{p},\mathbf{q}):\Delta(\mathbf{q},\mathbf{r}):\Delta(\mathbf{r},\mathbf{p}).$$

[STEP 3, 2000Q6]

Given that

$$x^4+px^2+qx+r=(x^2-ax+b)(x^2+ax+c),$$

express $p$, $q$ and $r$ in terms of $a$, $b$ and $c$.

Show also that $a^2$ is a root of the cubic equation

$$u^3+2pu^2+(p^2-4r)u-q^2=0.$$

Explain why this equation always has a non-negative root, and verify that $u=9$ is a root in the case $p=-1$, $q=-6$, $r=15$.

Hence, or otherwise, express

$$y^4-8y^3+23y^2-34y+39$$

as a product of two quadratic factors.

## [STEP 3, 2000Q7]

Given that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots,$$

Use the binomial theorem to show that

$$\left(1 + \frac{1}{n}\right)^n < e$$

for any positive integer $n$.

The product $P(n)$ is defined, for any positive integer $n$, by

$$P(n) = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \cdots \cdot \frac{2^n + 1}{2^n}.$$

Use the arithmetic-geometric mean inequality,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}},$$

to show that $P(n) < e$ for all $n$.

Explain briefly why $P(n)$ tends to a limit as $n \to \infty$. Show that this limit, $L$, satisfies $2 < L \leq e$.

## [STEP 3, 2000Q8]



The sequence $a_n$ is defined by $a_0 = 1$, $a_1 = 1$, and

$$a_n = \frac{1 + a_{n-1}^2}{a_{n-2}} \qquad (n \geq 2).$$

Prove by induction that

$$a_n = 3a_{n-1} - a_{n-2} \qquad (n \geq 2).$$

Hence show that

$$a_n = \frac{\alpha^{2n-1} + \alpha^{-(2n-1)}}{\sqrt{5}} \qquad (n \geq 1),$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

## Section B: Mechanics

### [STEP 3, 2000Q9]

Two small discs of masses $m$ and $\mu m$ lie on a smooth horizontal surface. The disc of mass $\mu m$ is at rest, and the disc of mass $m$ is projected towards it with velocity $\mathbf{u}$. After the collision, the disc of mass $\mu m$ moves in the direction given by unit vector $\hat{\mathbf{n}}$. The collision is perfectly elastic.

(i) Show that the speed of the disc of mass $\mu m$ after the collision is $\frac{2\mathbf{u}.\hat{\mathbf{n}}}{1+\mu}$.

(ii) Given that the two discs have equal kinetic energy after the collision, find an expression for the cosine of the angle between $\hat{\mathbf{n}}$ and $\mathbf{u}$ and show that $3-\sqrt{8} \leq \mu \leq 3+\sqrt{8}$.

### [STEP 3, 2000Q10]

A sphere of radius $a$ and weight $W$ rests on horizontal ground. A thin uniform beam of weight $3\sqrt{3}W$ and length $2a$ is freely hinged to the ground at $X$, which is a distance $\sqrt{3}a$ from the point of contact of the sphere with the ground. The beam rests on the sphere, lying in the same vertical plane as the centre of the sphere. The coefficients of friction between the beam and the sphere and between the sphere and the ground are $\mu_1$ and $\mu_2$ respectively.

Given that the sphere is on the point of slipping at its contacts with both the ground and the beam, find the values of $\mu_1$ and $\mu_2$.

### [STEP 3, 2000Q11]

A thin beam is fixed at a height $2a$ above a horizontal plane. A uniform straight rod $ACB$ of length $9a$ and mass $m$ is supported by the beam at $C$. Initially, the rod is held so that it is horizontal and perpendicular to the beam. The distance $AC$ is $3a$, and the coefficient of friction between the beam and the rod is $\mu$.

The rod is now released. Find the minimum value of $\mu$ for which $B$ strikes the horizontal plane before slipping takes place at $C$.

## Section C: Probability and Statistics

### [STEP 3, 2000Q12]

In a lottery, any one of $N$ numbers, where $N$ is large, is chosen at random and independently for each player by machine. Each week there are $2N$ players and one winning number is drawn. Write down an exact expression for the probability that there are three or fewer winners in a week, given that you hold a winning ticket that week. Using the fact that

$$\left(1-\frac{a}{n}\right)^n \approx e^{-a}$$

for $n$ much larger than $a$, or otherwise, show that this probability is approximately $\frac{2}{3}$.

Discuss briefly whether this probability would increase or decrease if the numbers were chosen by the players.

Show that the expected number of winners in a week, given that you hold a winning ticket that week, is $3 - N^{-1}$.

### [STEP 3, 2000Q13]

A set of $n$ dice is rolled repeatedly. For each die the probability of showing a six is $p$. Show that the probability that the first of the dice to show a six does so on the $r$th roll is

$$q^{nr}(q^{-n}-1)$$

where $q = 1 - p$.



Determine, and simplify, an expression for the probability generating function for this distribution, in terms of $q$ and $n$. The first of the dice to show a six does so on the $R$th roll. Find the expected value of $R$ and show that, in the case $n = 2$, $p = \frac{1}{6}$, this value is $\frac{36}{11}$.

Show that the probability that the last of the dice to show a six does so on the $r$th roll is

$$(1-q^r)^n - (1-q^{r-1})^n.$$

Find, for the case $n = 2$, the probability generating function. The last of the dice to show a six does so on the $S$th roll. Find the expected value of $S$ and evaluate this when $p = \frac{1}{6}$.

### [STEP 3, 2000Q14]

The random variable $X$ takes only the values $x_1$ and $x_2$ (where $x_1 \neq x_2$), and the random variable $Y$ takes only the values $y_1$ and $y_2$ (where $y_1 \neq y_2$). Their joint distribution is given by

$$\mathrm{P}(X = x_1, Y = y_1) = a; \quad \mathrm{P}(X = x_1, Y = y_2) = q - a; \quad \mathrm{P}(X = x_2, Y = y_1) = p - a.$$

Show that if $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$ then

$$(a - pq)(x_1 - x_2)(y_1 - y_2) = 0.$$

Hence show that two random variables each taking only two distinct values are independent if $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.

Give a joint distribution for two random variables $A$ and $B$, each taking the three values $-1$, $0$ and $1$ with probability $\frac{1}{3}$, which have $\mathrm{E}(AB) = \mathrm{E}(A)\mathrm{E}(B)$, but which are not independent.

# STEP 3 2001

STEP 3 2001

On-line Exam

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# Section A: Pure Mathematics

## [STEP 3, 2001Q1]

Given that $y = \ln\left(x + \sqrt{x^2+1}\right)$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}}$.

Prove by induction that, for $n \geq 0$,

$$(x^2+1)y^{n+2} + (2n+1)xy^{n+1} + n^2y^n = 0,$$

where $y^n = \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ and $y^0 = y$.

Using this result in the case $x = 0$, or otherwise, show that the Maclaurin series for $y$ begins

$$x - \frac{x^3}{6} + \frac{3x^5}{40}$$

and find the next non-zero term.

## [STEP 3, 2001Q2]

Show that $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$.

Show that the area of the region defined by the inequalities $y^2 \geq x^2 - 8$ and $x^2 \geq 25y^2 - 16$ is $\frac{72}{5}\ln 2$.

## [STEP 3, 2001Q3]

Consider the equation

$$x^2 - bx + c = 0,$$

where $b$ and $c$ are real numbers.

(i) Show that the roots of the equation are real and positive if and only if $b > 0$ and $b^2 \geq 4c > 0$, and sketch the region of the $b$-$c$ plane in which these conditions hold.

(ii) Sketch the region of the $b$-$c$ plane in which the roots of the equation are real and less than 1 in magnitude.

[STEP 3, 2001Q4]

In this question, the function $\sin^{-1}$ is defined to have domain $-1 \le x \le 1$ and range $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and the function $\tan^{-1}$ is defined to have the real numbers as its domain and range $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(i) Let

$$g(x) = \frac{2x}{1+x^2}, \qquad -\infty < x < \infty.$$

Sketch the graph of $g(x)$ and state the range of $g$.

(ii) Let

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \qquad -\infty < x < \infty.$$

Show that $f(x) = 2\tan^{-1} x$ for $-1 \le x \le 1$ and $f(x) = \pi - 2\tan^{-1} x$ for $x \ge 1$.

Sketch the graph of $f(x)$.

[STEP 3, 2001Q5]

Show that the equation $x^3 + px + q = 0$ has exactly one real solution if $p \ge 0$.

A parabola $C$ is given parametrically by

$$x = at^2, \qquad y = 2at \quad (a > 0).$$

Find an equation which must be satisfied by $t$ at points on $C$ at which the normal passes through the point $(h, k)$. Hence show that, if $h \le 2a$, exactly one normal to $C$ will pass through $(h, k)$.

Find, in Cartesian form, the equation of the locus of the points from which exactly two normal can be drawn to $C$. Sketch the locus.



[STEP 3, 2001Q6]

The plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

meets the co-ordinate axes at the points $A$, $B$ and $C$. The point $M$ has coordinates $\left(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c\right)$ and $O$ is the origin.

Show that $OM$ meets the plane at the centroid $\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right)$ of triangle $ABC$. Show also that the perpendiculars to the plane from $O$ and from $M$ meet the plane at the orthocentre and at the circumcentre of triangle $ABC$ respectively.

Hence prove that the centroid of a triangle lies on the line segment joining its orthocentre and circumcentre, and that it divides this line segment in the ratio $2:1$.

[The *orthocentre* of a triangle is the point at which the three altitudes intersect; the *circumcentre* of a triangle is the point equidistant from the three vertices.]

## [STEP 3, 2001Q7]

Sketch the graph of the function $\ln x - \frac{1}{2}x^2$.

Show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2 - 1}$$

describes a family of parabolas each of which passes through the points $(1, 0)$ and $(-1, 0)$ and has its vertex on the $y$-axis.

Hence find the equation of the curve that passes through the point $(1, 1)$ and intersects each of the above parabolas orthogonally. Sketch this curve.

[Two curves intersect *orthogonally* if their tangents at the point of intersection are perpendicular.]

## [STEP 3, 2001Q8]

(i) Prove that the equations

$$|z - (1 + \mathrm{i})|^2 = 2 \qquad (*)$$

and

$$|z - (1 - \mathrm{i})|^2 = 2|z - 1|^2$$

describe the same locus in the complex $z$-plane. Sketch this locus.

(ii) Prove that the equation

$$\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{4} \qquad (**)$$

describes part of this same locus, and show on your sketch which part.

(iii) The complex number $w$ is related to $z$ by

$$w = \frac{2}{z}.$$

Determine the locus produced in the complex $w$-plane if $z$ satisfies $(*)$. Sketch this locus and indicate the part of this locus that corresponds to $(**)$.

## Section B: Mechanics

### [STEP 3, 2001Q9]

$B_1$ and $B_2$ are parallel, thin, horizontal fixed beams. $B_1$ is a vertical distance $d\sin\alpha$ above $B_2$, and a horizontal distance $d\cos\alpha$ from $B_2$, where $0<\alpha<\frac{\pi}{2}$. A long heavy plank is held so that it rests on the two beams, perpendicular to each, with its centre of gravity at $B_1$. The coefficients of friction between the plank and $B_1$ and $B_2$ are $\mu_1$ and $\mu_2$, respectively, where $\mu_1<\mu_2$ and $\mu_1+\mu_2=2\tan\alpha$.

The plank is released and slips over the beams experiencing a force of resistance from each beam equal to the limiting frictional force (i.e. the product of the appropriate coefficient of friction and the normal reaction). Show that it will come to rest with its centre of gravity over $B_2$ in a time

$$\pi\left(\frac{d}{g(\mu_2-\mu_1)\cos\alpha}\right)^{\frac{1}{2}}.$$

### [STEP 3, 2001Q10]

Three ships $A$, $B$ and $C$ move with velocities $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{u}$ respectively. The velocities of $A$ and $B$ relative to $C$ are equal in magnitude and perpendicular. Write down conditions that $\mathbf{u}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy and show that

$$\left|\mathbf{u}-\frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)\right|^2=\left|\frac{1}{2}(\mathbf{v}_1-\mathbf{v}_2)\right|^2$$

and

$$\left(\mathbf{u}-\frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)\right)\cdot(\mathbf{v}_1-\mathbf{v}_2)=0.$$

Explain why these equations determine, for given $\mathbf{v}_1$ and $\mathbf{v}_2$, two possible velocities for $C$, provided $\mathbf{v}_1\neq\mathbf{v}_2$.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are equal in magnitude and perpendicular, show that if $\mathbf{u}\neq\mathbf{0}$ then $\mathbf{u}=\mathbf{v}_1+\mathbf{v}_2$.



### [STEP 3, 2001Q11]

A uniform cylinder of radius $a$ rotates freely about its axis, which is fixed and horizontal. The moment of inertia of the cylinder about its axis is $I$. A light string is wrapped around the cylinder and supports a mass $m$ which hangs freely. A particle of mass $M$ is fixed to the surface of the cylinder. The system is held at rest with the particle vertically below the axis of the cylinder, and then released. Find, in terms of $I$, $a$, $M$, $m$, $g$ and $\theta$, the angular velocity of the cylinder when it has rotated through angle $\theta$.

Show that the cylinder will rotate without coming to a halt if $\frac{m}{M}>\sin\alpha$, where $\alpha$ satisifes $\alpha=\tan\frac{1}{2}\alpha$ and $0<\alpha<\pi$.

## Section C: Probability and Statistics

### [STEP 3, 2001Q12]

A bag contains $b$ black balls and $w$ white balls. Balls are drawn at random from the bag and when a white ball is drawn it is put aside.

(i) If the black balls drawn are also put aside, find an expression for the expected number of black balls that have been drawn when the last white ball is removed.

(ii) If instead the black balls drawn are put back into the bag, prove that the expected number of times a black ball has been drawn when the first white ball is removed is $\frac{b}{w}$. Hence write down, in the form of a sum, an expression for the expected number of times a black ball has been drawn when the last white ball is removed.

### [STEP 3, 2001Q13]

In a game for two players, a fair coin is tossed repeatedly. Each player is assigned a sequence of heads and tails and the player whose sequence appears first wins. Four players, $A$, $B$, $C$ and $D$ take turns to play the game. Each time they play, $A$ is assigned the sequence TTH (i.e. Tail then Tail then Head), $B$ is assigned THH, $C$ is assigned HHT and $D$ is assigned HTT.

(i) $A$ and $B$ play the game. Let $p_{HH}$, $p_{HT}$, $p_{TH}$ and $p_{TT}$ be the probabilities of $A$ winning the game given that the first two tosses of the coin show HH, HT, TH and TT respectively. Explain why $p_{TT} = 1$, and why $p_{HT} = \frac{1}{2}p_{TH} + \frac{1}{2}p_{TT}$. Show that $p_{HH} = p_{HT} = \frac{2}{3}$ and that $p_{TH} = \frac{1}{3}$. Deduce that the probability that $A$ wins the game is $\frac{2}{3}$.



(ii) $B$ and $C$ play the game. Find the probability that $B$ wins.

(iii) Show that if $C$ plays $D$, then $C$ is more likely to win than $D$, but that if $D$ plays $A$, then $D$ is more likely to win than $A$.

### [STEP 3, 2001Q14]

A random variable $X$ is distributed uniformly on $[0, a]$. Show that the variance of $X$ is $\frac{1}{12}a^2$.

A sample, $X_1$ and $X_2$ of two independent values of the random variable is drawn, and the variance $V$ of the sample is determined. Show that $V = \frac{1}{4}(X_1 - X_2)^2$, and hence prove that $2V$ is an unbiased estimator of the variance of $X$.

Find an exact expression for the probability that the value of $V$ is less than $\frac{1}{12}a^2$ and estimate the value of this probability correct to one significant figure.

# STEP 3 2002

STEP 3 2002
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

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| **Section A** | Pure Mathematics |
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| **Section C** | Probability and Statistics |

There are 14 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 2002Q1]

Find the area of the region between the curve $y = \frac{\ln x}{x}$ and the $x$-axis, for $1 \leq x \leq a$. What happens to this area as $a$ tends to infinity?

Find the volume of the solid obtained when the region between the curve $y = \frac{\ln x}{x}$ and the $x$-axis, for $1 \leq x \leq a$, is rotated through $2\pi$ radians about the $x$-axis. What happens to this volume as $a$ tends to infinity?

## [STEP 3, 2002Q2]

Prove that $\arctan a + \arctan b = \arctan\left(\frac{a+b}{1-ab}\right)$ when $0 < a < 1$ and $0 < b < 1$.

Prove by induction that, for $n \geq 1$,

$$\sum_{r=1}^{n} \arctan\left(\frac{1}{r^2 + r + 1}\right) = \arctan\left(\frac{n}{n+2}\right)$$

and hence find

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{r^2 + r + 1}\right).$$

Hence prove that

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{r^2 - r + 1}\right) = \frac{\pi}{2}.$$

## [STEP 3, 2002Q3]

Let

$$f(x) = a\sqrt{x} - \sqrt{x - b},$$

Where $x \geq b > 0$ and $a > 1$. Sketch the graph of $f(x)$. Hence show that the equation $f(x) = c$, where $c > 0$, has no solution when $c^2 < b(a^2 - 1)$. Find conditions on $c^2$ in terms of $a$ and $b$ for the equation to have exactly one or exactly two solutions.

Solve the equations (i) $3\sqrt{x} - \sqrt{x-2} = 4$ and (ii) $3\sqrt{x} - \sqrt{x-3} = 5$.

[STEP 3, 2002Q4]

Show that if $x$ and $y$ are positive and $x^3 + x^2 = y^3 - y^2$ then $x < y$.

Show further that if $0 < x \le y - 1$, then $x^3 + x^2 < y^3 - y^2$.

Prove that there does not exist a pair of *positive* integers such that the difference of their cubes is equal to the sum of their squares.

Find all the pairs of integers such that the difference of their cubes is equal to the sum of their squares.

[STEP 3, 2002Q5]

Give a condition that must be satisfied by $p$, $q$ and $r$ for it to be possible to write the quadratic polynomial $px^2 + qx + r$ in the form $p(x+h)^2$, for some $h$.

Obtain an equation, which you need not simplify, that must be satisfied by $t$ if it is possible to write

$$\left(x^2 + \frac{1}{2}bx + t\right)^2 - (x^4 + bx^3 + cx^2 + dx + e)$$

in the form $k(x+h)^2$, for some $k$ and $h$.

Hence, or otherwise, write $x^4 + 6x^3 + 9x^2 - 2x - 7$ as a product of two quadratic factors.



[STEP 3, 2002Q6]

Find all the solution curves of the differential equation

$$y^4\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^4 = (y^2 - 1)^2$$

that pass through either of the points

(i) $\left(0, \frac{\sqrt{3}}{2}\right)$,

(ii) $\left(0, \frac{\sqrt{5}}{2}\right)$.

Show also that $y = 1$ and $y = -1$ are solutions of the differential equation. Sketch all these solution curves on a single set of axes.

## [STEP 3, 2002Q7]

Given that $\alpha$ and $\beta$ are acute angles, show that $\alpha + \beta = \frac{\pi}{2}$ if and only if $\cos^2 \alpha + \cos^2 \beta = 1$.

In the $x$-$y$ plane, the point $A$ has coordinates $(0, s)$ and the point $C$ has coordinates $(s, 0)$, where $s > 0$. The point $B$ lies in the first quadrant $(x > 0, y > 0)$. The lengths of $AB$, $OB$ and $CB$ are respectively $a$, $b$ and $c$.

Show that

$$(s^2 + b^2 - a^2)^2 + (s^2 + b^2 - c^2)^2 = 4s^2b^2$$

and hence that

$$(2s^2 - a^2 - c^2)^2 + (2b^2 - a^2 - c^2)^2 = 4a^2c^2.$$

Deduce that

$$(a - c)^2 \leq 2b^2 \leq (a + c)^2.$$

## [STEP 3, 2002Q8]

Four complex numbers $u_1, u_2, u_3$ and $u_4$ have unit modulus, and arguments $\theta_1, \theta_2, \theta_3$ and $\theta_4$, respectively, with $-\pi < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \pi$.

Show that

$$\arg(u_1 - u_2) = \frac{1}{2}(\theta_1 + \theta_2 - \pi) + 2n\pi$$

where $n = 0$ or $1$. Deduce that

$$\arg\big((u_1 - u_2)(u_4 - u_3)\big) = \arg\big((u_1 - u_4)(u_3 - u_2)\big) + 2n\pi$$

for some integer $n$.

Prove that

$$|(u_1 - u_2)(u_4 - u_3)| + |(u_1 - u_4)(u_3 - u_2)| = |(u_1 - u_3)(u_4 - u_2)|.$$

## Section B: Mechanics

### [STEP 3, 2002Q9]

A tall container made of light material of negligible thickness has the form of a prism, with a square base of area $a^2$. It contains a volume $ka^3$ of fluid of uniform density. The container is held so that it stands on a rough plane, which is inclined at angle $\theta$ to the horizontal, with two of the edges of the base of the container horizontal. In the case $k > \frac{1}{2}\tan\theta$, show that the centre of mass of the fluid is at a distance $x$ from the lower side of the container and at a distance $y$ from the base of the container, where

$$\frac{x}{a} = \frac{1}{2} - \frac{\tan\theta}{12k}, \qquad \frac{y}{a} = \frac{k}{2} + \frac{\tan^2\theta}{24k}.$$

Determine the corresponding coordinates in the case $k < \frac{1}{2}\tan\theta$.

The container is now released. Given that $k < \frac{1}{2}$, show that the container will topple if $\theta > 45°$.

### [STEP 3, 2002Q10]

A light hollow cylinder of radius $a$ can rotate freely about its axis of symmetry, which is fixed and horizontal. A particle of mass $m$ is fixed to the cylinder, and a second particle, also of mass $m$, moves on the rough inside surface of the cylinder. Initially, the cylinder is at rest, with the fixed particle on the same horizontal level as its axis and the second particle at rest vertically below this axis. The system is then released. Show that, if $\theta$ is the angle through which the cylinder has rotated, then

$$\ddot{\theta} = \frac{g}{2a}(\cos\theta - \sin\theta),$$

provided that the second particle does not slip.

Given that the coefficient of friction is $\frac{3+\sqrt{3}}{6}$, show that the second particle starts to slip when the cylinder has rotated through $60°$.

### [STEP 3, 2002Q11]

A particle moves on a smooth triangular horizontal surface $AOB$ with angle $AOB = 30°$. The surface is bounded by two vertical walls $OA$ and $OB$ and the coefficient of restitution between the particle and the walls is $e$, where $e < 1$. The particle, which is initially at point $P$ on the surface and moving with velocity $u_1$, strikes the wall $OA$ at $M_1$, with angle $PM_1A = \theta$, and rebounds, with velocity $v_1$, to strike the wall $OB$ at $N_1$, with angle $M_1N_1B = \theta$. Find $e$ and $\frac{v_1}{u_1}$ in terms of $\theta$.

The motion continues, with the particle striking side $OA$ at $M_2, M_3, \ldots$ and striking side $OB$ at $N_2, N_3, \ldots$. Show that, if $\theta < 60°$, the particle reaches $O$ in a finite time.

## Section C: Probability and Statistics

### [STEP 3, 2002Q12]

In a game, a player tosses a biased coin repeatedly until two successive tails occur, when the game terminates. For each head which occurs the player wins £1. If $E$ is the expected number of tosses of the coin in the course of a game, and $p$ is the probability of a head, explain why

$$E = p(1+E) + (1-p)p(2+E) + 2(1-p)^2$$

and hence determine $E$ in terms of $p$. Find also, in terms of $p$, the expected winnings in the course of a game.

A second game is played, with the same rules, except that the player continues to toss the coin until $r$ successive tails occur. Show that the expected number of tosses in the course of a game is given by the expression $\frac{1-q^r}{pq^r}$, where $q = 1-p$.

### [STEP 3, 2002Q13]

A continuous random variable is said to have an exponential distribution with parameter $\lambda$ if its density function is $f(t) = \lambda \mathrm{e}^{-\lambda t}$ $(0 \le t < \infty)$. If $X_1$ and $X_2$, which are independent random variables, have exponential distributions with parameters $\lambda_1$ and $\lambda_2$ respectively, find an expression for the probability that either $X_1$ or $X_2$ (or both) is less than $x$. Prove that if $X$ is the random variable whose value is the lesser of the values of $X_1$ and $X_2$, then $X$ also has an exponential distribution.

Route A and Route B buses run from my house to my college. The time between buses on each route has an exponential distribution and the mean time between buses is 15 minutes for Route A and 30 minutes for Route B. The timings of the buses on the two routes are independent. If I emerge from my house one day to see a Route A bus and a Route B bus just leaving the stop, show that the median wait for the next bus to my college will be approximately 7 minutes.

### [STEP 3, 2002Q14]

Prove that, for any two discrete random variables $X$ and $Y$,

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}[X,Y],$$

where $\mathrm{Var}[X]$ is the variance of $X$ and $\mathrm{Cov}[X,Y]$ is the covariance of $X$ and $Y$.

When a Grandmaster plays a sequence of $m$ games of chess, she is, independently, equally likely to win, lose or draw each game. If the values of the random variables $W$, $L$ and $D$ are the numbers of her wins, losses and draws respectively, justify briefly the following claims:

(i) $W + L + D$ has variance 0.

(ii) $W + L$ has a binomial distribution.

Find the value of $\frac{\mathrm{Cov}[W,L]}{\sqrt{\mathrm{Var}[W]\mathrm{Var}[L]}}$.

# STEP 3 2003

STEP 3 2003
On-line Exam

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Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

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**Calculators are not permitted.**

## Section A: Pure Mathematics

### [STEP 3, 2003Q1]

Given that $x + a > 0$ and $x + b > 0$, and that $b > a$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x+a}{x+b}\right) = \frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}}$$

and find $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh}\left(\frac{x+a}{x+b}\right)$.

Hence, or otherwise, integrate, for $x > -1$,

(i) $\int \frac{1}{(x+1)\sqrt{x+3}}\,\mathrm{d}x$,

(ii) $\int \frac{1}{(x+3)\sqrt{x+1}}\,\mathrm{d}x$.

[You may use the results $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$.]

### [STEP 3, 2003Q2]

Show that ${}^{2r}C_r = \frac{1\times 3\times\cdots\times(2r-1)}{r!}\times 2^r$, for $r \geq 1$.

(i) Give the first four terms of the binomial series for $(1-p)^{-\frac{1}{2}}$.

By choosing a suitable value for $p$ in this series, or otherwise, show that

$$\sum_{r=0}^{\infty}\frac{{}^{2r}C_r}{8^r} = \sqrt{2}.$$

(ii) Show that

$$\sum_{r=0}^{\infty}\frac{(2r+1)\,{}^{2r}C_r}{5^r} = \left(\sqrt{5}\right)^3.$$

[**Note:** ${}^{n}C_r$ is an alternative notation for $\binom{n}{r}$ for $r \geq 1$, and ${}^{0}C_0 = 1$.]

### [STEP 3, 2003Q3]

If $m$ is a positive integer, show that $(1+x)^m + (1-x)^m \neq 0$ for any real $x$.

The function $f$ is defined by

$$f(x) = \frac{(1+x)^m - (1-x)^m}{(1+x)^m + (1-x)^m}.$$

Find and simplify an expression for $f'(x)$.

In the case $m = 5$, sketch the curves $y = f(x)$ and $y = \frac{1}{f(x)}$.

[STEP 3, 2003Q4]

A curve is defined parametrically by

$$x = t^2, \qquad y = t(1+t^2).$$

The tangent at the point with parameter $t$, where $t \neq 0$, meets the curve again at the point with parameter $T$, where $T \neq t$. Show that

$$T = \frac{1-t^2}{2t} \quad \text{and} \quad 3t^2 \neq 1.$$

Given a point $P_0$ on the curve, with parameter $t_0$, a sequence of points $P_0, P_1, P_2, \ldots$ on the curve is constructed such that the tangent at $P_i$ meets the curve again at $P_{i+1}$. If $t_0 = \tan\frac{7\pi}{18}$, show that $P_3 = P_0$ but $P_1 \neq P_0$. Find a second value of $t_0$, with $t_0 > 0$, for which $P_3 = P_0$ but $P_1 \neq P_0$.

[STEP 3, 2003Q5]

Find the coordinates of the turning point on the curve $y = x^2 - 2bx + c$. Sketch the curve in the case that the equation $x^2 - 2bx + c = 0$ has two distinct real roots. Use your sketch to determine necessary and sufficient conditions on $b$ and $c$ for the equation $x^2 - 2bx + c = 0$ to have two distinct real roots. Determine necessary and sufficient conditions on $b$ and $c$ for this equation to have two distinct positive roots.

Find the coordinates of the turning points on the curve $y = x^3 - 3b^2x + c$ (with $b > 0$) and hence determine necessary and sufficient conditions on $b$ and $c$ for the equation $x^3 - 3b^2x + c = 0$ to have three distinct real roots. Determine necessary and sufficient conditions on $a$, $b$ and $c$ for the equation $(x-a)^3 - 3b^2(x-a) + c = 0$ to have three distinct positive roots.

Show that the equation $2x^3 - 9x^2 + 7x - 1 = 0$ has three distinct positive roots.

[STEP 3, 2003Q6]

Show that

$$2\sin\frac{1}{2}\theta\cos r\theta = \sin\left(r+\frac{1}{2}\right)\theta - \sin\left(r-\frac{1}{2}\right)\theta.$$

Hence, or otherwise, find all solutions of the equation

$$\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0,$$

where $a$ and $b$ are positive integers with $a < b-1$.

[STEP 3, 2003Q7]

In the $x$-$y$ plane, the point $A$ has coordinates $(a, 0)$ and the point $B$ has coordinates $(0, b)$, where $a$ and $b$ are positive. The point $P$, which is distinct from $A$ and $B$, has coordinates $(s, t)$. $X$ and $Y$ are the feet of the perpendiculars from $P$ to the $x$-axis and $y$-axis respectively, and $N$ is the foot of the perpendicular from $P$ to the line $AB$. Show that the coordinates $(x, y)$ of $N$ are given by

$$x = \frac{ab^2 - a(bt - as)}{a^2 + b^2}, \qquad y = \frac{a^2 b + b(bt - as)}{a^2 + b^2}.$$

Show that, if $\left(\frac{t-b}{s}\right)\left(\frac{t}{s-a}\right) = -1$, then $N$ lies on the line $XY$.

Give a geometrical interpretation of this result.

[STEP 3, 2003Q8]

(i) Show that the gradient at a point $(x, y)$ on the curve

$$(y + 2x)^3(y - 4x) = c,$$

where $c$ is a constant, is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16x - y}{2y - 5x}.$$

(ii) By considering the derivative with respect to $x$ of $(y + ax)^n(y + bx)$, or otherwise, find the general solutions of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{10x - 4y}{3x - y}.$$

# Section B: Mechanics

## [STEP 3, 2003Q9]

A particle $P$ of mass $m$ is constrained to move on a vertical circle of smooth wire with centre $O$ and of radius $a$. $L$ is the lowest point of the circle and $H$ the highest and $\angle LOP = \theta$. The particle is attached to $H$ by an elastic string of natural length $a$ and modulus of elasticity $\alpha mg$ where $\alpha > 1$. Show that, if $\alpha > 2$, there is an equilibrium position with $0 < \theta < \pi$.

Given that $\alpha = 2 + \sqrt{2}$, and that $\theta = \frac{\pi}{2} + \phi$, show that

$$\ddot{\phi} \approx \frac{g(\sqrt{2} + 1)}{2a}\phi$$

when $\phi$ is small.

For this value of $\alpha$, explain briefly what happens to the particle if it is given a small displacement when $\theta = \frac{\pi}{2}$.

## [STEP 3, 2003Q10]

A particle moves along the $x$-axis in such a way that its acceleration is $kx\dot{x}$ where $k$ is a positive constant. When $t = 0$, $x = d$ (where $d > 0$) and $\dot{x} = U$.

(i) Find $x$ as a function of $t$ in the case $U = kd^2$ and show that $x$ tends to infinity as $t$ tends to $\frac{\pi}{2dk}$.

(ii) If $U < 0$, find $x$ as a function of $t$ and show that it tends to a limit, which you should state in terms of $d$ and $U$, as $t$ tends to infinity.



## [STEP 3, 2003Q11]

Point $B$ is a distance $d$ due south of point $A$ on a horizontal plane. Particle $P$ is at rest at $B$ at $t = 0$, when it begins to move with constant acceleration $a$ in a straight line with fixed bearing $\beta$. Particle $Q$ is projected from point $A$ at $t = 0$ and moves in a straight line with constant speed $v$. Show that if the direction of projection of $Q$ can be chosen so that $Q$ strikes $P$, then

$$v^2 \geq ad(1 - \cos\beta).$$

Show further that if $v^2 > ad(1 - \cos\beta)$ then the direction of projection of $Q$ can be chosen so that $Q$ strikes $P$ before $P$ has moved a distance $d$.

## Section C: Probability and Statistics

### [STEP 3, 2003Q12]

Brief interruptions to my work occur on average every ten minutes and the number of interruptions in any given time period has a Poisson distribution. Given that an interruption has just occurred, find the probability that I will have less than $t$ minutes to work before the next interruption. If the random variable $T$ is the time I have to work before the next interruption, find the probability density function of $T$.

I need an uninterrupted half hour to finish an important paper. Show that the expected number of interruptions before my first uninterrupted period of half an hour or more is $\mathrm{e}^3 - 1$. Find also the expected length of time between interruptions that are less than half an hour apart. Hence write down the expected wait before my first uninterrupted period of half an hour or more.

### [STEP 3, 2003Q13]

In a rabbit warren, underground chambers $A$, $B$, $C$ and $D$ are at the vertices of a square, and burrows join $A$ to $B$, $B$ to $C$, $C$ to $D$ and $D$ to $A$. Each of the chambers also has a tunnel to the surface. A rabbit finding itself in any chamber runs along one of the two burrows to a neighbouring chamber, or leaves the burrow through the tunnel to the surface. Each of these three possibilities is equally likely.

Let $p_A$, $p_B$, $p_C$ and $p_D$ be the probabilities of a rabbit leaving the burrow through the tunnel from chamber $A$, given that it is currently in chamber $A$, $B$, $C$ or $D$, respectively.



(i) Explain why $p_A = \frac{1}{3} + \frac{1}{3}p_B + \frac{1}{3}p_D$.

(ii) Determine $p_A$.

(iii) Find the probability that a rabbit which starts in chamber $A$ does not visit chamber $C$, given that it eventually leaves the burrow through the tunnel in chamber $A$.

[STEP 3, 2003Q14]

Write down the probability generating function for the score on a standard, fair six-faced die whose faces are labelled 1, 2, 3, 4, 5, 6. Hence show that the probability generating function for the sum of the scores on two standard, fair six-faced dice, rolled independently, can be written as

$$\frac{1}{36}t^2(1+t^2)(1-t+t^2)^2(1+t+t^2)^2.$$

Write down, in factorised form, the probability generating functions for the scores on two fair six-faced dice whose faces are labelled with the numbers 1, 2, 2, 3, 3, 4 and 1, 3, 4, 5, 6, 8, and hence show that when these dice are rolled independently, the probability of any given sum of the scores is the same as for the two standard fair six-faced dice.

Standard, fair four-faced dice are tetrahedra whose faces are labelled 1, 2, 3, 4, the score being taken from the face which is not visible after throwing, and each score being equally likely. Find all the ways in which two fair four-faced dice can have their faces labelled with positive integers if the probability of any given sum of the scores is to be the same as for the two standard fair four-faced dice.

# STEP 3 2004

STEP 3 2004

On-line Exam

Scan the QR code or click on the link to take an on-line exam.

Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

**INSTRUCTIONS TO CANDIDATES**

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

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**INFORMATION FOR CANDIDATES**

This paper contains **three** sections: **A**, **B**, and **C**.

| | |
|---|---|
| **Section A** | Pure Mathematics |
| **Section B** | Mechanics |
| **Section C** | Probability and Statistics |

There are 14 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 2004Q1]

Show that

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,dx = \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

and find

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,dx.$$

Hence show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,dx = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

By substituting $u = e^x$ in this result, or otherwise, find

$$\int_1^\infty \frac{1}{1+u^4}\,du.$$

## [STEP 3, 2004Q2]

The equation of a curve is $y = f(x)$ where

$$f(x) = x - 4 - \frac{16(2x+1)^2}{x^2(x-4)}.$$

(i) Write down the equations of the vertical and oblique asymptotes to the curve and show that the oblique asymptote is a tangent to the curve.

(ii) Show that the equation $f(x) = 0$ has a double root.

(iii) Sketch the curve.

[STEP 3, 2004Q3]

Given that $f''(x) > 0$ when $a \le x \le b$, explain with the aid of a sketch why

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,\mathrm{d}x < (b-a)\frac{f(a)+f(b)}{2}.$$

By choosing suitable $a$, $b$ and $f(x)$, show that

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2}\right),$$

where $n$ is an integer greater than 1.

Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

and hence show that

$$\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}.$$



[STEP 3, 2004Q4]

The triangle $OAB$ is isosceles, with $OA = OB$ and angle $AOB = 2\alpha$ where $0 < \alpha < \frac{\pi}{2}$. The semi-circle $C_0$ has its centre at the midpoint of the base $AB$ of the triangle, and the sides $OA$ and $OB$ of the triangle are both tangent to the semi-circle. $C_1, C_2, C_3, \ldots$ are circles such that $C_n$ is tangent to $C_{n-1}$ and to sides $OA$ and $OB$ of the triangle.

Let $r_n$ be the radius of $C_n$. Show that

$$\frac{r_{n+1}}{r_n} = \frac{1 - \sin\alpha}{1 + \sin\alpha}.$$



Let $S$ be the total area of the semi-circle $C_0$ and circles $C_1, C_2, C_3, \ldots$. Show that

$$S = \frac{1 + \sin^2\alpha}{4\sin\alpha}\pi r_0^2.$$

Show that there are values of $\alpha$ for which $S$ is more than four fifths of the area of triangle $OAB$.

[STEP 3, 2004Q5]

Show that if $\cos(x-\alpha)=\cos\beta$ then either $\tan x=\tan(\alpha+\beta)$ or $\tan x=\tan(\alpha-\beta)$. By choosing suitable values of $x$, $\alpha$ and $\beta$, give an example to show that if $\tan x=\tan(\alpha+\beta)$, then $\cos(x-\alpha)$ need not equal to $\cos\beta$.

Let $\omega$ be the acute angle such that $\tan\omega=\frac{4}{3}$.

(i) For $0\le x\le 2\pi$, solve the equation

$$\cos x-7\sin x=5$$

giving both solutions in terms of $\omega$.

(ii) For $0\le x\le 2\pi$, solve the equation

$$2\cos x+11\sin x=10$$

showing that one solution is twice the other and giving both in terms of $\omega$.

[STEP 3, 2004Q6]

Given a sequence $w_0, w_1, w_2, \dots$, the sequence $F_1, F_2, \dots$ is defined by

$$F_n=w_n^2+w_{n-1}^2-4w_nw_{n-1}.$$

Show that $F_n-F_{n-1}=(w_n-w_{n-2})(w_n+w_{n-2}-4w_{n-1})$ for $n\ge 2$.

(i) The sequence $u_0, u_1, u_2, \dots$ has $u_0=1$, and $u_1=2$ and satisfies

$$u_n=4u_{n-1}-u_{n-2}\qquad(n\ge 2).$$

Prove that $u_n^2+u_{n-1}^2=4u_nu_{n-1}-3$ for $n\ge 1$.

(ii) A sequence $v_0, v_1, v_2, \dots$ has $v_0=1$ and satisfies

$$v_n^2+v_{n-1}^2=4v_nv_{n-1}-3\qquad(n\ge 1).\qquad(*)$$

(a) Find $v_1$ and prove that, for each $n\ge 2$, either $v_n=4v_{n-1}-v_{n-2}$ or $v_n=v_{n-2}$.

(b) Show that the sequence, with period 2, defined by

$$v_n=\begin{cases}1 & \text{for } n \text{ even}\\ 2 & \text{for } n \text{ odd}\end{cases}$$

satisfies $(*)$.

(c) Find a sequence $v_n$ with period 4 which has $v_0=1$, and satisfies $(*)$.

## [STEP 3, 2004Q7]

For $n = 1, 2, 3, \ldots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t.$$

By considering the greatest value taken by $\frac{t}{t+1}$ for $0 \le t \le 1$ show that $I_{n+1} < \frac{1}{2} I_n$.

Show also that $I_{n+1} = -\frac{1}{n2^n} + I_n$.

Deduce that $I_n < \frac{1}{n2^{n-1}}$.

Prove that

$$\ln 2 = \sum_{r=1}^{n} \frac{1}{r2^r} + I_{n+1}$$

and hence show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.

## [STEP 3, 2004Q8]

Show that if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y + \frac{g(x)}{y}$$

then the substitution $u = y^2$ gives a linear differential equation for $u(x)$.

Hence or otherwise solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} - \frac{1}{y}.$$

Determine the solution curves of this equation which pass through $(1, 1)$, $(2, 2)$ and $(4, 4)$ and sketch graphs of all three curves on the same axes.

# Section B: Mechanics

## [STEP 3, 2004Q9]

A circular hoop of radius $a$ is free to rotate about a fixed horizontal axis passing through a point $P$ on its circumference. The plane of the hoop is perpendicular to this axis. The hoop hangs in equilibrium with its centre, $O$, vertically below $P$. The point $A$ on the hoop is vertically below $O$, so that $POA$ is a diameter of the hoop.

A mouse $M$ runs at constant speed $u$ round the rough inner surface of the lower part of the hoop. Show that the mouse can choose its speed so that the hoop remains in equilibrium with diameter $POA$ vertical.

Describe what happens to the hoop when the mouse passes the point at which angle $AOM = 2\arctan\mu$, where $\mu$ is the coefficient of friction between mouse and hoop.

## [STEP 3, 2004Q10]

A particle $P$ of mass $m$ is attached to points $A$ and $B$, where $A$ is a distance $9a$ vertically above $B$, by elastic strings, each of which has modulus of elasticity $6mg$. The string $AP$ has natural length $6a$ and the string $BP$ has natural length $2a$. Let $x$ be the distance $AP$.

The system is released from rest with $P$ on the vertical line $AB$ and $x = 6a$. Show that the acceleration $\ddot{x}$ of $P$ is $\frac{4g}{a}(7a - x)$ for $6a < x < 7a$ and $\frac{g}{a}(7a - x)$ for $7a < x < 9a$.

Find the time taken for the particle to reach $B$.



## [STEP 3, 2004Q11]

Particles $P$, of mass 2, and $Q$, of mass 1, move along a line. Their distances from a fixed point are $x_1$ and $x_2$, respectively where $x_2 > x_1$. Each particle is subject to a repulsive force from the other of magnitude $\frac{2}{z^3}$, where $z = x_2 - x_1$.

Initially, $x_1 = 0$, $x_2 = 1$, $Q$ is at rest and $P$ moves towards $Q$ with speed 1. Show that $z$ obeys the equation $\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{3}{z^3}$.

By first writing $\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}z}$, where $v = \frac{\mathrm{d}z}{\mathrm{d}t}$, show that $z = \sqrt{4t^2 - 2t + 1}$.

By considering the equation satisfied by $2x_1 + x_2$, find $x_1$ and $x_2$ in terms of $t$.

## Section C: Probability and Statistics

### [STEP 3, 2004Q12]

A team of $m$ players, numbered from 1 to $m$, puts on a set of a $m$ shirts, similarly numbered from 1 to $m$. The players change in a hurry, so that the shirts are assigned to them randomly, one to each player.

Let $C_i$ be the random variable that takes the value 1 if player $i$ is wearing shirt $i$, and 0 otherwise. Show that $\mathrm{E}[C_1] = \frac{1}{m}$ and find $\mathrm{Var}[C_1]$ and $\mathrm{Cov}[C_1, C_2]$.

Let $N = C_1 + C_2 + \cdots + C_m$ be the random variable whose value is the number of players who are wearing the correct shirt. Show that $\mathrm{E}[N] = \mathrm{Var}[N] = 1$.

Explain why a Normal approximation to $N$ is not likely to be appropriate for any $m$, but that a Poisson approximation might be reasonable.

In the case $m = 4$, find, by listing equally likely possibilities or otherwise, the probability that no player is wearing the correct shirt and verify that an appropriate Poisson approximation to $N$ gives this probability with a relative error of about 2%. [Use $\mathrm{e} \approx 2\frac{72}{100}$.]

### [STEP 3, 2004Q13]

A men's endurance competition has an unlimited number of rounds. In each round, a competitor has, independently, a probability $p$ of making it through the round; otherwise, he fails the round. Once a competitor fails a round, he drops out of the competition; before he drops out, he takes part in every round. The grand prize is awarded to any competitor who makes it through a round which all the other remaining competitors fail; if all the remaining competitors fail at the same round the grand prize is not awarded.

If the competition begins with three competitors, find the probability that:

(i) all three drop out in the same round.

(ii) two of them drop out in round $r$ (with $r \geq 2$) and the third in an earlier round.

(iii) the grand prize is awarded.

[STEP 3, 2004Q14]

In this question, $\Phi(z)$ is the cumulative distribution function of a standard normal random variable.

A random variable is known to have a Normal distribution with mean $\mu$ and standard deviation either $\sigma_0$ or $\sigma_1$, where $\sigma_0 < \sigma_1$. The mean, $\bar{X}$, of a random sample of $n$ values of $X$ is to be used to test the hypothesis $H_0 : \sigma = \sigma_0$ against the alternative $H_1 : \sigma = \sigma_1$.

Explain carefully why it is appropriate to use a two sided test of the form: accept $H_0$ if $\mu - c < \bar{X} < \mu + c$, otherwise accept $H_1$.

Given that the probability of accepting $H_1$ when $H_0$ is true is $\alpha$, determine $c$ in terms of $n$, $\sigma_0$ and $z_\alpha$, where $z_\alpha$ is defined by $\Phi(z_\alpha) = 1 - \frac{\alpha}{2}$.

The probability of accepting $H_0$ when $H_1$ is true is denoted by $\beta$. Show that $\beta$ is independent of $n$.

Given that $\Phi(1.960) \approx 0.975$ and that $\Phi(0.063) \approx 0.525$, determine, approximately, the minimum value of $\frac{\sigma_1}{\sigma_0}$ if $\alpha$ and $\beta$ are both to be less than 0.05.

# STEP 3 2005

STEP 3 2005
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

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# Section A: Pure Mathematics

## [STEP 3, 2005Q1]

Show that $\sin A = \cos B$ if and only if $A = (4n+1)\frac{\pi}{2} \pm B$ for some integers $n$.

Show also that $|\sin x \pm \cos x| \leq \sqrt{2}$ for all values of $x$ and deduce that there are no solutions to the equation $\sin(\sin x) = \cos(\cos x)$.

Sketch, on the same axes, the graphs of $y = \sin(\sin x)$ and $y = \cos(\cos x)$. Sketch, not on the previous axes, the graph of $y = \sin(2\sin x)$.

## [STEP 3, 2005Q2]

Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{xy}{x^2+a^2}$, where $a \neq 0$, and show that it can be written in the form $y^2(x^2+a^2) = c^2$, where $c$ is an arbitrary constant. Sketch this curve.

Find an expression for $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2)$ and show that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2) = 2\left(1 - \frac{c^2}{(x^2+a^2)^2}\right) + \frac{8c^2x^2}{(x^2+a^2)^3}.$$

(i) Show that, if $0 < c < a^2$, the points on the curve whose distance from the origin is least are $\left(0, \pm\frac{c}{a}\right)$.

(ii) If $c > a^2$, determine the points on the curve whose distance from the origin is least.



## [STEP 3, 2005Q3]

Let $f(x) = x^2 + px + q$ and $g(x) = x^2 + rx + s$. Find an expression for $f\big(g(x)\big)$ and hence find a necessary and sufficient condition on $a$, $b$ and $c$ for it to be possible to write the quartic expression $x^4 + ax^3 + bx^2 + cx + d$ in the form $f\big(g(x)\big)$, for some choice of values of $p, q, r$ and $s$.

Show further that this condition holds if and only if it is possible to write the quartic expression $x^4 + ax^3 + bx^2 + cx + d$ in the form $(x^2 + vx + w)^2 - k$, for some choice of values of $v$, $w$ and $k$.

Find the roots of the quartic equation $x^4 - 4x^3 + 10x^2 - 12x + 4 = 0$.

[STEP 3, 2005Q4]

The sequence $u_n$ $(n = 1,2, \ldots)$ satisfies the recurrence relation

$$u_{n+2} = \frac{u_{n+1}}{u_n}(ku_n - u_{n+1})$$

where $k$ is a constant.

If $u_1 = a$ and $u_2 = b$, where $a$ and $b$ are non-zero and $b \neq ka$, prove by induction that

$$u_{2n} = \left(\frac{b}{a}\right) u_{2n-1}$$
$$u_{2n+1} = cu_{2n}$$

for $n \geq 1$, where $c$ is a constant to be found in terms of $k$, $a$ and $b$. Hence express $u_{2n}$ and $u_{2n-1}$ in terms of $a$, $b$, $c$ and $n$.

Find conditions on $a$, $b$ and $k$ in the three cases:

(i) the sequence $u_n$ is geometric.

(ii) the sequence $u_n$ has period 2.

(iii) the sequence $u_n$ has period 4.

[STEP 3, 2005Q5]

Let $P$ be the point on the curve $y = ax^2 + bx + c$ (where $a$ is non-zero) at which the gradient is $m$. Show that the equation of the tangent at $P$ is

$$y - mx = c - \frac{(m-b)^2}{4a}.$$

Show that the curves $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$ (where $a_1$ and $a_2$ are non-zero) have a common tangent with gradient $m$ if and only if

$$(a_2 - a_1)m^2 + 2(a_1b_2 - a_2b_1)m + 4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2 = 0.$$

Show that, in the case $a_1 \neq a_2$, the two curves have exactly one common tangent if and only if they touch each other. In the case $a_1 = a_2$, find a necessary and sufficient condition for the two curves to have exactly one common tangent.

[STEP 3, 2005Q6]

In this question, you may use without proof the results

$$4\cosh^3 y - 3\cosh y = \cosh(3y) \qquad \text{and} \qquad \operatorname{arcosh} y = \ln\left(y + \sqrt{y^2-1}\right).$$

[**Note:** $\operatorname{arcosh} y$ is another notation for $\cosh^{-1} y$]

Show that the equation $x^3 - 3a^2x = 2a^3\cosh T$ is satisfied by $2a\cosh\left(\frac{1}{3}T\right)$ and hence that, if $c^2 \geq b^3$ one of the roots of the equation $x^3 - 3bx = 2c$ is $u + \frac{b}{u}$ where $u = \left(c + \sqrt{c^2 - b^3}\right)^{\frac{1}{3}}$.

Show that the other two roots of the equation $x^3 - 3bx = 2c$ are the roots of the quadratic equation $x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b = 0$, and fine these roots in terms of $u$, $b$ and $\omega$, where $\omega = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3})$.

Solve completely the equation $x^3 - 6x = 6$.

[STEP 3, 2005Q7]

Show that if $\int \frac{1}{uf(u)}\,\mathrm{d}u = F(u) + c$, then $\int \frac{m}{xf(x^m)}\,\mathrm{d}x = F(x^m) + c$, where $m \neq 0$.

Find:

(i) $\int \frac{1}{x^n - x}\,\mathrm{d}x$.

(ii) $\int \frac{1}{\sqrt{x^n + x^2}}\,\mathrm{d}x$.



[STEP 3, 2005Q8]

In this question, $a$ and $c$ are distinct non-zero complex numbers. The complex conjugate of any complex number $z$ is denoted by $z^*$.

Show that

$$|a - c|^2 = aa^* + cc^* - ac^* - ca^*$$

and hence prove that the triangle $OAC$ in the Argand diagram, whose vertices are represented by 0, $a$ and $c$ respectively, is right angled at $A$ if and only if $2aa^* = ac^* + ca^*$.

Points $P$ and $P'$ in the Argand diagram are represented by the complex numbers $ab$ and $\frac{a}{b^*}$, where $b$ is a non-zero complex number. A circle in the Argand diagram has centre $C$ and passes through the point $A$, and is such that $OA$ is tangent to the circle. Show that the point $P$ lies on the circle if and only if point $P'$ lies on the circle.

Conversely, show that if the points represented by the complex numbers $ab$ and $\frac{a}{b^*}$, for some non-zero complex number $b$ with $bb^* \neq 1$, both lie on a circle centre $C$ in the Argand diagram which passes through $A$, then $OA$ is a tangent to the circle.

## Section B: Mechanics

### [STEP 3, 2005Q9]

Two particles, $A$ and $B$, move without friction along a horizontal line which is perpendicular to a vertical wall. The coefficient of restitution between the two particles is $e$ and the coefficient of restitution between particle $B$ and the wall is also $e$, where $0 < e < 1$. The mass of particle $A$ is $4em$ (with $m > 0$), and the mass of particle $B$ is $(1 - e^2)m$.

Initially, $A$ is moving towards the wall with speed $(1 - e)v$ (where $v > 0$) and $B$ is moving away from the wall and towards $A$ with speed $2ev$. The two particles collide at a distance $d$ from the wall. Find the speeds of $A$ and $B$ after the collision.

When $B$ strikes the wall, it rebounds along the same line. Show that a second collision will take place, at a distance $de$ from the wall.

Deduce that further collisions will take place. Find the distance from the wall at which the $n$th collision takes place, and show that the times between successive collisions are equal.

### [STEP 3, 2005Q10]

Two thin discs, each of radius $r$ and mass $m$, are held on a rough horizontal surface with their centres a distance $6r$ apart. A thin light elastic band, of natural length $2\pi r$ and modulus $\frac{\pi mg}{12}$, is wrapped once round the discs, its straight sections being parallel. The contact between the elastic band and the discs is smooth. The coefficient of friction between each disc and the horizontal surface is $\mu$, so that each disc experiences a force due to friction equal to $\mu mg$, whether the disc is at rest or sliding.

The discs are released simultaneously. If the discs collide, they rebound and a half of their total kinetic energy is lost in the collision.

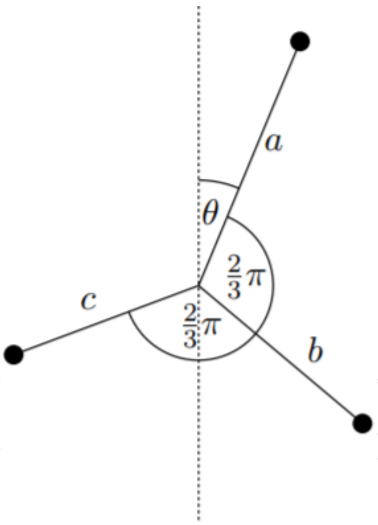
(i) Show that the discs start sliding, but come to rest before colliding, if and only if $\frac{2}{3} < \mu < 1$.

(ii) Show that, if the discs collide at least once, their total kinetic energy just before the first collision is $\frac{4}{3}mg(2 - 3\mu)$.

(iii) Show that if $\frac{4}{9} > \mu^2 > \frac{5}{27}$ the discs come to rest exactly once after the first collision.

## [STEP 3, 2005Q11]

A horizontal spindle rotates freely in a fixed bearing. Three light rods are each attached by one end to the spindle so that they rotate in vertical plane. A particle of mass $m$ is fixed to the other end of each of the three rods. The rods have lengths $a$, $b$ and $c$, with $a > b > c$ and the angle between any pair of rods is $\frac{2}{3}\pi$. The angle between the rod of length $a$ and the vertical axis is $\theta$, as shown in the diagram.




Find an expression for the energy of the system and show that, if the system is in equilibrium, then

$$\tan\theta = -\frac{(b-c)\sqrt{3}}{2a-b-c}.$$

Deduce that there are exactly two equilibrium positions and determine which of the two equilibrium positions is stable.

Show that, for the system to make complete revolutions, it must pass through its position of stable equilibrium with an angular velocity of at least

$$\sqrt{\frac{4gR}{a^2+b^2+c^2}},$$

where $2R^2 = (a-b)^2 + (b-c)^2 + (c-a)^2$.

## Section C: Probability and Statistics

### [STEP 3, 2005Q12]

Five independent timers time a runner as she runs four laps of a track. Four of the timers measure the individual lap times, the results of the measurements being the random variables $T_1$ to $T_4$, each of which has variance $\sigma^2$ and expectation equal to the true time for the lap. The fifth timer measures the total time for the race, the result of the measurement being the random variable $T$ which has variance $\sigma^2$ and expectation equal to the true race time (which is equal to the sum of the four true lap times).

Find a random variable $X$ of the form $aT + b(T_1 + T_2 + T_3 + T_4)$, where $a$ and $b$ are constants independent of the true lap times, with the properties that:

(i) whatever the true lap times, the expectation of $X$ is equal to the true race time,

(ii) the variance of $X$ is as small as possible.

Find also a random variable $Y$ of the form $cT + d(T_1 + T_2 + T_3 + T_4)$, where $c$ and $d$ are constants independent of the true lap times, with the property that, whatever the true lap times, the expectation of $Y^2$ is equal to $\sigma^2$.

In one particular race, $T$ takes the value 220 seconds and $(T_1 + T_2 + T_3 + T_4)$ takes the value 220.5 seconds. Use the random variables $X$ and $Y$ to estimate an interval in which true race time lies.

### [STEP 3, 2005Q13]



A pack of cards consists of $n + 1$ cards, which are printed with the integers from 0 to $n$. A game consists of drawing cards repeatedly at random from the pack until the card printed with 0 is drawn, at which point the game ends. After each draw, the player receives £1 if the card drawn shows any of the integers from 1 to $w$ inclusive but receives nothing if the card drawn shows any of the integers from $w + 1$ to $n$ inclusive.

(i) In one version of the game, each card drawn is replaced immediately and randomly in the pack. Explain clearly why the probability that the player wins a total of exactly £3 is equal to the probability of the following event occurring: out of the first four cards drawn which show numbers in the range 0 to $w$, the numbers on the first three are non-zero and the number on the fourth is zero. Hence show that the probability that the player wins a total of exactly £3 is equal to $\frac{w^3}{(w+1)^4}$.

Write down the probability that the player wins a total of exactly £$r$ and hence find the expected total win.

(ii) In another version of the game, each card drawn is removed from the pack. Show that the expected total win in this version is half of the expected total win in the other version.

**[STEP 3, 2005Q14]**

In this question, you may use the result

$$\int_0^\infty \frac{t^m}{(t+k)^{n+2}}\,\mathrm{d}t = \frac{m!\,(n-m)!}{(n+1)!\,k^{n-m+1}},$$

where $m$ and $n$ are positive integers with $n \geq m$, and where $k > 0$.

The random variable $V$ has density function

$$f(x) = \frac{Ck^{a+1}x^a}{(x+k)^{2a+2}} \qquad (0 \leq x < \infty),$$

where $a$ is a positive integer. Show that $C = \frac{(2a+1)!}{a!a!}$.

Show, by means of a suitable substitution, that

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}}\,\mathrm{d}x = \int_{\frac{k^2}{v}}^\infty \frac{u^a}{(u+k)^{2a+2}}\,\mathrm{d}u$$

and deduce that the median value of $V$ is $k$. Find the expected value of $V$.

The random variable $V$ represents the speed of randomly chosen gas molecule. The time taken for such a particle to travel a fixed distance $s$ is given by the random variable $T = \frac{s}{V}$.

Show that

$$\mathrm{P}(T < t) = \int_{\frac{s}{t}}^\infty \frac{Ck^{a+1}x^a}{(x+k)^{2a+2}}\,\mathrm{d}x \qquad (*)$$

and hence find the density function of $T$. You may find it helpful to make the substitution $u = \frac{s}{x}$ in the integral $(*)$.



Hence show that the product of the median time and the median speed is equal to the distance $s$, but that the product of the expected time and the expected speed is greater than $s$.

# STEP 3 2006

STEP 3 2006
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

| | |
|---|---|
| **Section A** | Pure Mathematics |
| **Section B** | Mechanics |
| **Section C** | Probability and Statistics |

There are 14 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

## Section A: Pure Mathematics

### [STEP 3, 2006Q1]

Sketch the curve with cartesian equation

$$y = \frac{2x(x^2-5)}{x^2-4}$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence determine the number of real roots of the following equations:

(i) $3x(x^2-5) = (x^2-4)(x+3)$.

(ii) $4x(x^2-5) = (x^2-4)(5x-2)$.

(iii) $4x^2(x^2-5)^2 = (x^2-4)^2(x^2+1)$.

### [STEP 3, 2006Q2]

Let

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1-\sin\theta\sin 2\alpha}\,\mathrm{d}\theta \quad \text{and} \quad J = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2\theta}{1+\tan^2\theta\cos^2 2\alpha}\,\mathrm{d}\theta$$

where $0 < \alpha < \frac{1}{4}\pi$.

(i) Show that $I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1+\sin\theta\sin 2\alpha}\,\mathrm{d}\theta$ and hence that $2I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{2}{1+\tan^2\theta\cos^2 2\alpha}\,\mathrm{d}\theta$.

(ii) Find $J$.

(iii) By considering $I\sin^2 2\alpha + J\cos^2 2\alpha$, or otherwise, show that $I = \frac{1}{2}\pi\sec^2\alpha$.

(iv) Evaluate $I$ in the case $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$.

[STEP 3, 2006Q3]

(i) Let

$$\tan x = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad \cot x = \frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n$$

for $0 < x < \frac{1}{2}\pi$. Explain why $a_n = 0$ for even $n$.

Prove the identity

$$\cot x - \tan x \equiv 2\cot 2x$$

and show that

$$a_n = (1 - 2^{n+1})b_n.$$

(ii) Let $\operatorname{cosec} x = \frac{1}{x} + \sum_{n=0}^{\infty} c_n x^n$ for $0 < x < \frac{1}{2}\pi$. By considering $\cot x + \tan x$, or otherwise, show that

$$c_n = (2^{-n} - 1)b_n.$$

(iii) Show that

$$\left(1 + x\sum_{n=0}^{\infty} b_n x^n\right)^2 + x^2 = \left(1 + x\sum_{n=0}^{\infty} c_n x^n\right)^2.$$

Deduce from this and the previous results that $a_1 = 1$, and find $a_3$.



[STEP 3, 2006Q4]

The function $f$ satisfies the identity

$$f(x) + f(y) \equiv f(x+y) \qquad (*)$$

for all $x$ and $y$. Show that $2f(x) \equiv f(2x)$ and deduce that $f''(0) = 0$. By considering the Maclaurin Series for $f(x)$, find the most general function that satisfies $(*)$.

[Do not consider issues of existence or convergence of Maclaurin series in this question.]



(i) By considering the function $G$, defined by $\ln(g(x)) = G(x)$, find the most general function that, for all $x$ and $y$, satisfies the identity

$$g(x)g(y) \equiv g(x+y).$$

(ii) By considering the function $H$, defined by $h(\mathrm{e}^u) = H(u)$, find the most general function that satisfies, for all positive $x$ and $y$, the identity

$$h(x) + h(y) \equiv h(xy).$$

(iii) Find the most general function $t$ that, for all $x$ and $y$, satisfies the identity

$$t(x) + t(y) \equiv t(z),$$

where $z = \frac{x+y}{1-xy}$.

[STEP 3, 2006Q5]

Show that the distinct complex numbers $\alpha$, $\beta$ and $\gamma$ represent the vertices of an equilateral triangle (in clockwise or anti-clockwise order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.$$

Show that the roots of the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

represent the vertices of an equilateral triangle if and only if $a^2 = 3b$.

Under the transformation $z = pw + q$, where $p$ and $q$ are given complex numbers with $p \neq 0$, the equation $(*)$ becomes

$$w^3 + Aw^2 + Bw + C = 0. \qquad (**)$$

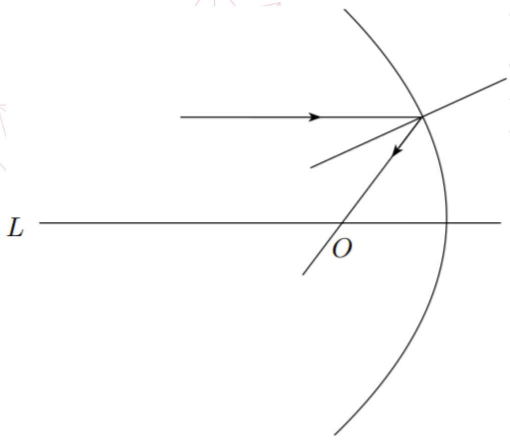
Show that if the roots of equation $(*)$ represent the vertices of an equilateral triangle, then the roots of equation $(**)$ also represent the vertices of an equilateral triangle.

[STEP 3, 2006Q6]

Show that in polar coordinates the gradient of any curve at the point $(r, \theta)$ is

$$\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r}{\frac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta}.$$

A mirror is designed so that if an incident ray of light is parallel to a fixed line $L$ the reflected ray passes through a fixed point $O$ on $L$. Prove that the mirror intersects any plane containing $L$ in a parabola. You should assume that the angle between the incident ray and the normal to the mirror is the same as the angle between the reflected ray and the normal.



[STEP 3, 2006Q7]

(i) Solve the equation $u^2 + 2u\sinh x - 1 = 0$ giving $u$ in terms of $x$.

Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

that satisfies $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ at $x = 0$.

(ii) Find the solution, not identically zero, of the differential equation

$$\sinh y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh y = 0$$

that satisfies $y = 0$ at $x = 0$, expressing your solution in the form $\cosh y = f(x)$. Show that the asymptotes to the solution curve are $y = \pm(-x + \ln 4)$.

[STEP 3, 2006Q8]

$\Delta$ is an operation that takes polynomials in $x$ to polynomials in $x$; that is, given any polynomial $h(x)$, there is a polynomial called $\Delta h(x)$ which is obtained from $h(x)$ using the rules that define $\Delta$. These rules are as follows:

(i) $\Delta x = 1$.

(ii) $\Delta\big(f(x) + g(x)\big) = \Delta f(x) + \Delta g(x)$ for any polynomials $f(x)$ and $g(x)$.

(iii) $\Delta\big(\lambda f(x)\big) = \lambda\Delta f(x)$ for any constant $\lambda$ and any polynomial $f(x)$.

(iv) $\Delta\big(f(x)g(x)\big) = f(x)\Delta g(x) + g(x)\Delta f(x)$ for any polynomials $f(x)$ and $g(x)$.



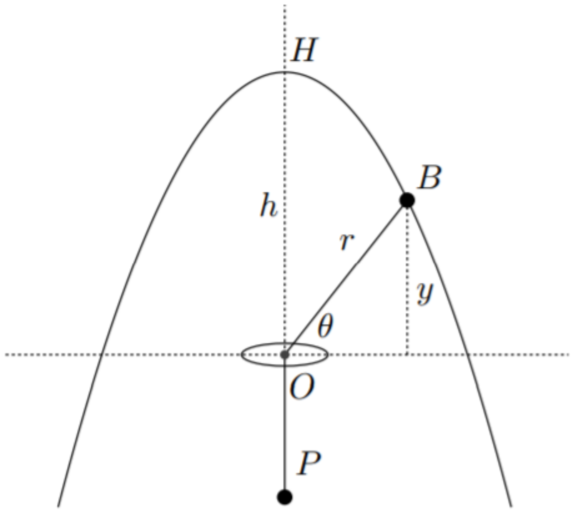
Using these rules show that, if $f(x)$ is a polynomial of degree zero (that is, a constant), then $\Delta f(x) = 0$. Calculate $\Delta x^2$ and $\Delta x^3$.

Prove that $\Delta h(x) = \frac{\mathrm{d}h(x)}{\mathrm{d}x}$ for any polynomial $h(x)$. You should make it clear whenever you use one of the above rules in your proof.

## Section B: Mechanics

### [STEP 3, 2006Q9]

A long, light, inextensible string passes through a small, smooth ring fixed at the point $O$. One end of the string is attached to a particle $P$ of mass $m$ which hangs freely below $O$. The other end is attached to a bead, $B$, also of mass $m$, which is threaded on a smooth rigid wire fixed in the same vertical plane as $O$. The distance $OB$ is $r$, the distance $OH$ is $h$ and the height of the bead above the horizontal plane through $O$ is $y$, as shown in the diagram.




The shape of the wire is such that the system can be in static equilibrium for all positions of the bead. By considering potential energy, show that the equation of the wire is $y + r = 2h$.

The bead is initially at $H$. It is then projected along the wire with initial speed $V$. Show that, in the subsequent motion,

$$\dot{\theta} = -\frac{h\dot{r}}{r\sqrt{rh - h^2}}$$

where $\theta$ is given by $\theta = \arcsin\left(\frac{y}{r}\right)$.

Hence show that the speed of the particle $P$ is $V\left(\frac{r-h}{2r-h}\right)^{\frac{1}{2}}$.

[Note that $\arcsin\theta$ is another notation for $\sin^{-1}\theta$.]

[STEP 3, 2006Q10]

A disc rotates freely in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is $mk^2$ (where $k > 0$). Along one diameter is a smooth narrow groove in which a particle of mass $m$ slides freely. At time $t = 0$, the disc is rotating with angular speed $\Omega$, and the particle is a distance $a$ from the axis and is moving with speed $V$ along the groove, towards the axis, where $k^2V^2 = \Omega^2a^2(k^2 + a^2)$.

Show that, at a later time $t$, while the particle is still moving towards the axis, the angular speed $\omega$ of the disc and the distance $r$ of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \qquad \text{and} \qquad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{\Omega^2r^2(k^2 + a^2)^2}{k^2(k^2 + r^2)}.$$

Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}},$$

where $\theta$ is the angle through which the disc has turned by time $t$.

By making the substitution $u = \frac{k}{r}$, or otherwise, show that $r\sinh(\theta + \alpha) = k$, where $\sinh\alpha = \frac{k}{a}$. Deduce that the particle never reaches the axis.

[STEP 3, 2006Q11]

A lift of mass $M$ and its counterweight of mass $M$ are connected by a light inextensible cable which passes over a fixed frictionless pulley. The lift is constrained to move vertically between smooth guides. The distance between the floor and the ceiling of the lift is $h$. Initially, the lift is at rest, and the distance between the top of the lift and the pulley is greater than $h$. A small tile of mass $m$ becomes detached from the ceiling of the lift and falls to the floor of the lift. Show that the speed of the tile just before the impact is

$$\sqrt{\frac{(2M - m)gh}{M}}.$$

The coefficient of restitution between the tile and the floor of the lift is $e$. Given that the magnitude of the impulsive force on the lift due to tension in the cable is equal to the magnitude of the impulsive force on the counterweight due to tension in the cable, show that the loss of energy of the system due to the impact is $mgh(1 - e^2)$. Comment on this result.

## Section C: Probability and Statistics

### [STEP 3, 2006Q12]

Fifty times a year, 1024 tourists disembark from a cruise liner at a port. From there they must travel to the city centre either by bus or by taxi. Tourists are equally likely to be directed to the bus station or to the taxi rank. Each bus of the bus company holds 32 passengers, and the company currently runs 15 buses. The company makes a profit of £1 for each passenger carried. It carries as many passengers as it can, with any excess being (eventually) transported by taxi. Show that the largest annual licence fee, in pounds, that the company should consider paying to be allowed to run an extra bus is approximately

$$1600\Phi(2) - \frac{800}{\sqrt{2\pi}}(1 - \mathrm{e}^{-2}),$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t$.

[You should not consider continuity corrections.]

### [STEP 3, 2006Q13]

Two points are chosen independently at random on the perimeter (including the diameter) of a semicircle of unit radius. The area of the triangle whose vertices are these two points and the midpoint of the diameter is denoted by the random variable $A$. Show that the expected value of $A$ is $(2 + \pi)^{-1}$.



### [STEP 3, 2006Q14]

For any random variables $X_1$ and $X_2$ state the relationship between $\mathrm{E}(aX_1 + bX_2)$ and $\mathrm{E}(X_1)$ and $\mathrm{E}(X_2)$, where $a$ and $b$ are constants. If $X_1$ and $X_2$ are independent, state the relationship between $\mathrm{E}(X_1X_2)$ and $\mathrm{E}(X_1)$ and $\mathrm{E}(X_2)$.

An industrial process produces rectangular plates. The length and the breadth of the plates are modelled by independent random variables $X_1$ and $X_2$ with non-zero means $\mu_1$ and $\mu_2$ and non-zero standard deviations $\sigma_1$ and $\sigma_2$, respectively. Using the results in the paragraph above, and without quoting a formula for $\mathrm{Var}(aX_1 + bX_2)$, find the means and standard deviations of the perimeter $P$ and area $A$ of the plates. Show that $P$ and $A$ are not independent.

The random variable $Z$ is defined by $Z = P - \alpha A$, where $\alpha$ is a constant. Show that $Z$ and $A$ are not independent if

$$\alpha \neq \frac{2(\mu_1\sigma_2^2 + \mu_2\sigma_1^2)}{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2}.$$

Given that $X_1$ and $X_2$ can each take values 1 and 3 only, and that they each take these values with probability $\frac{1}{2}$, show that $Z$ and $A$ are not independent for any value of $\alpha$.

# STEP 3 2007

STEP 3 2007
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

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## INFORMATION FOR CANDIDATES

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| **Section A** | Pure Mathematics |
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There are 14 questions in this paper.

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**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 2007Q1]

In this question, do not consider the special cases in which the denominators of any of your expressions are zero.

Express $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ in terms of $t_i$, where $t_1 = \tan\theta_1$, etc.

Given that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are the four roots of the equation

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

(where $a \neq 0$), find an expression in terms of $a, b, c, d$ and $e$ for $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$.

The four real numbers $\theta_1, \theta_2, \theta_3$ and $\theta_4$ lie in the range $0 \leq \theta_i < 2\pi$ and satisfy the equation

$$p\cos 2\theta + \cos(\theta - \alpha) + p = 0,$$

where $p$ and $\alpha$ are independent of $\theta$. Show that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$ for some integer $n$.

## [STEP 3, 2007Q2]

(i) Show that $1.3.5.7.\ldots.(2n-1) = \frac{(2n)!}{2^n n!}$ and that, for $|x| < \frac{1}{4}$,

$$\frac{1}{\sqrt{1-4x}} = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n.$$

(ii) By differentiating the above result, deduce that

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,(n-1)!} \left(\frac{6}{25}\right)^n = 60.$$

(iii) Show that

$$\sum_{n=1}^{\infty} \frac{2^{n+1}(2n)!}{3^{2n}(n+1)!\,n!} = 1.$$

## [STEP 3, 2007Q3]

A sequence of numbers, $F_1, F_2, \ldots$, is defined by $F_1 = 1$, $F_2 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 3.$$

(i) Write down the values of $F_3, F_4, \ldots, F_8$.

(ii) Prove that $F_{2k+3}F_{2k+1} - F_{2k+2}^2 = -F_{2k+2}F_{2k} + F_{2k+1}^2$.

(iii) Prove by induction or otherwise that $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$ and deduce that $F_{2n}^2 + 1$ is divisible by $F_{2n+1}$.

(iv) Prove that $F_{2n-1}^2 + 1$ is divisible by $F_{2n+1}$.

[STEP 3, 2007Q4]

A curve is given parametrically by

$$x = a\left(\cos t + \ln\tan\frac{1}{2}t\right),$$
$$y = a\sin t,$$

where $0 < t < \frac{1}{2}\pi$ and $a$ is a positive constant. Show that $\frac{dy}{dx} = \tan t$ and sketch the curve.

Let $P$ be the point with parameter $t$ and let $Q$ be the point where the tangent to the curve at $P$ meets the $x$-axis. Show that $PQ = a$.

The *radius of curvature*, $\rho$, at $P$ is defined by

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|},$$

where the dots denote differentiation with respect to $t$. Show that $\rho = a\cot t$.

The point $C$ lies on the normal to the curve at $P$, a distance $\rho$ from $P$ and above the curve. Show that $CQ$ is parallel to the $y$-axis.

[STEP 3, 2007Q5]

Let $y = \ln(x^2 - 1)$, where $x > 1$, and let $r$ and $\theta$ be functions of $x$ determined by $r = \sqrt{x^2 - 1}$ and $\coth\theta = x$. Show that

$$\frac{dy}{dx} = \frac{2\cosh\theta}{r} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{2\cosh 2\theta}{r^2},$$

and find an expression in terms of $r$ and $\theta$ for $\frac{d^3y}{dx^3}$.

Find, with proof, a similar formula for $\frac{d^ny}{dx^n}$ in terms of $r$ and $\theta$.

[STEP 3, 2007Q6]

The distinct points $P$, $Q$, $R$ and $S$ in the Argand diagram lie on a circle of radius $a$ centred at the origin and are represented by the complex numbers $p$, $q$, $r$ and $s$, respectively. Show that

$$pq = -a^2\frac{p - q}{p^* - q^*}.$$

Deduce that, if the chords $PQ$ and $RS$ are perpendicular, then $pq + rs = 0$.

The distinct points $A_1, A_2, \dots, A_n$ (where $n \geq 3$) lie on a circle. The points $B_1, B_2, \dots, B_n$ lie on the same circle and are chosen so that the chords $B_1B_2, B_2B_3, \dots, B_nB_1$ are perpendicular, respectively, to the chords $A_1A_2, A_2A_3, \dots, A_nA_1$. Show that, for $n = 3$, there are only two choices of $B_1$ for which this is possible. What is the corresponding result for $n = 4$? State the corresponding results for values of $n$ greater than 4.

## [STEP 3, 2007Q7]

The functions $s(x)$ $(0 \le x < 1)$ and $t(x)$ $(x \ge 0)$, and the real number $p$, are defined by

$$s(x) = \int_0^x \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u, \qquad t(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u, \qquad p = 2\int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u.$$

For this question, do not evaluate any of the above integrals explicitly in terms of inverse trigonometric functions or the number π.

(i) Use the substitution $u = v^{-1}$ to show that $t(x) = \int_{\frac{1}{x}}^{\infty} \frac{1}{1+v^2}\mathrm{d}v$. Hence evaluate $t\left(\frac{1}{x}\right) + t(x)$ in terms of $p$ and deduce that $2t(1) = \frac{1}{2}p$.

(ii) Let $y = \frac{u}{\sqrt{1+u^2}}$. Express $u$ in terms of $y$, and show that $\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{\sqrt{(1-y^2)^3}}$.

By making a substitution in the integral for $t(x)$, show that

$$t(x) = s\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Deduce that $s\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}p$.

(iii) Let $z = \frac{u+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}u}$. Show that $t\left(\frac{1}{\sqrt{3}}\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2}\mathrm{d}z$, and hence that $3t\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2}p$.

## [STEP 3, 2007Q8]

(i) Find functions $a(x)$ and $b(x)$ such that $u = x$ and $u = \mathrm{e}^{-x}$ both satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + a(x)\frac{\mathrm{d}u}{\mathrm{d}x} + b(x)u = 0.$$

For these functions $a(x)$ and $b(x)$, write down the general solution of the equation.

Show that the substitution $y = \frac{1}{3u}\frac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \frac{1-\mathrm{e}^{-x}}{3(x+\mathrm{e}^{-x})}$.

(ii) Find the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.

## Section B: Mechanics

### [STEP 3, 2007Q9]

Two small beads, $A$ and $B$, each of mass $m$, are threaded on a smooth horizontal circular hoop of radius $a$ and centre $O$. The angle $\theta$ is the acute angle determined by $2\theta = \angle AOB$.

The beads are connected by a light straight spring. The energy stored in the spring is

$$mk^2a^2(\theta - \alpha)^2,$$

where $k$ and $\alpha$ are constants satisfying $k > 0$ and $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$.

The spring is held in compression with $\theta = \beta$ and then released. Find the period of oscillations in the two cases that arise according to the value of $\beta$ and state the value of $\beta$ for which oscillations do not occur.

### [STEP 3, 2007Q10]

A particle is projected from a point on a plane that is inclined at an angle $\phi$ to the horizontal. The position of the particle at time $t$ after it is projected is $(x, y)$, where $(0, 0)$ is the point of projection, $x$ measures distance up the line of greatest slope and $y$ measures perpendicular distance from the plane. Initially, the velocity of the particle is given by $(\dot{x}, \dot{y}) = (V\cos\theta, V\sin\theta)$, where $V > 0$ and $\phi + \theta < \frac{\pi}{2}$. Write down expressions for $x$ and $y$.

The particle bounces on the plane and returns along the same path to the point of projection. Show that



$$2\tan\phi\tan\theta = 1$$

and that

$$R = \frac{V^2\cos^2\theta}{2g\sin\phi},$$

where $R$ is the range along the plane.

Show further that



$$\frac{2V^2}{gR} = 3\sin\phi + \operatorname{cosec}\phi$$

and deduce that the largest possible value of $R$ is $\frac{V^2}{\sqrt{3}g}$.

[STEP 3, 2007Q11]

(i) A wheel consists of a thin light circular rim attached by light spokes of length $a$ to a small hub of mass $m$. The wheel rolls without slipping on a rough horizontal table directly towards a straight edge of the table. The plane of the wheel is vertical throughout the motion. The speed of the wheel is $u$, where $u^2 < ag$.

Show that, after the wheel reaches the edge of the table and while it is still in contact with the table, the frictional force on the wheel is zero. Show also that the hub will fall a vertical distance $\frac{ag-u^2}{3g}$ before the rim loses contact with the table.

(ii) Two particles, each of mass $\frac{m}{2}$, are attached to a light circular hoop of radius $a$, at the ends of a diameter. The hoop rolls without slipping on a rough horizontal table directly towards a straight edge of the table. The plane of the hoop is vertical throughout the motion. When the centre of the hoop is vertically above the edge of the table it has speed $u$, where $u^2 < ag$, and one particle is vertically above the other.

Show that, after the hoop reaches the edge of the table and while it is still in contact with the table, the frictional force on the hoop is non-zero and deduce that the hoop will slip before it loses contact with the table.

## Section C: Probability and Statistics

### [STEP 3, 2007Q12]

I choose a number from the integers $1, 2, \ldots, (2n-1)$ and the outcome is the random variable $N$. Calculate $\mathrm{E}(N)$ and $\mathrm{E}(N^2)$.

I then repeat a certain experiment $N$ times, the outcome of the $i$th experiment being the random variable $X_i$ $(1 \le i \le N)$. For each $i$, the random variable $X_i$ has mean $\mu$ and variance $\sigma^2$, and $X_i$ is independent of $X_j$ for $i \neq j$ and also independent of $N$. The random variable $Y$ is defined by $Y = \sum_{i=1}^{N} X_i$.Show that $\mathrm{E}(Y) = n\mu$ and that $\mathrm{Cov}(Y, N) = \frac{1}{3}n(n-1)\mu$.

Find $\mathrm{Var}(Y)$ in terms of $n$, $\sigma^2$ and $\mu$.

### [STEP 3, 2007Q13]

A frog jumps towards a large pond. Each jump takes the frog either 1 m or 2 m nearer to the pond. The probability of a 1 m jump is $p$ and the probability of a 2 m jumps is $q$, where $p + q = 1$, the occurrence of long and short jumps being independent.

(i) Let $p_n(j)$ be the probability that the frog, starting at a point $\left(n - \frac{1}{2}\right)$ m away from the edge of the pond, lands in the pond for the first time on its $j$th jump. Show that $p_2(2) = p$.

(ii) Let $u_n$ be the expected number of jumps, starting at a point $\left(n - \frac{1}{2}\right)$ m away from the edge of the pond, required to land in the pond for the first time. Write down the values of $u_1$. By finding first the relevant values of $p_n(m)$, calculate $u_2$ and show that $u_3 = 3 - 2q + q^2$.

(iii) Given that $u_n$ can be expressed in the form $u_n = A(-q)^{n-1} + B + Cn$, where $A$, $B$ and $C$ are constants (independent of $n$), show that $C = (1+q)^{-1}$ and find $A$ and $B$ in terms of $q$. Hence show that, for large $n$, $u_n \approx \frac{n}{p+2q}$ and explain carefully why this result is to be expected.

[STEP 3, 2007Q14]

(i) My favourite dartboard is a disc of unit radius and centre $O$. I never miss the board, and the probability of my hitting any given area of the dartboard is proportional to the area. Each throw is independent of any other throw. I throw a dart $n$ times (where $n > 1$). Find the expected area of the smallest circle, with centre $O$, that encloses all the $n$ holes made by my dart.

Find also the expected area of the smallest circle, with centre $O$, that encloses all the $(n-1)$ holes nearest to $O$.

(ii) My other dartboard is a square of side 2 units, with centre $Q$. I never miss the board, and the probability of my hitting any given area of the dartboard is proportional to the area. Each throw is independent of any other throw. I throw a dart $n$ times (where $n > 1$). Find the expected area of the smallest square, with centre $Q$, that encloses all the $n$ holes made by my dart.

(iii) Determine, without detailed calculations, whether the expected area of the smallest circle, with centre $Q$, on my square dartboard that encloses all the $n$ holes made by my darts is larger or smaller than that for my circular dartboard.

# STEP 3 2008

STEP 3 2008

On-line Exam

Scan the QR code or click on the link to take an on-line exam.

Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

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Read the additional instructions on the front of the answer booklet.

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## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

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| **Section C** | Probability and Statistics |

There are 13 questions in this paper.

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**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 2008Q1]

Find all values of $a$, $b$, $x$ and $y$ that satisfy the simultaneous equations

$$\begin{aligned} a + b &= 1 \\ ax + by &= \frac{1}{3} \\ ax^2 + by^2 &= \frac{1}{5} \\ ax^3 + by^3 &= \frac{1}{7}. \end{aligned}$$

[**Hint:** you may wish to start by multiplying the second equation by $x + y$.]

## [STEP 3, 2008Q2]

Let $S_k(n) \equiv \sum_{r=0}^{n} r^k$, where $k$ is a positive integer, so that

$$S_1(n) \equiv \frac{1}{2}n(n+1) \quad \text{and} \quad S_2(n) \equiv \frac{1}{6}n(n+1)(2n+1).$$

(i) By considering $\sum_{r=0}^{n}[(r+1)^k - r^k]$, show that

$$kS_{k-1}(n) = (n+1)^k - (n+1) - \binom{k}{2}S_{k-2}(n) - \binom{k}{3}S_{k-3}(n) - \cdots - \binom{k}{k-1}S_1(n). \qquad (*)$$

Obtain simplified expressions for $S_3(n)$ and $S_4(n)$.

(ii) Explain, using $(*)$, why $S_k(n)$ is a polynomial of degree $k+1$ in $n$. Show that in this polynomial the constant term is zero and the sum of the coefficients is 1.



## [STEP 3, 2008Q3]

The point $P(a\cos\theta, b\sin\theta)$, where $a > b > 0$, lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The point $S(-\mathrm{e}a, 0)$, where $b^2 = a^2(1 - \mathrm{e}^2)$, is a focus of the ellipse. The point $N$ is the foot of the perpendicular from the origin, $O$, to the tangent to the ellipse at $P$. The lines $SP$ and $ON$ intersect at $T$. Show that the $y$-coordinate of $T$ is

$$\frac{b\sin\theta}{1 + \mathrm{e}\cos\theta}.$$

Show that $T$ lies on the circle with centre $S$ and radius $a$.

[STEP 3, 2008Q4]

(i) Show, with the aid of a sketch, that $y > \tanh\left(\frac{y}{2}\right)$ for $y > 0$ and deduce that

$$\operatorname{arcosh} x > \frac{x-1}{\sqrt{x^2-1}} \quad \text{for} \quad x > 1. \qquad (*)$$

(ii) By integrating $(*)$, show that $\operatorname{arcosh} x > 2\frac{x-1}{\sqrt{x^2-1}}$ for $x > 1$.

(iii) Show that $\operatorname{arcosh} x > 3\frac{\sqrt{x^2-1}}{x+2}$ for $x > 1$.

[**Note:** arcosh $x$ is another notation for $\cosh^{-1} x$.]

[STEP 3, 2008Q5]

The functions $T_n(x)$, for $n = 0, 1, 2, \ldots$, satisfy the recurrence relation

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \quad (n \geq 1). \qquad (*)$$

Show by induction that

$$\left(T_n(x)\right)^2 - T_{n-1}(x)T_{n+1}(x) = f(x),$$

where $f(x) = \left(T_1(x)\right)^2 - T_0(x)T_2(x)$.

In the case $f(x) \equiv 0$, determine (with proof) an expression for $T_n(x)$ in terms of $T_0(x)$ (assumed to be non-zero) and $r(x)$, where $r(x) = \frac{T_1(x)}{T_0(x)}$. Find the two possible expressions for $r(x)$ in terms of $x$.

[STEP 3, 2008Q6]

In this question, $p$ denotes $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Given that

$$y = p^2 + 2xp,$$

show by differentiating with respect to $x$ that

$$\frac{\mathrm{d}x}{\mathrm{d}p} = -2 - \frac{2x}{p}.$$

Hence show that $x = -\frac{2}{3}p + Ap^{-2}$, where $A$ is an arbitrary constant.

Find $y$ in terms of $x$ if $p = -3$ when $x = 2$.

(ii) Given instead that

$$y = 2xp + p\ln p,$$

and that $p = 1$ when $x = -\frac{1}{4}$, show that $x = -\frac{1}{2}\ln p - \frac{1}{4}$ and find $y$ in terms of $x$.

[STEP 3, 2008Q7]

The points $A$, $B$ and $C$ in the Argand diagram are the vertices of an equilateral triangle described anticlockwise. Show that the complex numbers $a$, $b$ and $c$ representing $A$, $B$ and $C$ satisfy

$$2c = (a+b) + \mathrm{i}\sqrt{3}(b-a).$$

Find a similar relation in the case that $A$, $B$ and $C$ are the vertices of an equilateral triangle described clockwise.

(i) The quadrilateral $DEFG$ lies in the Argand diagram. Show that points $P$, $Q$, $R$ and $S$ can be chosen so that $PDE$, $QEF$, $RFG$ and $SGD$ are equilateral triangles and $PQRS$ is a parallelogram.

(ii) The triangle $LMN$ lies in the Argand diagram. Show that the centroids $U$, $V$ and $W$ of the equilateral triangles drawn externally on the sides of $LMN$ are the vertices of an equilateral triangle.

[**Note:** The *centroid* of a triangle with vertices represented by the complex numbers $x$, $y$ and $z$ is the point represented by $\frac{1}{3}(x+y+z)$].

[STEP 3, 2008Q8]

(i) The coefficients in the series

$$S = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \cdots + a_r x^r + \cdots$$

satisfy a recurrence relation of the form $a_{r+1} + pa_r = 0$. Write down the value of $p$.

By considering $(1+px)S$, find an expression for the sum to infinity of $S$ (assuming that it exists). Find also an expression for the sum of the first $n+1$ terms of $S$.

(ii) The coefficients in the series

$$T = 2 + 8x + 18x^2 + 37x^3 + \cdots + a_r x^r + \cdots$$

satisfy a recurrence relation of the form $a_{r+2} + pa_{r+1} + qa_r = 0$. Find an expression for the sum to infinity of $T$ (assuming that it exists). By expressing $T$ in partial fractions, or otherwise, find an expression for the sum of the first $n+1$ terms of $T$.

## Section B: Mechanics

### [STEP 3, 2008Q9]

A particle of mass $m$ is initially at rest on a rough horizontal surface. The particle experiences a force $mg \sin \pi t$, where $t$ is time, acting in a fixed horizontal direction. The coefficient of friction between the particle and the surface is $\mu$. Given that the particle starts to move first at $t = T_0$, state the relation between $T_0$ and $\mu$.

(i) For $\mu = \mu_0$, the particle comes to rest for the first time at $t = 1$. Sketch the acceleration-time graph for $0 \le t \le 1$. Show that

$$1 + (1 - \mu_0^2)^{\frac{1}{2}} - \mu_0 \pi + \mu_0 \arcsin \mu_0 = 0.$$

(ii) For $\mu = \mu_0$ sketch the acceleration-time graph for $0 \le t \le 3$. Describe the motion of the particle in this case and in the case $\mu = 0$.

[**Note:** $\arcsin x$ is another notation for $\sin^{-1} x$.]

### [STEP 3, 2008Q10]

A long string consists of $n$ short light strings joined together, each of natural length $l$ and modulus of elasticity $\lambda$. It hangs vertically at rest, suspended from one end. Each of the short strings has a particle of mass $m$ attached to its lower end. The short strings are numbered 1 to $n$, the $n$th short string being at the top. By considering the tension in the $r$th short string, determine the length of the long string. Find also the elastic energy stored in the long string.

A uniform heavy rope of mass $M$ and natural length $L_0$ has modulus of elasticity $\lambda$. The rope hangs vertically at rest, suspended from one end. Show that the length, $L$, of the rope is given by

$$L = L_0\left(1 + \frac{Mg}{2\lambda}\right),$$

and find an expression in terms of $L$, $L_0$ and $\lambda$ for the elastic energy stored in the rope.

## [STEP 3, 2008Q11]

A circular wheel of radius $r$ has moment of inertia $I$ about its axle, which is fixed in a horizontal position. A light string is wrapped around the circumference of the wheel and a particle of mass $m$ hangs from the free end. The system is released from rest and the particle descends. The string does not slip on the wheel.

As the particle descends, the wheel turns through $n_1$ revolutions, and the string then detaches from the wheel. At this moment, the angular speed of the wheel is $\omega_0$. The wheel then turns through a further $n_2$ revolutions, in time $T$, before coming to rest. The couple on the wheel due to resistance is constant.

Show that

$$\frac{1}{2}\omega_0 T = 2\pi n_2$$

and

$$I = \frac{mgrn_1T^2 - 4\pi mr^2n_2^2}{4\pi n_2(n_1 + n_2)}.$$

## Section C: Probability and Statistics

### [STEP 3, 2008Q12]

Let $X$ be a random variable with a Laplace distribution, so that its probability density function is given by

$$f(x) = \frac{1}{2}\mathrm{e}^{-|x|}, \quad -\infty < x < \infty. \qquad (*)$$

Sketch $f(x)$. Show that its moment generating function $M_X(\theta)$ is given by $M_X(\theta) = (1-\theta^2)^{-1}$ and hence find the variance of $X$.

A frog is jumping up and down, attempting to land on the same spot each time. In fact, in each of $n$ successive jumps he always lands on a fixed straight line but when he lands from the $i$th jump $(i = 1,2,\ldots,n)$ his displacement from the point from which he jumped is $X_i$ cm, where $X_i$ has the distribution $(*)$. His displacement from his starting point after $n$ jumps is $Y$cm (so that $Y = \sum_{i=1}^{n} X_i$). Each jump is independent of the others.

Obtain the moment generating function for $\frac{Y}{\sqrt{2n}}$ and, by considering its logarithm, show that this moment generating function tends to $\exp\left(\frac{1}{2}\theta^2\right)$ as $n \to \infty$.

Given that $\exp\left(\frac{1}{2}\theta^2\right)$ is the moment generating function of the standard Normal random variable, estimate the least number of jumps such that there is a 5% chance that the frog lands 25 cm or more from his starting point.



### [STEP 3, 2008Q13]

A box contains $n$ pieces of string, each of which has two ends. I select two string ends at random and tie them together. This creates either a ring (if the two ends are from the same string) or a longer piece of string. I repeat the process of tying together string ends chosen at random until there are none left.

Find the expected number of rings created at the first step and hence obtain an expression for the expected number of rings created by the end of the process. Find also an expression for the variance of the number of rings created.

Given that $\ln 20 \approx 3$ and that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n$ for large $n$, determine approximately the expected number of rings created in the case $n = 40\,000$.

# STEP 3 2009

STEP 3 2009
On-line Exam

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**TIME ALLOWED: 180 MINUTES**

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# Section A: Pure Mathematics

## [STEP 3, 2009Q1]

The points $S$, $T$, $U$ and $V$ have coordinates $(s, ms)$, $(t, mt)$, $(u, nu)$ and $(v, nv)$, respectively. The lines $SV$ and $UT$ meet the line $y = 0$ at the points with coordinates $(p, 0)$ and $(q, 0)$, respectively. Show that

$$p = \frac{(m-n)sv}{ms - nv},$$

and write down a similar expression for $q$.

Given that $S$ and $T$ lie on the circle $x^2 + (y-c)^2 = r^2$, find a quadratic equation satisfied by $s$ and by $t$, and hence determine $st$ and $s + t$ in terms of $m$, $c$ and $r$.

Given that $S$, $T$, $U$ and $V$ lie on the above circle, show that $p + q = 0$.

## [STEP 3, 2009Q2]

(i) Let $y = \sum_{n=0}^{\infty} a_n x^n$, where the coefficients $a_n$ are independent of $x$ and are such that this series and all others in this question converge. Show that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and write down a similar expression for $y''$.

Write out explicitly each of the three series as far as the term containing $a_3$.

(ii) It is given that $y$ satisfies the differential equation

$$xy'' - y' + 4x^3 y = 0.$$

By substituting the series of part (i) into the differential equation and comparing coefficients, show that $a_1 = 0$.

Show that, for $n \geq 4$,

$$a_n = -\frac{4}{n(n-2)} a_{n-4},$$

and that, if $a_0 = 1$ and $a_2 = 0$, then $y = \cos(x^2)$.

Find the corresponding result when $a_0 = 0$ and $a_2 = 1$.

[STEP 3, 2009Q3]

The function $f(t)$ is defined, for $t \neq 0$, by

$$f(t) = \frac{t}{e^t - 1}.$$

(i) By expanding $e^t$, show that $\lim_{t\to 0} f(t) = 1$. Find $f'(t)$ and evaluate $\lim_{t\to 0} f'(t)$.

(ii) Show that $f(t) + \frac{1}{2}t$ is an even function. [**Note:** A function $g(t)$ is said to be *even* if $g(t) \equiv g(-t)$.]

(iii) Show with the aid of a sketch that $e^t(1-t) \leq 1$ and deduce that $f'(t) \neq 0$ for $t \neq 0$.

Sketch the graph of $f(t)$.

[STEP 3, 2009Q4]

For any given (suitable) function $f$, the *Laplace transform* of $f$ is the function $F$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad (s > 0).$$

(i) Show that the Laplace transform of $e^{-bt}f(t)$, where $b > 0$, is $F(s+b)$.

(ii) Show that the Laplace transform of $f(at)$, where $a > 0$, is $a^{-1}F\left(\frac{s}{a}\right)$.

(iii) Show that the Laplace transform of $f'(t)$ is $sF(s) - f(0)$.

(iv) In the case $f(t) = \sin t$, show that $F(s) = \frac{1}{s^2+1}$.

Using only these four results, find the Laplace transform of $e^{-pt}\cos qt$, where $p > 0$ and $q > 0$.

[STEP 3, 2009Q5]

The numbers $x$, $y$ and $z$ satisfy

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 2 \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

Show that

$$yz + zx + xy = -\frac{1}{2}.$$

Show also that $x^2y + x^2z + y^2z + y^2x + z^2x + z^2y = -1$, and hence that

$$xyz = \frac{1}{6}.$$

Let $S_n = x^n + y^n + z^n$. Use the above results to find numbers $a$, $b$ and $c$ such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2},$$

holds for all $n$.

[STEP 3, 2009Q6]

Show that $\left|e^{i\beta} - e^{i\alpha}\right| = 2\sin\frac{1}{2}(\beta - \alpha)$ for $0 < \alpha < \beta < 2\pi$. Hence show that

$$\left|e^{i\alpha} - e^{i\beta}\right|\left|e^{i\gamma} - e^{i\delta}\right| + \left|e^{i\beta} - e^{i\gamma}\right|\left|e^{i\alpha} - e^{i\delta}\right| = \left|e^{i\alpha} - e^{i\gamma}\right|\left|e^{i\beta} - e^{i\delta}\right|,$$

where $0 < \alpha < \beta < \gamma < \delta < 2\pi$.

Interpret this result as a theorem about cyclic quadrilaterals.

[STEP 3, 2009Q7]

(i) The functions $f_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$f_0(x) = \frac{1}{1+x^2} \quad \text{and} \quad f_{n+1}(x) = \frac{df_n(x)}{dx}.$$

Prove, for $n \geq 1$, that

$$(1+x^2)f_{n+1}(x) + 2(n+1)xf_n(x) + n(n+1)f_{n-1}(x) = 0.$$

(ii) The functions $P_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$P_n(x) = (1+x^2)^{n+1}f_n(x).$$

Find expressions for $P_0(x)$, $P_1(x)$ and $P_2(x)$.

Prove, for $n \geq 0$, that

$$P_{n+1}(x) - (1+x^2)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x) = 0,$$

and that $P_n(x)$ is a polynomial of degree $n$.



[STEP 3, 2009Q8]

Let $m$ be a positive integer and let $n$ be a non-negative integer.

(i) Use the result $\lim_{t\to\infty} e^{-mt}t^n = 0$ to show that

$$\lim_{x\to 0} x^m(\ln x)^n = 0.$$

By writing $x^x$ as $e^{x\ln x}$ show that

$$\lim_{x\to 0} x^x = 1.$$

(ii) Let $I_n = \int_0^1 x^m(\ln x)^n\, dx$. Show that

$$I_{n+1} = -\frac{n+1}{m+1}I_n$$

and hence evaluate $I_n$.

(iii) Show that

$$\int_0^1 x^x\, dx = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \cdots.$$

## Section B: Mechanics

### [STEP 3, 2009Q9]

A particle is projected under gravity from a point $P$ and passes through a point $Q$. The angles of the trajectory with the positive horizontal direction at $P$ and at $Q$ are $\theta$ and $\phi$, respectively. The angle of elevation of $Q$ from $P$ is $\alpha$.

(i) Show that $\tan\theta + \tan\phi = 2\tan\alpha$.

(ii) It is given that there is a second trajectory from $P$ to $Q$ with the same speed of projection. The angles of this trajectory with the positive horizontal direction at $P$ and at $Q$ are $\theta'$ and $\phi'$, respectively. By considering a quadratic equation satisfied by $\tan\theta$, show that $\tan(\theta + \theta') = -\cot\alpha$. Show also that $\theta + \theta' = \pi + \phi + \phi'$.

### [STEP 3, 2009Q10]

A light spring is fixed at its lower end and its axis is vertical. When a certain particle $P$ rests on the top of the spring, the compression is $d$. When, instead, $P$ is dropped onto the top of the spring from a height $h$ above it, the compression at time $t$ after $P$ hits the top of the spring is $x$. Obtain a second-order differential equation relating $x$ and $t$ for $0 \le t \le T$, where $T$ is the time at which $P$ first loses contact with the spring.

Find the solution of this equation in the form

$$x = A + B\cos(\omega t) + C\sin(\omega t),$$

where the constants $A$, $B$, $C$ and $\omega$ are to be given in terms of $d$, $g$ and $h$ as appropriate.

Show that

$$T = \sqrt{\frac{d}{g}}\left(2\pi - 2\arctan\sqrt{\frac{2h}{d}}\right).$$

[STEP 3, 2009Q11]

A comet in deep space picks up mass as it travels through a large stationary dust cloud. It is subject to a gravitational force of magnitude $Mf$ acting in the direction of its motion. When it entered the cloud, the comet had mass $M$ and speed $V$. After a time $t$, it has travelled a distance $x$ through the cloud, its mass is $M(1+bx)$, where $b$ is a positive constant, and its speed is $v$.

(i) In the case when $f = 0$, write down an equation relating $V$, $x$, $v$ and $b$. Hence find an expression for $x$ in terms of $b$, $V$ and $t$.

(ii) In the case when $f$ is a non-zero constant, use Newton's second law in the form

$$\text{force} = \text{rate of change of momentum}$$

to show that

$$v = \frac{ft + V}{1 + bx}.$$

Hence find an expression for $x$ in terms of $b$, $V$, $f$ and $t$.

Show that it is possible, if $b$, $V$ and $f$ are suitably chosen, for the comet to move with constant speed. Show also that, if the comet does not move with constant speed, its speed tends to a constant as $t \to \infty$.

## Section C: Probability and Statistics

### [STEP 3, 2009Q12]

(i) Albert tosses a fair coin $k$ times, where $k$ is a given positive integer. The number of heads he gets is $X_1$. He then tosses the coin $X_1$ times, getting $X_2$ heads. He then tosses the coin $X_2$ times, geting $X_3$ heads. The random variables $X_4, X_5, \ldots$ are defined similarly. Write down $\mathrm{E}(X_1)$.

By considering $\mathrm{E}(X_2|X_1 = x_1)$, or otherwise, show that $\mathrm{E}(X_2) = \frac{1}{4}k$.

Find $\sum_{i=1}^{\infty} \mathrm{E}(X_i)$.

(ii) Bertha has $k$ fair coins. She tosses the first coin until she gets a tail. The number of heads she gets before the first tail is $Y_1$. She then tosses the second coin until she gets a tail and the number of heads she gets with this coin before the first tail is $Y_2$. The random variables $Y_3, Y_4, \ldots, Y_k$ are defined similarly, and $Y = \sum_{i=1}^{k} Y_i$.

Obtain the probability generating function of $Y$, and use it to find $\mathrm{E}(Y)$, $\mathrm{Var}(Y)$ and $\mathrm{P}(Y = r)$.

### [STEP 3, 2009Q13]

(i) The point $P$ lies on the circumference of a circle of unit radius and centre $O$. The angle, $\theta$, between $OP$ and the positive $x$-axis is a random variable, uniformly distributed on the interval $0 \leq \theta < 2\pi$. The cartesian coordinates of $P$ with respect to $O$ are $(X, Y)$. Find the probability density function for $X$, and calculate $\mathrm{Var}(X)$.

Show that $X$ and $Y$ are uncorrelated and discuss briefly whether they are independent.

(ii) The points $P_i$ $(i = 1, 2, \ldots, n)$ are chosen independently on the circumference of the circle, as in part (i), and have cartesian coordinates $(X_i, Y_i)$. The point $\bar{P}$ has coordinates $(\bar{X}, \bar{Y})$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Show that $\bar{X}$ and $\bar{Y}$ are uncorrelated.

Show that, for large $n$, $P\left(\left|\bar{X}\right| \leq \sqrt{\frac{2}{n}}\right) \approx 0.95$.

# STEP 3 2010

STEP 3 2010
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

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Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

| | |
|---|---|
| **Section A** | Pure Mathematics |
| **Section B** | Mechanics |
| **Section C** | Probability and Statistics |

There are 13 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

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**Calculators are not permitted.**

## Section A: Pure Mathematics

### [STEP 3, 2010Q1]

Let $x_1, x_2, \dots, x_n$ and $x_{n+1}$ be any fixed real numbers. The numbers $A$ and $B$ are defined by

$$A = \frac{1}{n}\sum_{k=1}^{n} x_k\,, \qquad B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2,$$

and the numbers $C$ and $D$ are defined by

$$C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k\,, \qquad D = \frac{1}{n+1}\sum_{k=1}^{n+1}(x_k - C)^2.$$

(i) Express $C$ in terms of $A$, $x_{n+1}$ and $n$.

(ii) Show that

$$B = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - A^2.$$

(iii) Express $D$ in terms of $B$, $A$, $x_{n+1}$ and $n$.

Hence show that $(n+1)D \geq nB$ for all values of $x_{n+1}$, but that $D < B$ if and only if

$$A - \sqrt{\frac{(n+1)B}{n}} < x_{n+1} < A + \sqrt{\frac{(n+1)B}{n}}.$$



### [STEP 3, 2010Q2]

In this question, $a$ is a positive constant.

(i) Express $\cosh a$ in terms of exponentials.

By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,dx = \frac{a}{2\sinh a}.$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,dx$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2\cosh a + 1}\,dx.$$

[STEP 3, 2010Q3]

For any given positive integer $n$, a number $a$ (which may be complex) is said to be a *primitive $n$th root of unity* if $a^n = 1$ and there is no integer $m$ such that $0 < m < n$ and $a^m = 1$. Write down the two primitive 4th roots of unity.

Let $C_n(x)$ be the polynomial such that the roots of the equation $C_n(x) = 0$ are the primitive $n$th roots of unity, the coefficient of the highest power of $x$ is one and the equation has no repeated roots. Show that $C_4(x) = x^2 + 1$.

(i) Find $C_1(x)$, $C_2(x)$, $C_3(x)$, $C_5(x)$ and $C_6(x)$, giving your answers as unfactorised polynomials.

(ii) Find the value of $n$ for which $C_n(x) = x^4 + 1$.

(iii) Given that $p$ is prime, find an expression for $C_p(x)$, giving your answer as an unfactorised polynomial.

(iv) Prove that there are no positive integers $q$, $r$ and $s$ such that $C_q(x) \equiv C_r(x)C_s(x)$.

[STEP 3, 2010Q4]

(i) The number $\alpha$ is a common root of the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ (that is, $\alpha$ satisfies both equations). Given that $a \neq c$, show that

$$\alpha = -\frac{b-d}{a-c}.$$

Hence, or otherwise, show that the equations have at least one common root if and only if

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0.$$

Does this result still hold if the condition $a \neq c$ is not imposed?

(ii) Show that the equations $x^2 + ax + b = 0$ and $x^3 + (a+1)x^2 + qx + r = 0$ have at least one common root if and only if

$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0.$$

Hence, or otherwise, find the values of $b$ for which the equations $2x^2 + 5x + 2b = 0$ and $2x^3 + 7x^2 + 5x + 1 = 0$ have at least one common root.

[STEP 3, 2010Q5]

The vertices $A$, $B$, $C$ and $D$ of a square have coordinates $(0,0)$, $(a,0)$, $(a,a)$ and $(0,a)$, respectively. The points $P$ and $Q$ have coordinates $(an, 0)$ and $(0, am)$ respectively, where $0 < m < n < 1$. The line $CP$ produced meets $DA$ produced at $R$ and the line $CQ$ produced meets $BA$ produced at $S$. The line $PQ$ produced meets the line $RS$ produced at $T$. Show that $TA$ is perpendicular to $AC$.

Explain how, given a square of area $a^2$, a square of area $2a^2$ may be constructed using only a straight-edge.

[**Note:** a straight-edge is a ruler with no markings on it; no measurements (and no use of compasses) are allowed in the construction.]

## [STEP 3, 2010Q6]

The points $P$, $Q$ and $R$ lie on a sphere of unit radius centred at the origin, $O$, which is fixed. Initially, $P$ is at $P_0(1,0,0)$, $Q$ is at $Q_0(0,1,0)$ and $R$ is at $R_0(0,0,1)$.

(i) The sphere is then rotated about the $z$-axis, so that the line $OP$ turns directly towards the positive $y$-axis through an angle $\phi$. The position of $P$ after this rotation is denoted by $P_1$. Write down the coordinates of $P_1$.

(ii) The sphere is now rotated about the line in the $x$-$y$ plane perpendicular to $OP_1$, so that the line $OP$ turns directly towards the positive $z$-axis through an angle $\lambda$. The position of $P$ after this rotation is denoted by $P_2$. Find the coordinates of $P_2$. Find also the coordinates of the points $Q_2$ and $R_2$, which are the positions of $Q$ and $R$ after the two rotations.

(iii) The sphere is now rotated for a third time, so that $P$ returns from $P_2$ to its original position $P_0$. During the rotation, $P$ remains in the plane containing $P_0$, $P_2$ and $O$. Show that the angle of this rotation, $\theta$, satisfies

$$\cos\theta = \cos\phi\cos\lambda,$$

and find a vector in the direction of the axis about which this rotation takes place.

## [STEP 3, 2010Q7]

Given that $y = \cos(m\arcsin x)$, for $|x| < 1$, prove that

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y = 0.$$

Obtain a similar equation relating $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$, and a similar equation relating $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Conjecture and prove a relation between $\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}$, $\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}$ and $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$.

Obtain the first three non-zero terms of the Maclaurin series for $y$. Show that, if $m$ is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin\theta$ beginning

$$1 - \frac{m^2\sin^2\theta}{2!} + \frac{m^2(m^2-2^2)\sin^4\theta}{4!} - \cdots. \qquad \left(|\theta| < \frac{1}{2}\pi\right)$$

State the degree of the polynomial.

[STEP 3, 2010Q8]

Given that $P(x) = Q(x)R'(x) - Q'(x)R(x)$, write down an expression for

$$\int \frac{P(x)}{\left(Q(x)\right)^2}\,\mathrm{d}x$$

(i) By choosing the function $R(x)$ to be of the form $a + bx + cx^2$, find

$$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2}\,\mathrm{d}x.$$

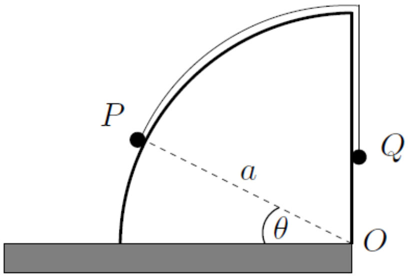
Show that the choice of $R(x)$ is not unique and, by comparing the two functions $R(x)$ corresponding to two different values of $a$, explain how the different choices are related.

(ii) Find the general solution of

$$(1 + \cos x + 2\sin x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\sin x - 2\cos x)y = 5 - 3\cos x + 4\sin x.$$

## Section B: Mechanics

### [STEP 3, 2010Q9]




The diagram shows two particles, $P$ and $Q$, connected by a light inextensible string which passes over a smooth block fixed to a horizontal table. The cross-section of the block is a quarter circle with centre $O$, which is at the edge of the table, and radius $a$. The angle between $OP$ and the table is $\theta$. The masses of $P$ and $Q$ are $m$ and $M$, respectively, where $m < M$.

Initially, $P$ is held at rest on the table and in contact with the block, $Q$ is vertically above $O$, and the string is taut. Then $P$ is released. Given that, in the subsequent motion, $P$ remains in contact with the block as $\theta$ increases from 0 to $\frac{1}{2}\pi$, find an expression, in terms of $m$, $M$, $\theta$ and $g$, for the normal reaction of the block on $P$ and show that

$$\frac{m}{M} \geq \frac{\pi - 1}{3}.$$

### [STEP 3, 2010Q10]



A small bead $B$, of mass $m$, slides without friction on a fixed horizontal ring of radius $a$. The centre of the ring is at $O$. The bead is attached by a light elastic string to a fixed point $P$ in the plane of the ring such that $OP = b$, where $b > a$. The natural length of the elastic string is $c$, where $c < b - a$, and its modulus of elasticity is $\lambda$. Show that the equation of motion of the bead is

$$ma\ddot{\phi} = -\lambda\left(\frac{a\sin\phi}{c\sin\theta} - 1\right)\sin(\theta + \phi),$$

where $\theta = \angle BPO$ and $\phi = \angle BOP$.

Given that $\theta$ and $\phi$ are small, show that $a(\theta + \phi) \approx b\theta$. Hence find the period of small oscillations about the equilibrium position $\theta = \phi = 0$.

[STEP 3, 2010Q11]

A bullet of mass $m$ is fired horizontally with speed $u$ into a wooden block of mass $M$ at rest on a horizontal surface. The coefficient of friction between the block and the surface is $\mu$. While the bullet is moving through the block, it experiences a constant force of resistance to its motion of magnitude $R$, where $R > (M+m)\mu g$. The bullet moves horizontally in the block and does not emerge from the other side of the block.

(i) Show that the magnitude, $a$, of the deceleration of the bullet relative to the block while the bullet is moving through the block is given by

$$a = \frac{R}{m} + \frac{R-(M+m)\mu g}{M}.$$

(ii) Show that the common speed, $v$, of the block and bullet when the bullet stops moving through the block satisfies

$$av = \frac{Ru-(M+m)\mu g u}{M}.$$

(iii) Obtain an expression, in terms of $u$, $v$ and $a$, for the distance moved by the block while the bullet is moving through the block.

(iv) Show that the total distance moved by the block is

$$\frac{muv}{2(M+m)\mu g}.$$

Describe briefly what happens if $R < (M+m)\mu g$.

## Section C: Probability and Statistics

### [STEP 3, 2010Q12]

The infinite series $S$ is given by

$$S = 1 + (1 + d)r + (1 + 2d)r^2 + \cdots + (1 + nd)r^n + \cdots,$$

for $|r| < 1$. By considering $S - rS$, or otherwise, prove that

$$S = \frac{1}{1-r} + \frac{rd}{(1-r)^2}.$$

Arthur and Boadicea shoot arrows at a target. The probability that an arrow shot by Arthur hits the target is $a$; the probability that an arrow shot by Boadicea hits the target is $b$. Each shot is independent of all others. Prove that the expected number of shots it takes Arthur to hit the target is $\frac{1}{a}$.

Arthur and Boadicea now have a contest. They take alternate shots, with Arthur going first. The winner is the one who hits the target first. The probability that Arthur wins the contest is $\alpha$ and the probability that Boadicea wins is $\beta$. Show that

$$\alpha = \frac{a}{1 - a'b'},$$

where $a' = 1 - a$ and $b' = 1 - b$, and find $\beta$.

Show that the expected number of shots in the contest is $\frac{\alpha}{a} + \frac{\beta}{b}$.



### [STEP 3, 2010Q13]

In this question, $\mathrm{Corr}(U, V)$ denotes the product moment correlation coefficient between the random variables $U$ and $V$, defined by

$$\mathrm{Corr}(U, V) \equiv \frac{\mathrm{Cov}(U, V)}{\sqrt{\mathrm{Var}(U)\mathrm{Var}(V)}}.$$

The independent random variables $Z_1$, $Z_2$ and $Z_3$ each have expectation 0 and variance 1. What is the value of $\mathrm{Corr}(Z_1, Z_2)$?

Let $Y_1 = Z_1$ and let

$$Y_2 = \rho_{12} Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}} Z_2,$$

where $\rho_{12}$ is a given constant with $-1 < \rho_{12} < 1$. Find $\mathrm{E}(Y_2)$, $\mathrm{Var}(Y_2)$ and $\mathrm{Corr}(Y_1, Y_2)$.

Now let $Y_3 = aZ_1 + bZ_2 + cZ_3$, where $a$, $b$ and $c$ are real constants and $c \geq 0$. Given that $\mathrm{E}(Y_3) = 0$, $\mathrm{Var}(Y_3) = 1$, $\mathrm{Corr}(Y_1, Y_3) = \rho_{13}$ and $\mathrm{Corr}(Y_2, Y_3) = \rho_{23}$, express $a$, $b$ and $c$ in terms of $\rho_{23}$, $\rho_{13}$ and $\rho_{12}$.

Given constants $\mu_i$ and $\sigma_i$, for $i = 1, 2$ and $3$, give expressions in terms of the $Y_i$, for random variables $X_i$ such that $\mathrm{E}(X_i) = \mu_i$, $\mathrm{Var}(X_i) = \sigma_i^2$ and $\mathrm{Corr}(X_i, X_j) = \rho_{ij}$.

# STEP 3 2011

STEP 3 2011
On-line Exam

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Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

**INSTRUCTIONS TO CANDIDATES**

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# Section A: Pure Mathematics

## [STEP 3, 2011Q1]

(i) Find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \left(\frac{x+2}{x+1}\right)u = 0.$$

(ii) Show that substituting $y = ze^{-x}$ (where $z$ is a function of $x$) into the second order differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0 \qquad (*)$$

leads to a first order differential equation for $\frac{\mathrm{d}z}{\mathrm{d}x}$. Find $z$ and hence show that the general solution of $(*)$ is

$$y = Ax + B\mathrm{e}^{-x},$$

where $A$ and $B$ are arbitrary constants.

(iii) Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = (x+1)^2.$$

## [STEP 3, 2011Q2]



The polynomial $f(x)$ is defined by

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

where $n \geq 2$ and the coefficients $a_0, \ldots, a_{n-1}$ are integers, with $a_0 \neq 0$. Suppose that the equation $f(x) = 0$ has a rational root $\frac{p}{q}$, where $p$ and $q$ are integers with no common factor greater than 1, and $q > 0$. By considering $q^{n-1}f\left(\frac{p}{q}\right)$, find the value of $q$ and deduce that any rational root of the equation $f(x) = 0$ must be an integer.

(i) Show that the $n$th root of 2 is irrational for $n \geq 2$.

(ii) Show that the cubic equation

$$x^3 - x + 1 = 0$$

has no rational roots.

(iii) Show that the polynomial equation

$$x^n - 5x + 7 = 0$$

has no rational roots for $n \geq 2$.

[STEP 3, 2011Q3]

Show that, provided $q^2 \neq 4p^3$, the polynomial

$$x^3 - 3px + q \qquad (p \neq 0, q \neq 0)$$

can be written in the form

$$a(x-\alpha)^3 + b(x-\beta)^3,$$

where $\alpha$ and $\beta$ are the roots of the quadratic equation $pt^2 - qt + p^2 = 0$, and $a$ and $b$ are constants which you should express in terms of $\alpha$ and $\beta$.

Hence show that one solution of the equation $x^3 - 24x + 48 = 0$ is

$$x = \frac{2\left(2 - 2^{\frac{1}{3}}\right)}{1 - 2^{\frac{1}{3}}}$$

and obtain similar expressions for the other two solutions in terms of $\omega$, where $\omega = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$.

Find also the roots of $x^3 - 3px + q = 0$ when $p = r^2$ and $q = 2r^3$ for some non-zero constant $r$.

[STEP 3, 2011Q4]

The following result applies to any function $f$ which is continuous, has positive gradient and satisfies $f(0) = 0$:

$$ab \leq \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y, \qquad (*)$$

where $f^{-1}$ denotes the inverse function of $f$, and $a \geq 0$ and $b \geq 0$.

(i) By considering the graph of $y = f(x)$, explain briefly why the inequality $(*)$ holds.

In the case $a > 0$ and $b > 0$, state a condition on $a$ and $b$ under which equality holds.

(ii) By taking $f(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Verify that equality holds under the condition you stated above.

(iii) Show that, for $0 \leq a \leq \frac{1}{2}\pi$ and $0 \leq b \leq 1$,

$$ab \leq b \arcsin b + \sqrt{1-b^2} - \cos a.$$

Deduce that, for $t \geq 1$,

$$\arcsin(t^{-1}) \geq t - \sqrt{t^2 - 1}.$$

[STEP 3, 2011Q5]

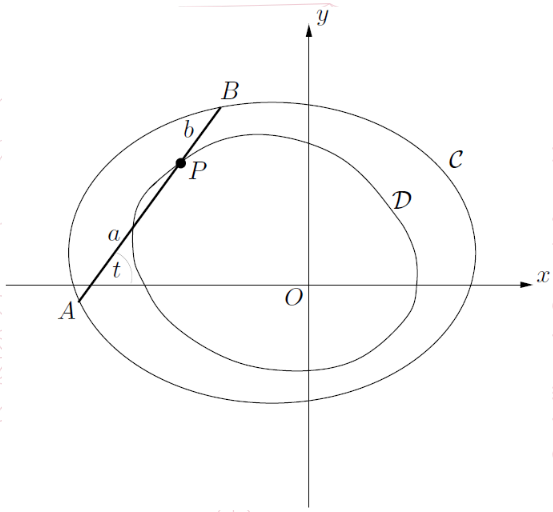
A movable point $P$ has cartesian coordinates $(x, y)$, where $x$ and $y$ are functions of $t$. The polar coordinates of $P$ with respect to the origin $O$ are $r$ and $\theta$. Starting with the expression

$$\frac{1}{2}\int r^2\,\mathrm{d}\theta$$

for the area swept out by $OP$, obtain the equivalent expression

$$\frac{1}{2}\int\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t. \qquad (*)$$

The ends of a thin straight rod $AB$ lie on a closed convex curve $\mathcal{C}$. The point $P$ on the rod is a fixed distance $a$ from $A$ and a fixed distance $b$ from $B$. The angle between $AB$ and the positive $x$ direction is $t$. As $A$ and $B$ move anticlockwise round $\mathcal{C}$, the angle $t$ increases from 0 to $2\pi$ and $P$ traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin $O$ lying inside $\mathcal{D}$, as shown in the diagram.






Let $(x, y)$ be the coordinates of $P$. Write down the coordinates of $A$ and $B$ in terms of $a$, $b$, $x$, $y$ and $t$.

The areas swept out by $OA$, $OB$ and $OP$ are denoted by $[A]$, $[B]$ and $[P]$, respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \frac{1}{2}\int_0^{2\pi}\left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t}\right)\cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\sin t\right)\mathrm{d}t.$$

Obtain a corresponding expression for $[B]$ involving $b$. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is $\pi ab$.

[STEP 3, 2011Q6]

The definite integrals $T$, $U$, $V$ and $X$ are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t, \qquad U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,\mathrm{d}u,$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1-v^2}\,\mathrm{d}v, \qquad X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\,\mathrm{d}x.$$

Show, without evaluating any of them, that $T$, $U$, $V$ and $X$ are all equal.

[STEP 3, 2011Q7]

Let

$$T_n = \left(\sqrt{a+1} + \sqrt{a}\right)^n,$$

where $n$ is a positive integer and $a$ is any given positive integer.

(i) In the case when $n$ is even, show by induction that $T_n$ can be written in the form

$$A_n + B_n\sqrt{a(a+1)},$$

where $A_n$ and $B_n$ are integers (depending on $a$ and $n$) and $A_n^2 = a(a+1)B_n^2 + 1$.

(ii) In the case when $n$ is odd, show by considering $\left(\sqrt{a+1} + \sqrt{a}\right)T_m$ where $m$ is even, or otherwise, that $T_n$ can be written in the form

$$C_n\sqrt{a+1} + D_n\sqrt{a},$$

where $C_n$ and $D_n$ are integers (depending on $a$ and $n$) and $(a+1)C_n^2 = aD_n^2 + 1$.

(iii) Deduce that, for each $n$, $T_n$ can be written as the sum of the square roots of two consecutive integers.

[STEP 3, 2011Q8]

The complex numbers $z$ and $w$ are related by

$$w = \frac{1 + \mathbf{i}z}{\mathbf{i} + z}.$$

Let $z = x + \mathbf{i}y$ and $w = u + \mathbf{i}v$, where $x$, $y$, $u$ and $v$ are real. Express $u$ and $v$ in terms of $x$ and $y$.

(i) By setting $x = \tan\left(\frac{\theta}{2}\right)$, or otherwise, show that if the locus of $z$ is the real axis $y = 0$, $-\infty < x < \infty$, then the locus of $w$ is the circle $u^2 + v^2 = 1$ with one point omitted.

(ii) Find the locus of $w$ when the locus of $z$ is the line segment $y = 0$, $-1 < x < 1$.

(iii) Find the locus of $w$ when the locus of $z$ is the line segment $x = 0$, $-1 < y < 1$.

(iv) Find the locus of $w$ when the locus of $z$ is the line $y = 1$, $-\infty < x < \infty$.

## Section B: Mechanics

### [STEP 3, 2011Q9]

Particles $P$ and $Q$ have masses $3m$ and $4m$, respectively. They lie on the outer curved surface of a smooth circular cylinder of radius $a$ which is fixed with its axis horizontal. They are connected by a light inextensible string of length $\frac{1}{2}\pi a$, which passes over the surface of the cylinder. The particles and the string all lie in a vertical plane perpendicular to the axis of the cylinder, and the axis intersects this plane at $O$. Initially, the particles are in equilibrium.

Equilibrium is slightly disturbed and $Q$ begins to move downwards. Show that while the two particles are still in contact with the cylinder the angle $\theta$ between $OQ$ and the vertical satisfies

$$7a\dot{\theta}^2 + 8g\cos\theta + 6g\sin\theta = 10g.$$

(i) Given that $Q$ loses contact with the cylinder first, show that it does so when $\theta = \beta$, where $\beta$ satisfies

$$15\cos\beta + 6\sin\beta = 10.$$

(ii) Show also that while $P$ and $Q$ are still in contact with the cylinder the tension in the string is $\frac{12}{7}mg(\sin\theta + \cos\theta)$.

### [STEP 3, 2011Q10]

Particles $P$ and $Q$, each of mass $m$, lie initially at rest a distance $a$ apart on a smooth horizontal plane. They are connected by a light elastic string of natural length $a$ and modulus of elasticity $\frac{1}{2}ma\omega^2$, where $\omega$ is a constant.

Then $P$ receives an impulse which gives it a velocity $u$ directly away from $Q$. Show that when the string next returns to length $a$, the particles have travelled a distance $\frac{1}{2}\frac{\pi u}{\omega}$, and find the speed of each particle.

Find also the total time between the impulse and the subsequent collision of the particles.

[STEP 3, 2011Q11]

A thin uniform circular disc of radius $a$ and mass $m$ is held in equilibrium in a horizontal plane a distance $b$ below a horizontal ceiling, where $b > 2a$. It is held in this way by $n$ light inextensible vertical strings, each of length $b$; one end of each string is attached to the edge of the disc and the other end is attached to a point on the ceiling. The strings are equally spaced around the edge of the disc. One of the strings is attached to the point $P$ on the disc which has coordinates $(a, 0, -b)$ with respect to cartesian axes with origin on the ceiling directly above the centre of the disc.

The disc is then rotated through an angle $\theta$ (where $\theta < \pi$) about its vertical axis of symmetry and held at rest by a couple acting in the plane of the disc. Show that the string attached to $P$ now makes an angle $\phi$ with the vertical, where

$$b \sin\phi = 2a \sin\frac{1}{2}\theta.$$

Show further that the magnitude of the couple is

$$\frac{mga^2 \sin\theta}{\sqrt{b^2 - 4a^2 \sin^2\frac{1}{2}\theta}}.$$

The disc is now released from rest. Show that its angular speed, $\omega$, when the strings are vertical is given by

$$\frac{a^2\omega^2}{4g} = b - \sqrt{b^2 - 4a^2 \sin^2\frac{1}{2}\theta}.$$

## Section C: Probability and Statistics

### [STEP 3, 2011Q12]

The random variable $N$ takes positive integer values and has pgf (probability generating function) $G(t)$. The random variables $X_i$, where $i = 1, 2, 3, \ldots$, are independently and identically distributed, each with pgf $H(t)$. The random variables $X_i$ are also independent of $N$. The random variable $Y$ is defined by

$$Y = \sum_{i=1}^{N} X_i.$$

Given that the pgf of $Y$ is $G\big(H(t)\big)$, show that

$$\mathrm{E}(Y) = \mathrm{E}(N)\mathrm{E}(X_i) \quad \text{and} \quad \mathrm{Var}(Y) = \mathrm{Var}(N)\big(\mathrm{E}(X_i)\big)^2 + \mathrm{E}(N)\mathrm{Var}(X_i).$$

A fair coin is tossed until a head occurs. The total number of tosses is $N$. The coin is then tossed a further $N$ times and the total number of heads in these $N$ tosses is $Y$. Find in this particular case the pgf of $Y$, $\mathrm{E}(Y)$, $\mathrm{Var}(Y)$ and $\mathrm{P}(Y = r)$.

### [STEP 3, 2011Q13]

In this question, the notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$, so for example $\lfloor \pi \rfloor = 3$ and $\lfloor 3 \rfloor = 3$.

(i) A bag contains $n$ balls, of which $b$ are black. A sample of $k$ balls is drawn, one after another, at random *with* replacement. The random variable $X$ denotes the number of black balls in the sample. By considering

$$\frac{\mathrm{P}(X = r + 1)}{\mathrm{P}(X = r)},$$

show that, in the case that it is unique, the most probable number of black balls in the sample is

$$\left\lfloor \frac{(k+1)b}{n} \right\rfloor.$$

Under what circumstances is the answer not unique?

(ii) A bag contains $n$ balls, of which $b$ are black. A sample of $k$ balls (where $k \leq b$) is drawn, one after another, at random *without* replacement. Find, in the case that it is unique, the most probable number of black balls in the sample.

Under what circumstances is the answer not unique?

# STEP 3 2012

STEP 3 2012
On-line Exam

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# Section A: Pure Mathematics

## [STEP 3, 2012Q1]

Given that $z = y^n \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, show that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left(n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right).$$

(i) Use the above result to show that the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{y} \qquad (y > 0)$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ is $y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}}$.

(ii) Find the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^2 = 0$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

## [STEP 3, 2012Q2]

In this question, $|x| < 1$ and you may ignore issues of convergence.

(i) Simplify

$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots\left(1+x^{2^n}\right),$$

where $n$ is a positive integer, and deduce that

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)\cdots\left(1+x^{2^n}\right) + \frac{x^{2^{n+1}}}{1-x}.$$

Deduce further that

$$\ln(1-x) = -\sum_{r=0}^{\infty}\ln\left(1+x^{2^r}\right),$$

and hence that

$$\frac{1}{1-x} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots.$$

(ii) Show that

$$\frac{1+2x}{1+x+x^2} = \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \cdots.$$

[STEP 3, 2012Q3]

It is given that the two curves

$$y = 4 - x^2 \quad \text{and} \quad mx = k - y^2,$$

where $m > 0$, touch exactly once.

(i) In each of the following four cases, sketch the two curves on a single diagram, noting the coordinates of any intersections with the axes:

(a) $k < 0$.

(b) $0 < k < 16, \frac{k}{m} < 2$.

(c) $k > 16, \frac{k}{m} > 2$.

(d) $k > 16, \frac{k}{m} < 2$.

(ii) Now set $m = 12$.

Show that the $x$-coordinate of any point at which the two curves meet satisfies

$$x^4 - 8x^2 + 12x + 16 - k = 0.$$

Let $a$ be the value of $x$ at the point where the curves touch. Show that $a$ satisfies

$$a^3 - 4a + 3 = 0$$

and hence find the three possible values of $a$.

Derive also the equation

$$k = -4a^2 + 9a + 16.$$

Which of the four sketches in part (i) arise?



[STEP 3, 2012Q4]

(i) Show that

$$\sum_{n=1}^{\infty} \frac{n+1}{n!} = 2e - 1$$

and

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!} = 5e - 1.$$

Sum the series $\sum_{n=1}^{\infty} \frac{(2n-1)^3}{n!}$.

(ii) Sum the series $\sum_{n=0}^{\infty} \frac{(n^2+1)2^{-n}}{(n+1)(n+2)}$, giving your answer in terms of natural logarithms.

[STEP 3, 2012Q5]

(i) The point with coordinates $(a, b)$, where $a$ and $b$ are rational numbers, is called:

an *integer rational point* if both $a$ and $b$ are integers,

a *non-integer rational point* if neither $a$ nor $b$ is an integer.

(a) Write down an integer rational point and a non-integer rational point on the circle $x^2 + y^2 = 1$.

(b) Write down an integer rational point on the circle $x^2 + y^2 = 2$. Simplify

$$\left(\cos\theta + \sqrt{m}\sin\theta\right)^2 + \left(\sin\theta - \sqrt{m}\cos\theta\right)^2$$

and hence obtain a non-integer rational point on the circle $x^2 + y^2 = 2$.

(ii) The point with coordinates $\left(p + \sqrt{2}q, r + \sqrt{2}s\right)$, where $p$, $q$, $r$ and $s$ are rational numbers, is called:

an *integer 2-rational point* if all of $p$, $q$, $r$ and $s$ are integers,

a *non-integer 2-rational point* if none of $p$, $q$, $r$ and $s$ is an integer.

(a) Write down an integer 2-rational point, and obtain a non-integer 2-rational point, on the circle $x^2 + y^2 = 3$.

(b) Obtain a non-integer 2-rational point on the circle $x^2 + y^2 = 11$.

(c) Obtain a non-integer 2-rational point on the hyperbola $x^2 - y^2 = 7$.



[STEP 3, 2012Q6]

Let $x + \mathrm{i}y$ be a root of the quadratic equation $z^2 + pz + 1 = 0$, where $p$ is a real number. Show that $x^2 - y^2 + px + 1 = 0$ and $(2x + p)y = 0$. Show further that

$$\text{either } p = -2x \quad \text{or} \quad p = -\frac{(x^2+1)}{x} \quad \text{with} \quad x \neq 0.$$

Hence show that the set of points in the Argand diagram that can (as $p$ varies) represent roots of the quadratic equation consists of the real axis with one point missing and a circle. This set of points is called the *root locus* of the quadratic equation.

Obtain and sketch in the Argand diagram the root locus of the equation

$$pz^2 + z + 1 = 0$$

and the root locus of the equation

$$pz^2 + p^2 z + 2 = 0.$$

## [STEP 3, 2012Q7]

A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time $t$ of the drug in the blood and the brain respectively are $y(t)$ and $z(t)$. These satisfy

$$\dot{y} = -2(y - z), \qquad \dot{z} = -\dot{y} - 3z,$$

where the dot denotes differentiation with respect to $t$.

Obtain a second order differential equation for $y$ and hence derive the solution

$$y = Ae^{-t} + Be^{-6t}, \qquad z = \frac{1}{2}Ae^{-t} - 2Be^{-6t},$$

where $A$ and $B$ are arbitrary constants.

(i) Obtain the solution that satisfies $z(0) = 0$ and $y(0) = 5$. The quantity of the drug in the brain for this solution is denoted by $z_1(t)$.

(ii) Obtain the solution that satisfies $z(0) = z(1) = c$, where $c$ is a given constant. The quantity of the drug in the brain for this solution is denoted by $z_2(t)$.

(iii) Show that for $0 \le t \le 1$,

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t - n),$$

provided $c$ takes a particular value that you should find.



## [STEP 3, 2012Q8]

The sequence $F_0, F_1, F_2, \ldots$ is defined by $F_0 = 0$, $F_1 = 1$ and, for $n \ge 0$,

$$F_{n+2} = F_{n+1} + F_n.$$

(i) Show that $F_0F_3 - F_1F_2 = F_2F_5 - F_3F_4$.

(ii) Find the values of $F_nF_{n+3} - F_{n+1}F_{n+2}$ in the two cases that arise.

(iii) Prove that, for $r = 1, 2, 3, \ldots$,

$$\arctan\left(\frac{1}{F_{2r}}\right) = \arctan\left(\frac{1}{F_{2r+1}}\right) + \arctan\left(\frac{1}{F_{2r+2}}\right)$$

and hence evaluate the following sum (which you may assume converges):

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+1}}\right).$$

## Section B: Mechanics

### [STEP 3, 2012Q9]

A pulley consists of a disc of radius $r$ with centre $O$ and a light thin axle through $O$ perpendicular to the plane of the disc. The disc is non-uniform, its mass is $M$ and its centre of mass is at $O$. The axle is fixed and horizontal.

Two particles, of masses $m_1$ and $m_2$ where $m_1 > m_2$, are connected by a light inextensible string which passes over the pulley. The contact between the string and the pulley is rough enough to prevent the string sliding. The pulley turns and the vertical force on the axle is found, by measurement, to be $P + Mg$.

(i) The moment of inertia of the pulley about its axle is calculated assuming that the pulley rotates without friction about its axle. Show that the calculated value is

$$\frac{\big((m_1+m_2)P - 4m_1m_2g\big)r^2}{(m_1+m_2)g - P}. \qquad (*)$$

(ii) Instead, the moment of inertia of the pulley about its axle is calculated assuming that a couple of magnitude $C$ due to friction acts on the axle of the pulley. Determine whether this calculated value is greater or smaller than $(*)$.

Show that $C < (m_1 - m_2)rg$.

### [STEP 3, 2012Q10]

A small ring of mass $m$ is free to slide without friction on a hoop of radius $a$. The hoop is fixed in a vertical plane. The ring is connected by a light elastic string of natural length $a$ to the highest point of the hoop. The ring is initially at rest at the lowest point of the hoop and is then slightly displaced. In the subsequent motion the angle of the string to the downward vertical is $\phi$. Given that the ring first comes to rest just as the string becomes slack, find an expression for the modulus of elasticity of the string in terms of $m$ and $g$.

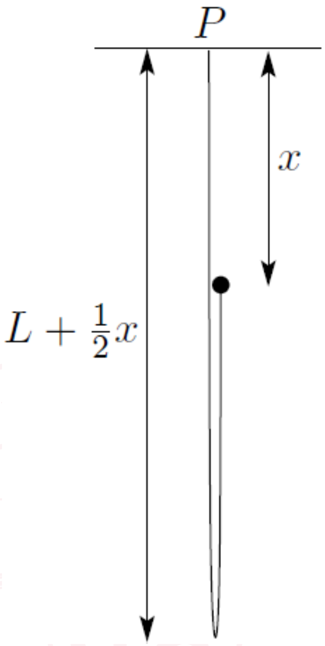
Show that, throughout the motion, the magnitude $R$ of the reaction between the ring and the hoop is given by

$$R = (12\cos^2\phi - 15\cos\phi + 5)mg$$

and that $R$ is non-zero throughout the motion.

[STEP 3, 2012Q11]

One end of a thin heavy uniform inextensible perfectly flexible rope of length $2L$ and mass $2M$ is attached to a fixed point $P$. A particle of mass $m$ is attached to the other end. Initially, the particle is held at $P$ and the rope hangs vertically in a loop below $P$. The particle is then released so that it and a section of the rope (of decreasing length) fall vertically as shown in the diagram.




You may assume that each point on the moving section of the rope falls at the same speed as the particle. Given that energy is conserved, show that, when the particle has fallen a distance $x$ (where $x < 2L$), its speed $v$ is given by

$$v^2 = \frac{2gx\left(mL + ML - \frac{1}{4}Mx\right)}{mL + ML - \frac{1}{2}Mx}.$$

Hence show that the acceleration of the particle is

$$g + \frac{Mgx\left(mL + ML - \frac{1}{4}Mx\right)}{2\left(mL + ML - \frac{1}{2}Mx\right)^2}.$$

Deduce that the acceleration of the particle after it is released is greater than $g$.

## Section C: Probability and Statistics

### [STEP 3, 2012Q12]

(i) A point $P$ lies in an equilateral triangle $ABC$ of height 1. The perpendicular distances from $P$ to the sides $AB$, $BC$ and $CA$ are $x_1$, $x_2$ and $x_3$, respectively. By considering the areas of triangles with one vertex at $P$, show that $x_1 + x_2 + x_3 = 1$.

Suppose now that $P$ is placed at random in the equilateral triangle (so that the probability of it lying in any given region of the triangle is proportional to the area of that region). The perpendicular distances from $P$ to the sides $AB$, $BC$ and $CA$ are random variables $X_1$, $X_2$ and $X_3$, respectively. In the case $X_1 = \min(X_1, X_2, X_3)$, give a sketch showing the region of the triangle in which $P$ lies.

Let $X = \min(X_1, X_2, X_3)$. Show that the probability density function for $X$ is given by

$$f(x) = \begin{cases} 6(1-3x) & 0 \le x \le \dfrac{1}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of $X$.

(ii) A point is chosen at random in a regular tetrahedron of height 1. Find the expected value of the distance from the point to the closest face.

[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ height and its centroid is a distance $\frac{1}{4} \times$ height from the base.]



### [STEP 3, 2012Q13]

(i) The random variable $Z$ has a Normal distribution with mean 0 and variance 1. Show that the expectation of $Z$ given that $a < Z < b$ is

$$\frac{\exp\left(-\frac{1}{2}a^2\right) - \exp\left(-\frac{1}{2}b^2\right)}{\sqrt{2\pi}\big(\Phi(b) - \Phi(a)\big)},$$

where $\Phi$ denotes the cumulative distribution function for $Z$.

(ii) The random variable $X$ has a Normal distribution with mean $\mu$ and variance $\sigma^2$. Show that

$$\mathrm{E}(X|X > 0) = \mu + \sigma \mathrm{E}\left(Z \middle| Z > -\frac{\mu}{\sigma}\right).$$

Hence, or otherwise, show that the expectation, $m$, of $|X|$ is given by

$$m = \mu\left(1 - 2\Phi\left(-\frac{\mu}{\sigma}\right)\right) + \sigma\sqrt{\frac{2}{\pi}}\exp\left(-\frac{\frac{1}{2}\mu^2}{\sigma^2}\right).$$

Obtain an expression for the variance of $|X|$ in terms of $\mu$, $\sigma$ and $m$.

# STEP 3 2013

STEP 3 2013

On-line Exam

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# Section A: Pure Mathematics

## [STEP 3, 2013Q1]

Given that $t = \tan\frac{1}{2}x$, show that $\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}(1+t^2)$ and $\sin x = \frac{2t}{1+t^2}$.

Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,\mathrm{d}x = \frac{2}{\sqrt{1-a^2}}\arctan\frac{\sqrt{1-a}}{\sqrt{1+a}} \qquad (0 < a < 1).$$

Let

$$I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x}{2+\sin x}\,\mathrm{d}x \qquad (n \geq 0).$$

By considering $I_{n+1} + 2I_n$, or otherwise, evaluate $I_3$.

## [STEP 3, 2013Q2]

In this question, you may ignore questions of convergence.

Let $y = \frac{\arcsin x}{\sqrt{1-x^2}}$. Show that

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1 = 0$$

and prove that, for any positive integer $n$,

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+3)x\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} - (n+1)^2\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0.$$

Hence obtain the Maclaurin series for $\frac{\arcsin x}{\sqrt{1-x^2}}$, giving the general term for odd and for even powers of $x$.

Evaluate the infinite sum

$$1 + \frac{1}{3!} + \frac{2^2}{5!} + \frac{2^2\times 3^2}{7!} + \cdots + \frac{2^2\times 3^2\times\cdots\times n^2}{(2n+1)!} + \cdots.$$

[STEP 3, 2013Q3]

The four vertices $P_i$ $(i = 1, 2, 3, 4)$ of a regular tetrahedron lie on the surface of a sphere with centre at $O$ and of radius 1. The position vector of $P_i$ with respect to $O$ is $\mathbf{p}_i (i = 1, 2, 3, 4)$. Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i . \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$.

Let $X$ be any point on the surface of the sphere, and let $XP_i$ denote the length of the line joining $X$ and $P_i$ $(i = 1, 2, 3, 4)$.

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}).(\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of $X$ with respect to $O$, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8.$$

(ii) Given that $P_1$ has coordinates $(0, 0, 1)$ and that the coordinates of $P_2$ are of the form $(a, 0, b)$, where $a > 0$, show that $a = \frac{2\sqrt{2}}{3}$ and $b = -\frac{1}{3}$, and find the coordinates of $P_3$ and $P_4$.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x}.\mathbf{p}_i)^2.$$

By letting the coordinates of $X$ be $(x, y, z)$, show further that $\sum_{i=1}^{4} (XP_i)^4$ is independent of the position of $X$.



[STEP 3, 2013Q4]

Show that $(z - \mathrm{e}^{\mathrm{i}\theta})(z - \mathrm{e}^{-\mathrm{i}\theta}) = z^2 - 2z\cos\theta + 1$.

Write down the $(2n)$th roots of $-1$ in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^{n} \left( z^2 - 2z\cos\left(\frac{(2k-1)\pi}{2n}\right) + 1 \right).$$

Here, $n$ is a positive integer, and the $\Pi$ notation denotes the product.

(i) By substituting $z = \mathrm{i}$ show that, when $n$ is even,

$$\cos\left(\frac{\pi}{2n}\right)\cos\left(\frac{3\pi}{2n}\right)\cos\left(\frac{5\pi}{2n}\right)\cdots\cos\left(\frac{(2n-1)\pi}{2n}\right) = (-1)^{\frac{1}{2}n}2^{1-n}.$$

(ii) Show that, when $n$ is odd,

$$\cos^2\left(\frac{\pi}{2n}\right)\cos^2\left(\frac{3\pi}{2n}\right)\cos^2\left(\frac{5\pi}{2n}\right)\cdots\cos^2\left(\frac{(n-2)\pi}{2n}\right) = n2^{1-n}.$$

You may use without proof the fact that $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when $n$ is odd.

## [STEP 3, 2013Q5]

In this question, you may assume that, if $a$, $b$ and $c$ are positive integers such that $a$ and $b$ are coprime and $a$ divides $bc$, then $a$ divides $c$. (Two positive integers are said to be *coprime* if their highest common factor is 1.)

(i) Suppose that there are positive integers $p$, $q$, $n$ and $N$ such that $p$ and $q$ are coprime and $q^n N = p^n$. Show that $N = kp^n$ for some positive integer $k$ and deduce the value of $q$.

Hence prove that, for any positive integers $n$ and $N$, $\sqrt[n]{N}$ is either a positive integer or irrational.

(ii) Suppose that there are positive integers $a$, $b$, $c$ and $d$ such that $a$ and $b$ are coprime and $c$ and $d$ are coprime, and $a^a d^b = b^a c^b$. Prove that $d^b = b^a$ and deduce that, if $p$ is a prime factor of $d$, then $p$ is also a prime factor of $b$.

If $p^m$ and $p^n$ are the highest powers of the prime number $p$ that divide $d$ and $b$, respectively, express $b$ in terms of $a$, $m$ and $n$ and hence show that $p^n \leq n$. Deduce the value of $b$. (You may assume that if $x > 0$ and $y \geq 2$ then $y^x > x$.)

Hence prove that, if $r$ is a positive rational number such that $r^r$ is rational, then $r$ is a positive integer.

## [STEP 3, 2013Q6]

Let $z$ and $w$ be complex numbers. Use a diagram to show that $|z - w| \leq |z| + |w|$.

For any complex numbers $z$ and $w$, $E$ is defined by

$$E = zw^* + z^*w + 2|zw|.$$

(i) Show that $|z - w|^2 = (|z| + |w|)^2 - E$, and deduce that $E$ is real and non-negative.

(ii) Show that $|1 - zw^*|^2 = (1 + |zw|)^2 - E$.

Hence show that, if both $|z| > 1$ and $|w| > 1$, then

$$\frac{|z - w|}{|1 - zw^*|} \leq \frac{|z| + |w|}{1 + |zw|}.$$

Does this inequality also hold if both $|z| < 1$ and $|w| < 1$?

[STEP 3, 2013Q7]

(i) Let $y(x)$ be a solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$ with $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$E(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{1}{2}y^4.$$

Show by differentiation that $E$ is constant and deduce that $|y(x)| \leq 1$ for all $x$.

(ii) Let $v(x)$ be a solution of the differential equation $\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + x\frac{\mathrm{d}v}{\mathrm{d}x} + \sinh v = 0$ with $v = \ln 3$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$E(x) = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + 2\cosh v.$$

Show that $\frac{\mathrm{d}E}{\mathrm{d}x} \leq 0$ for $x \geq 0$ and deduce that $\cosh v(x) \leq \frac{5}{3}$ for $x \geq 0$.

(iii) Let $w(x)$ be a solution of the differential equation

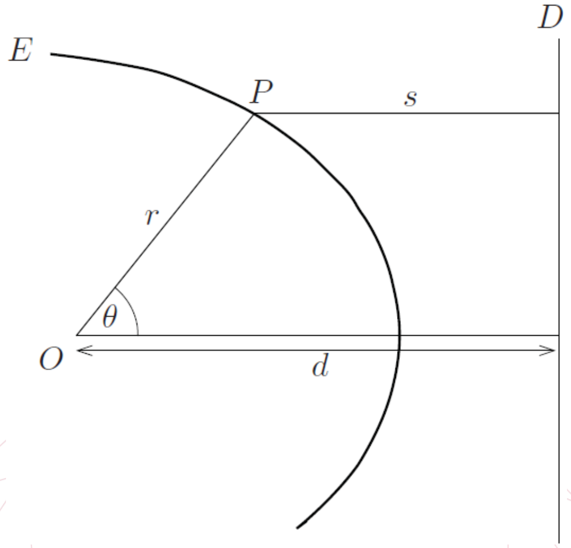
$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + (5\cosh x - 4\sinh x - 3)\frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) = 0$$

with $\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \leq \frac{5}{4}$ for $x \geq 0$.

[STEP 3, 2013Q8]

Evaluate $\sum_{r=0}^{n-1} \mathrm{e}^{2\mathrm{i}\left(\alpha+\frac{r\pi}{n}\right)}$ where $\alpha$ is a fixed angle and $n \geq 2$.

The fixed point $O$ is a distance $d$ from a fixed line $D$. For any point $P$, let $s$ be the distance from $P$ to $D$ and let $r$ be the distance from $P$ to $O$. Write down an expression for $s$ in terms of $d$, $r$ and the angle $\theta$, where $\theta$ is as shown in the diagram below.




The curve $E$ shown in the diagram is such that, for any point $P$ on $E$, the relation $r = ks$ holds, where $k$ is a fixed number with $0 < k < 1$.

Each of the $n$ lines $L_1, \ldots, L_n$ passes through $O$ and the angle between adjacent lines is $\frac{\pi}{n}$. The line $L_j$ $(j = 1, \ldots, n)$ intersects $E$ in two points forming a chord of length $l_j$. Show that, for $n \geq 2$,

$$\sum_{j=1}^{n} \frac{1}{l_j} = \frac{(2-k^2)n}{4kd}.$$

## Section B: Mechanics

### [STEP 3, 2013Q9]

A sphere of radius $R$ and uniform density $\rho_s$ is floating in a large tank of liquid of uniform density $\rho$. Given that the centre of the sphere is a distance $x$ above the level of the liquid, where $x < R$, show that the volume of liquid displaced is

$$\frac{\pi}{3}(2R^3 - 3R^2x + x^3).$$

The sphere is acted upon by two forces only: its weight and an upward force equal in magnitude to the weight of the liquid it has displaced. Show that

$$4R^3\rho_s(g + \ddot{x}) = (2R^3 - 3R^2x + x^3)\rho g.$$

Given that the sphere is in equilibrium when $x = \frac{1}{2}R$, find $\rho_s$ in terms of $\rho$. Find, in terms of $R$ and $g$, the period of small oscillations about this equilibrium position.

### [STEP 3, 2013Q10]

A uniform rod $AB$ has mass $M$ and length $2a$. The point $P$ lies on the rod a distance $a - x$ from $A$. Show that the moment of inertia of the rod about an axis through $P$ and perpendicular to the rod is

$$\frac{1}{3}M(a^2 + 3x^2).$$

The rod is free to rotate, in a horizontal plane, about a fixed vertical axis through $P$. Initially the rod is at rest. The end $B$ is struck by a particle of mass $m$ moving horizontally with speed $u$ in a direction perpendicular to the rod. The coefficient of restitution between the rod and the particle is $e$. Show that the angular velocity of the rod immediately after impact is

$$\frac{3mu(1 + e)(a + x)}{M(a^2 + 3x^2) + 3m(a + x)^2}.$$

In the case $m = 2M$, find the value of $x$ for which the angular velocity is greatest and show that this angular velocity is $\frac{u(1+e)}{a}$.

[STEP 3, 2013Q11]

An equilateral triangle, comprising three light rods each of length $\sqrt{3}a$, has a particle of mass $m$ attached to each of its vertices. The triangle is suspended horizontally from a point vertically above its centre by three identical springs, so that the springs and rods form a tetrahedron. Each spring has natural length $a$ and modulus of elasticity $kmg$, and is light. Show that when the springs make an angle $\theta$ with the horizontal the tension in each spring is

$$\frac{kmg(1-\cos\theta)}{\cos\theta}.$$

Given that the triangle is in equilibrium when $\theta = \frac{1}{6}\pi$, show that $k = 4\sqrt{3} + 6$.

The triangle is released from rest from the position at which $\theta = \frac{1}{3}\pi$. Show that when it passes through the equilibrium position its speed $V$ satisfies

$$V^2 = \frac{4ag}{3}(6+\sqrt{3}).$$

## Section C: Probability and Statistics

### [STEP 3, 2013Q12]

A list consists only of letters $A$ and $B$ arranged in a row. In the list, there are $a$ letter $A$s and $b$ letter $B$s, where $a \geq 2$ and $b \geq 2$, and $a + b = n$. Each possible ordering of the letters is equally probable. The random variable $X_1$ is defined by

$$X_1 = \begin{cases} 1 & \text{if the first letter in the row is } A; \\ 0 & \text{otherwise.} \end{cases}$$

The random variables $X_k$ $(2 \leq k \leq n)$ are defined by

$$X_k = \begin{cases} 1 & \text{if the } (k-1)\text{th letter is } B \text{ and the } k\text{th is A;} \\ 0 & \text{otherwise.} \end{cases}$$

The random variable $S$ is defined by $S = \sum_{i=1}^{n} X_i$.

(i) Find expressions for $\mathrm{E}(X_i)$, distinguishing between the cases $i = 1$ and $i \neq 1$, and show that $\mathrm{E}(S) = \frac{a(b+1)}{n}$.

(ii) Show that:

(a) for $j \geq 3$, $\mathrm{E}(X_1 X_j) = \frac{a(a-1)b}{n(n-1)(n-2)}$.

(b) $\sum_{i=2}^{n-2}\left(\sum_{j=i+2}^{n} \mathrm{E}(X_i X_j)\right) = \frac{a(a-1)b(b-1)}{2n(n-1)}$.

(c) $\mathrm{Var}(S) = \frac{a(a-1)b(b+1)}{n^2(n-1)}$.

[STEP 3, 2013Q13]

(i) The continuous random variable $X$ satisfies $0 \le X \le 1$, and has probability density function $f(x)$ and cumulative distribution function $F(x)$. The greatest value of $f(x)$ is $M$, so that $0 \le f(x) \le M$.

(a) Show that $0 \le F(x) \le Mx$ for $0 \le x \le 1$.

(b) For any function $g(x)$, show that

$$\int_0^1 2g(x)F(x)f(x)\,\mathrm{d}x = g(1) - \int_0^1 g'(x)\big(F(x)\big)^2\,\mathrm{d}x.$$

(ii) The continuous random variable $Y$ satisfies $0 \le Y \le 1$, and has probability density function $kF(y)f(y)$, where $f$ and $F$ are as above.

(a) Determine the value of the constant $k$.

(b) Show that

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \le \mathrm{E}(Y^n) \le 2M\mu_{n+1},$$

where $\mu_{n+1} = \mathrm{E}(X^{n+1})$ and $n \ge 0$.

(c) Hence show that, for $n \ge 1$,

$$\mu_n \ge \frac{n}{(n+1)M} - \frac{n-1}{n+1}.$$

# STEP 3 2014

STEP 3 2014

On-line Exam

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**TIME ALLOWED: 180 MINUTES**

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## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

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|---|---|
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| **Section B** | Mechanics |
| **Section C** | Probability and Statistics |

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Each question is marked out of 20. There is no restriction of choice.

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You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 2014Q1]

Let $a$, $b$ and $c$ be real numbers such that $a + b + c = 0$ and let

$$(1 + ax)(1 + bx)(1 + cx) = 1 + qx^2 + rx^3$$

for all real $x$. Show that $q = bc + ca + ab$ and $r = abc$.

(i) Show that the coefficient of $x^n$ in the series expansion (in ascending powers of $x$) of $\ln(1 + qx^2 + rx^3)$ is $(-1)^{n+1}S_n$ where

$$S_n = \frac{a^n + b^n + c^n}{n}, \qquad (n \geq 1).$$

(ii) Find, in terms of $q$ and $r$, the coefficients of $x^2$, $x^3$ and $x^5$ in the series expansion (in ascending powers of $x$) of $\ln(1 + qx^2 + rx^3)$ and hence show that $S_2 S_3 = S_5$.

(iii) Show that $S_2 S_5 = S_7$.

(iv) Give a proof of, or find a counterexample to, the claim that $S_2 S_7 = S_9$.

## [STEP 3, 2014Q2]

(i) Show, by means of the substitution $u = \cosh x$, that

$$\int \frac{\sinh x}{\cosh 2x} \mathrm{d}x = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| + C.$$

(ii) Use a similar substitution to find an expression for

$$\int \frac{\cosh x}{\cosh 2x} \mathrm{d}x.$$

(iii) Using parts (i) and (ii) above, show that

$$\int_0^1 \frac{1}{1 + u^4} \mathrm{d}u = \frac{\pi + 2\ln(\sqrt{2} + 1)}{4\sqrt{2}}.$$

[STEP 3, 2014Q3]

(i) The line $L$ has equation $y = mx + c$, where $m > 0$ and $c > 0$. Show that, in the case $mc > a > 0$, the shortest distance between $L$ and the parabola $y^2 = 4ax$ is

$$\frac{mc - a}{m\sqrt{m^2 + 1}}.$$

What is the shortest distance in the case that $mc \le a$?

(ii) Find the shortest distance between the point $(p, 0)$, where $p > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the value of $\frac{p}{a}$. [*You may wish to use the parametric coordinates* $(at^2, 2at)$ *of points on the parabola.*]

Hence find the shortest distance between the circle $(x - p)^2 + y^2 = b^2$, where $p > 0$ and $b > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the values of $p$, $a$ and $b$.

[STEP 3, 2014Q4]

(i) Let

$$I = \int_0^1 \left((y')^2 - y^2\right) \mathrm{d}x \qquad \text{and} \qquad I_1 = \int_0^1 (y' + y\tan x)^2 \,\mathrm{d}x,$$

where $y$ is a given function of $x$ satisfying $y = 0$ at $x = 1$. Show that $I - I_1 = 0$ and deduce that $I \ge 0$. Show further that $I = 0$ only if $y = 0$ for all $x$ $(0 \le x \le 1)$.



(ii) Let

$$J = \int_0^1 \left((y')^2 - a^2 y^2\right) \mathrm{d}x,$$

where $a$ is a given positive constant and $y$ is a given function of $x$, not identically zero satisfying $y = 0$ at $x = 1$. By considering an integral of the form

$$\int_0^1 (y' + ay\tan bx)^2 \,\mathrm{d}x,$$



where $b$ is suitably chosen, show that $J \ge 0$. You should state the range of values of $a$, in the form $a < k$, for which your proof is valid.

In the case $a = k$, find a function $y$ (not everywhere zero) such that $J = 0$.

## [STEP 3, 2014Q5]

A quadrilateral drawn in the complex plane has vertices $A$, $B$, $C$ and $D$, labelled anticlockwise. These vertices are represented, respectively, by the complex numbers $a$, $b$, $c$ and $d$. Show that $ABCD$ is a parallelogram (defined as a quadrilateral in which opposite sides are parallel and equal in length) if and only if $a + c = b + d$. Show further that, in this case, $ABCD$ is a square if and only if $\mathbf{i}(a - c) = b - d$.

Let $PQRS$ be a quadrilateral in the complex plane, with vertices labelled anticlockwise, the internal angles of which are all less than 180°. Squares with centres $X$, $Y$, $Z$ and $T$ are constructed externally to the quadrilateral on the sides $PQ$, $QR$, $RS$ and $SP$, respectively.

(i) If $P$ and $Q$ are represented by the complex numbers $p$ and $q$, respectively, show that $X$ can be represented by

$$\frac{1}{2}\big(p(1 + \mathbf{i}) + q(1 - \mathbf{i})\big).$$

(ii) Show that $XYZT$ is a square if and only if $PQRS$ is a parallelogram.

## [STEP 3, 2014Q6]

Starting from the result that

$$h(t) > 0 \text{ for } 0 < t < x \Longrightarrow \int_0^x h(t)\,\mathrm{d}t > 0,$$

show that, if $f''(t) > 0$ for $0 < t < x_0$ and $f(0) = f'(0) = 0$, then $f(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x \cosh x < 1.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}.$$

[STEP 3, 2014Q7]

The four distinct points $P_i$ $(i = 1, 2, 3, 4)$ are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines $P_1P_3$ and $P_2P_4$ intersect at $Q$.

(i) By considering the triangles $P_1QP_4$ and $P_2QP_3$ show that $(P_1Q)(QP_3) = (P_2Q)(QP_4)$.

(ii) Let $\mathbf{p}_i$ be the position vector of the point $P_i$ $(i = 1, 2, 3, 4)$. Show that there exist numbers $a_i$, not all zero, such that

$$\sum_{i=1}^{4} a_i = 0 \quad \text{and} \quad \sum_{i=1}^{4} a_i \mathbf{p}_i = \mathbf{0}. \qquad (*)$$

(iii) Let $a_i$ $(i = 1, 2, 3, 4)$ be any numbers, not all zero, that satisfy $(*)$. Show that $a_1 + a_3 \neq 0$ and that the lines $P_1P_3$ and $P_2P_4$ intersect at the point with position vector

$$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}.$$

Deduce that $a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$.

[STEP 3, 2014Q8]

The numbers $f(r)$ satisfy $f(r) > f(r + 1)$ for $r = 1, 2, \ldots$. Show that, for any non-negative integer $n$,

$$k^n(k-1)f(k^{n+1}) \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k^n(k-1)f(k^n)$$



where $k$ is an integer greater than 1.

(i) By taking $f(r) = \frac{1}{r}$, show that

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N + 1.$$

Deduce that the sum $\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge.



(ii) By taking $f(r) = \frac{1}{r^3}$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq 1\frac{1}{3}.$$

(iii) Let $S(n)$ be the set of positive integers less than $n$ which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all $n$.

# Section B: Mechanics

## [STEP 3, 2014Q9]

A particle of mass $m$ is projected with velocity $\mathbf{u}$. It is acted upon by the force $m\mathbf{g}$ due to gravity and by a resistive force $-mk\mathbf{v}$, where $\mathbf{v}$ is its velocity and $k$ is a positive constant.

Given that, at time $t$ after projection, its position $\mathbf{r}$ relative to the point of projection is given by

$$\mathbf{r} = \frac{kt - 1 + \mathrm{e}^{-kt}}{k^2}\mathbf{g} + \frac{1 - \mathrm{e}^{-kt}}{k}\mathbf{u},$$

find an expression for $\mathbf{v}$ in terms of $k$, $t$, $\mathbf{g}$ and $\mathbf{u}$. Verify that the equation of motion and the initial conditions are satisfied.

Let $\mathbf{u} = u\cos\alpha\,\hat{\mathbf{i}} + u\sin\alpha\,\hat{\mathbf{j}}$ and $\mathbf{g} = -g\hat{\mathbf{j}}$, where $0 < \alpha < 90°$, and let $T$ be the time after projection at which $\mathbf{r}.\hat{\mathbf{j}} = 0$. Show that

$$uk\sin\alpha = \left(\frac{kT}{1 - \mathrm{e}^{-kT}} - 1\right)g.$$

Let $\beta$ be the acute angle between $\mathbf{v}$ and $\hat{\mathbf{i}}$ at time $T$. Show that

$$\tan\beta = \frac{(\mathrm{e}^{kT} - 1)g}{uk\cos\alpha} - \tan\alpha.$$

Show further that $\tan\beta > \tan\alpha$ (you may assume that $\sinh kT > kT$) and deduce that $\beta > \alpha$.

## [STEP 3, 2014Q10]



Two particles $X$ and $Y$, of equal mass $m$, lie on a smooth horizontal table and are connected by a light elastic spring of natural length $a$ and modulus of elasticity $\lambda$. Two more springs, identical to the first, connect $X$ to a point $P$ on the table and $Y$ to a point $Q$ on the table. The distance between $P$ and $Q$ is $3a$.

Initially, the particles are held so that $XP = a$, $YQ = \frac{1}{2}a$, and $PXYQ$ is a straight line. The particles are then released.

At time $t$, the particle $X$ is a distance $a + x$ from $P$ and the particle $Y$ is a distance $a + y$ from $Q$. Show that

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\lambda}{a}(2x + y)$$

and find a similar expression involving $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$. Deduce that

$$x - y = A\cos\omega t + B\sin\omega t$$

where $A$ and $B$ are constants to be determined and $ma\omega^2 = \lambda$. Find a similar expression for $x + y$.

Show that $Y$ will never return to its initial position.

[STEP 3, 2014Q11]

A particle $P$ of mass $m$ is connected by two light inextensible strings to two fixed points $A$ and $B$, with $A$ vertically above $B$. The string $AP$ has length $x$. The particle is rotating about the vertical through $A$ and $B$ with angular velocity $\omega$, and both strings are taut. Angles $PAB$ and $PBA$ are $\alpha$ and $\beta$, respectively.

Find the tensions $T_A$ and $T_B$ in the strings $AP$ and $BP$ (respectively), and hence show that $\omega^2 x \cos\alpha \geq g$.

Consider now the case that $\omega^2 x \cos\alpha = g$. Given that $AB = h$ and $BP = d$, where $h > d$, show that $h\cos\alpha \geq \sqrt{h^2 - d^2}$. Show further that

$$mg < T_A \leq \frac{mgh}{\sqrt{h^2 - d^2}}.$$

Describe the geometry of the strings when $T_A$ attains its upper bound.

## Section C: Probability and Statistics

### [STEP 3, 2014Q12]

The random variable $X$ has probability density function $f(x)$ (which you may assume is differentiable) and cumulative distribution function $F(x)$ where $-\infty < x < \infty$. The random variable $Y$ is defined by $Y = \mathrm{e}^X$. You may assume throughout this question that $X$ and $Y$ have unique modes.

(i) Find the median value $y_m$ of $Y$ in terms of the median value $x_m$ of $X$.

(ii) Show that the probability density function of $Y$ is $\frac{f(\ln y)}{y}$, and deduce that the mode $\lambda$ of $Y$ satisfies $f'(\ln \lambda) = f(\ln \lambda)$.

(iii) Suppose now that $X \sim \mathrm{N}(\mu, \sigma^2)$, so that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Explain why

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-\mu-\sigma^2)^2}{2\sigma^2}} \mathrm{d}x = 1$$

and hence show that $\mathrm{E}(Y) = \mathrm{e}^{\mu + \frac{1}{2}\sigma^2}$.

(iv) Show that, when $X \sim \mathrm{N}(\mu, \sigma^2)$,

$$\lambda < y_m < \mathrm{E}(Y).$$

[STEP 3, 2014Q13]

I play a game which has repeated rounds. Before the first round, my score is 0. Each round can have three outcomes:

1. my score is unchanged and the game ends.
2. my score is unchanged and I continue to the next round.
3. my score is increased by one and I continue to the next round.

The probabilities of these outcomes are $a$, $b$ and $c$, respectively (the same in each round), where $a + b + c = 1$ and $abc \neq 0$. The random variable $N$ represents my score at the end of a randomly chosen game.

Let $G(t)$ be the probability generating function of $N$.

(i) Suppose in the first round, the game ends. Show that the probability generating function conditional on this happening is 1.

(ii) Suppose in the first round, the game continues to the next round with no change in score. Show that the probability generating function conditional on this happening is $G(t)$.

(iii) By comparing the coefficients of $t^n$, show that $G(t) = a + bG(t) + ctG(t)$. Deduce that, for $n \geq 0$,

$$\mathrm{P}(N = n) = \frac{ac^n}{(1-b)^{n+1}}.$$

(iv) Show further that, for $n \geq 0$,

$$\mathrm{P}(N = n) = \frac{\mu^n}{(1+\mu)^{n+1}},$$

where $\mu = \mathrm{E}(N)$.

# STEP 3 2015

STEP 3 2015
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You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 2015Q1]

(i) Let

$$I_n = \int_0^\infty \frac{1}{(1+u^2)^n}\,\mathrm{d}u,$$

where $n$ is a positive integer. Show that

$$I_n - I_{n+1} = \frac{1}{2n} I_n$$

and deduce that

$$I_{n+1} = \frac{(2n)!\,\pi}{2^{2n+1}(n!)^2}.$$

(ii) Let

$$J = \int_0^\infty f\big((x - x^{-1})^2\big)\,\mathrm{d}x,$$

where $f$ is any function for which the integral exists. Show that

$$J = \int_0^\infty x^{-2} f\big((x-x^{-1})^2\big)\,\mathrm{d}x = \frac{1}{2}\int_0^\infty (1+x^{-2}) f\big((x-x^{-1})^2\big)\,\mathrm{d}x = \int_0^\infty f(u^2)\,\mathrm{d}u.$$

(iii) Hence evaluate

$$\int_0^\infty \frac{x^{2n-2}}{(x^4 - x^2 + 1)^n}\,\mathrm{d}x,$$

where $n$ is a positive integer.



## [STEP 3, 2015Q2]

If $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ are sequences of positive numbers, we write

$$(s_n) \le (t_n)$$

to mean

"there exists a positive integer $m$ such that $s_n \le t_n$ whenever $n \ge m$".

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate $m$; in the case of a false statement, you should give a counterexample.

(i) $(1000n) \le (n^2)$.

(ii) If it is not the case that $(s_n) \le (t_n)$, then it is the case that $(t_n) \le (s_n)$.

(iii) If $(s_n) \le (t_n)$ and $(t_n) \le (u_n)$, then $(s_n) \le (u_n)$.

(iv) $(n^2) \le (2^n)$.

[STEP 3, 2015Q3]

In this question, $r$ and $\theta$ are polar coordinates with $r \geq 0$ and $-\pi < \theta \leq \pi$, and $a$ and $b$ are positive constants.

Let $L$ be a fixed line and let $A$ be a fixed point not lying on $L$. Then the locus of points that are a fixed distance (call it $d$) from $L$ measured along lines through $A$ is called a *conchoid of Nicomedes*.

(i) Show that if

$$|r - a\sec\theta| = b, \qquad (*)$$

where $a > b$, then $\sec\theta > 0$. Show that all points with coordinates satisfying $(*)$ lie on a certain conchoid of Nicomedes (you should identify $L$, $d$ and $A$). Sketch the locus of these points.

(ii) In the case $a < b$, sketch the curve (including the loop for which $\sec\theta < 0$) given by

$$|r - a\sec\theta| = b.$$

Find the area of the loop in the case $a = 1$ and $b = 2$.

[**Note:** $\int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.]

[STEP 3, 2015Q4]

(i) If $a$, $b$ and $c$ are all real, show that the equation

$$z^3 + az^2 + bz + c = 0 \quad (*)$$

has at least one real root.

(ii) Let

$$S_1 = z_1 + z_2 + z_3, \qquad S_2 = z_1^2 + z_2^2 + z_3^2, \qquad S_3 = z_1^3 + z_2^3 + z_3^3,$$

where $z_1$, $z_2$ and $z_3$ are the roots of the equation $(*)$. Express $a$ and $b$ in terms of $S_1$ and $S_2$, and show that

$$6c = -S_1^3 + 3S_1S_2 - 2S_3.$$

(iii) The six real numbers $r_k$ and $\theta_k$ $(k = 1, 2, 3)$, where $r_k > 0$ and $-\pi < \theta_k < \pi$, satisfy

$$\sum_{k=1}^{3} r_k \sin(\theta_k) = 0, \qquad \sum_{k=1}^{3} r_k^2 \sin(2\theta_k) = 0, \qquad \sum_{k=1}^{3} r_k^3 \sin(3\theta_k) = 0.$$

Show that $\theta_k = 0$ for at least one value of $k$.

Show further that if $\theta_1 = 0$ then $\theta_2 = -\theta_3$.

[STEP 3, 2015Q5]

(i) In the following argument to show that $\sqrt{2}$ is irrational, give proofs appropriate for steps 3, 5 and 6.

1. Assume that $\sqrt{2}$ is rational.
2. Define the set $S$ to be the set of positive integers with the following property:

$$n \text{ is in } S \text{ if and only if } n\sqrt{2} \text{ is an integer.}$$

3. Show that the set $S$ contains at least one positive integer.
4. Define the integer $k$ to be the smallest positive integer in $S$.
5. Show that $(\sqrt{2}-1)k$ is in $S$.
6. Show that steps 4 and 5 are contradictory and hence that $\sqrt{2}$ is irrational.

(ii) Prove that $2^{\frac{1}{3}}$ is rational if and only if $2^{\frac{2}{3}}$ is rational.

Use an argument similar to that of part (i) to prove that $2^{\frac{1}{3}}$ and $2^{\frac{2}{3}}$ are irrational.

[STEP 3, 2015Q6]

(i) Let $w$ and $z$ be complex numbers, and let $u = w + z$ and $v = w^2 + z^2$. Prove that $w$ and $z$ are real if and only if $u$ and $v$ are real and $u^2 \leq 2v$.

(ii) The complex numbers $u$, $w$ and $z$ satisfy the equations

$$w + z - u = 0$$
$$w^2 + z^2 - u^2 = -\frac{2}{3}$$
$$w^3 + z^3 - \lambda u = -\lambda$$

where $\lambda$ is a positive real number. Show that for all values of $\lambda$ except one (which you should find) there are three possible values of $u$, all real.

Are $w$ and $z$ necessarily real? Give a proof or counterexample.

[STEP 3, 2015Q7]

An operator D is defined, for any function $f$, by

$$\mathrm{D}f(x) = x\frac{\mathrm{d}f(x)}{\mathrm{d}x}.$$

The notation $\mathrm{D}^n$ means that D is applied $n$ times; for example

$$\mathrm{D}^2 f(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right).$$

Show that, for any constant $a$, $\mathrm{D}^2 x^a = a^2 x^a$.

(i) Show that if $P(x)$ is a polynomial of degree $r$ (where $r \geq 1$) then, for any positive integer $n$, $\mathrm{D}^n P(x)$ is also a polynomial of degree $r$.

(ii) Show that if $n$ and $m$ are positive integers with $n < m$, then $\mathrm{D}^n(1-x)^m$ is divisible by $(1-x)^{m-n}$.

(iii) Deduce that, if $m$ and $n$ are positive integers with $n < m$, then

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n = 0.$$

[STEP 3, 2015Q8]

(i) Show that under the changes of variable $x = r\cos\theta$ and $y = r\sin\theta$, where $r$ is a function of $\theta$ with $r > 0$, the differential equation

$$(y + x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$$

becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r = 0.$$

Sketch a solution in the $x$-$y$ plane.

(ii) Show that the solutions of

$$\big(y + x - x(x^2 + y^2)\big)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x - y(x^2 + y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + A\mathrm{e}^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of $A$.

## Section B: Mechanics

### [STEP 3, 2015Q9]

A particle $P$ of mass $m$ moves on a smooth fixed straight horizontal rail and is attached to a fixed peg $Q$ by a light elastic string of natural length $a$ and modulus $\lambda$. The peg $Q$ is a distance $a$ from the rail. Initially $P$ is at rest with $PQ = a$.

An impulse imparts to $P$ a speed $v$ along the rail. Let $x$ be the displacement at time $t$ of $P$ from its initial position. Obtain the equation

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2$$

where $k^2 = \frac{\lambda}{ma}$, $k > 0$ and the dot denotes differentiation with respect to $t$.

Find, in terms of $k$, $a$ and $v$, the greatest value, $x_0$, attained by $x$. Find also the acceleration of $P$ at $x = x_0$.

Obtain, in the form of an integral, an expression for the period of the motion. Show that, in the case $v \ll ka$ (that is, $v$ is much less than $ka$), this is approximately

$$\sqrt{\frac{32a}{kv}}\int_0^1 \frac{1}{\sqrt{1-u^4}}\,\mathrm{d}u.$$

### [STEP 3, 2015Q10]

A light rod of length $2a$ has a particle of mass $m$ attached to each end and it moves in a vertical plane. The midpoint of the rod has coordinates $(x, y)$, where the $x$-axis is horizontal (within the plane of motion) and $y$ is the height above a horizontal table. Initially, the rod is vertical, and at time $t$ later it is inclined at an angle $\theta$ to the vertical.

Show that the velocity of one particle can be written in the form

$$\begin{pmatrix} \dot{x} + a\dot{\theta}\cos\theta \\ \dot{y} - a\dot{\theta}\sin\theta \end{pmatrix}$$

and that

$$m\begin{pmatrix} \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where the dots denote differentiation with respect to time $t$ and $T$ is the tension in the rod. Obtain the corresponding equations for the other particle.

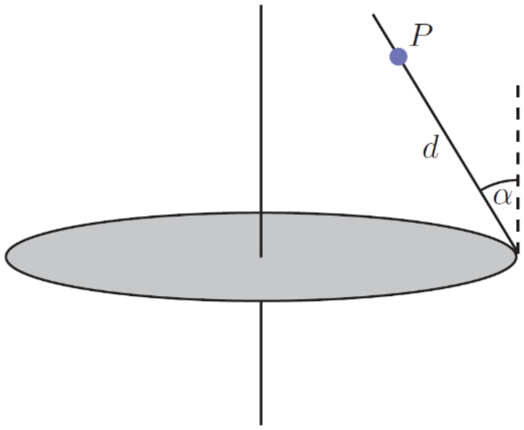
Deduce that $\ddot{x} = 0$, $\ddot{y} = -g$ and $\ddot{\theta} = 0$.

Initially, the midpoint of the rod is a height $h$ above the table, the velocity of the higher particle is $\binom{u}{v}$, and the velocity of the lower particle is $\binom{0}{v}$. Given that the two particles hit the table for the first time simultaneously, when the rod has rotated by $\frac{1}{2}\pi$, show that

$$2hu^2 = \pi^2 a^2 g - 2\pi uva.$$

[STEP 3, 2015Q11]

(i) A horizontal disc of radius $r$ rotates about a vertical axis through its centre with angular speed $\omega$. One end of a light rod is fixed by a smooth hinge to the edge of the disc so that it can rotate freely in a vertical plane through the centre of the disc. A particle $P$ of mass $m$ is attached to the rod at a distance $d$ from the hinge. The rod makes a constant angle $\alpha$ with the upward vertical, as shown in the diagram, and $d \sin\alpha < r$.




By considering moments about the hinge for the (light) rod, show that the force exerted on the rod by $P$ is parallel to the rod.

Show also that

$$r \cot\alpha = a + d\cos\alpha,$$

where $a = \frac{g}{\omega^2}$. State clearly the direction of the force exerted by the hinge on the rod, and find an expression for its magnitude in terms of $m$, $g$ and $\alpha$.

(ii) The disc and rod rotate as in part (i), but two particles (instead of $P$) are attached to the rod. The masses of the particles are $m_1$ and $m_2$ and they are attached to the rod at distances $d_1$ and $d_2$ from the hinge, respectively. The rod makes a constant angle $\beta$ with the upward vertical and $d_1 \sin\beta < d_2 \sin\beta < r$. Show that $\beta$ satisfies an equation of the form

$$r \cot\beta = a + b\cos\beta,$$

where $b$ should be expressed in terms of $d_1$, $d_2$, $m_1$ and $m_2$.

## Section C: Probability and Statistics

### [STEP 3, 2015Q12]

A 6-sided fair die has the numbers 1, 2, 3, 4, 5, 6 on its faces. The die is thrown $n$ times, the outcome (the number on the top face) of each throw being independent of the outcome of any other throw. The random variable $S_n$ is the sum of the outcomes.

(i) The random variable $R_n$ is the remainder when $S_n$ is divided by 6. Write down the probability generating function, $G(x)$, of $R_1$ and show that the probability generating function of $R_2$ is also $G(x)$. Use a generating function to find the probability that $S_n$ is divisible by 6.

(ii) The random variable $T_n$ is the remainder when $S_n$ is divided by 5. Write down the probability generating function, $G_1(x)$, of $T_1$ and show that $G_2(x)$, the probability generating function of $T_2$, is given by

$$G_2(x) = \frac{1}{36}(x^2 + 7y)$$

where $y = 1 + x + x^2 + x^3 + x^4$.

Obtain the probability generating function of $T_n$ and hence show that the probability that $S_n$ is divisible by 5 is

$$\frac{1}{5}\left(1 - \frac{1}{6^n}\right)$$

if $n$ is not divisible by 5. What is the corresponding probability if $n$ is divisible by 5?



### [STEP 3, 2015Q13]

Each of the two independent random variables $X$ and $Y$ is uniformly distributed on the interval $[0, 1]$.

(i) By considering the lines $x + y =$ constant in the $x$ - $y$ plane, find the cumulative distribution function of $X + Y$.

Hence show that the probability density function $f$ of $(X + Y)^{-1}$ is given by

$$f(t) = \begin{cases} 2t^{-2} - t^{-3} & \text{for } \frac{1}{2} \le t \le 1 \\ t^{-3} & \text{for } 1 \le t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate $\mathrm{E}\left(\frac{1}{X+Y}\right)$.

(ii) Find the cumulative distribution function of $\frac{Y}{X}$ and use this result to find the probability density function of $\frac{X}{X+Y}$.

Write down $\mathrm{E}\left(\frac{X}{X+Y}\right)$ and verify your result by integration.

# STEP 3 2016

STEP 3 2016
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

| | |
|---|---|
| **Section A** | Pure Mathematics |
| **Section B** | Mechanics |
| **Section C** | Probability and Statistics |

There are 13 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

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**Calculators are not permitted.**

## Section A: Pure Mathematics

### [STEP 3, 2016Q1]

Let

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2ax + b)^n} \, \mathrm{d}x,$$

where $a$ and $b$ are constants with $b > a^2$, and $n$ is a positive integer.

(i) By using the substitution $x + a = \sqrt{b - a^2} \tan u$, or otherwise, show that

$$I_1 = \frac{\pi}{\sqrt{b - a^2}}.$$

(ii) Show that $2n(b - a^2)I_{n+1} = (2n - 1)I_n$.

(iii) Hence prove by induction that

$$I_n = \frac{\pi}{2^{2n-2}(b - a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.$$

### [STEP 3, 2016Q2]

The distinct points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ and $R(ar^2, 2ar)$ lie on the parabola $y^2 = 4ax$, where $a > 0$. The points are such that the normal to the parabola at $Q$ and the normal to the parabola at $R$ both pass through $P$.



(i) Show that $q^2 + qp + 2 = 0$.

(ii) Show that $QR$ passes through a certain point that is independent of the choice of $P$.

(iii) Let $T$ be the point of intersection of $OP$ and $QR$, where $O$ is the coordinate origin. Show that $T$ lies on a line that is independent of the choice of $P$.

Show further that the distance from the $x$-axis to $T$ is less than $\frac{a}{\sqrt{2}}$.

[STEP 3, 2016Q3]

(i) Given that

$$\int \frac{x^3 - 2}{(x+1)^2} \mathrm{e}^x \, \mathrm{d}x = \frac{P(x)}{Q(x)} \mathrm{e}^x + \text{constant},$$

where $P(x)$ and $Q(x)$ are polynomials, show that $Q(x)$ has a factor of $x + 1$.

Show also that the degree of $P(x)$ is exactly one more than the degree of $Q(x)$, and find $P(x)$ in the case $Q(x) = x + 1$.

(ii) Show that there are no polynomials $P(x)$ and $Q(x)$ such that

$$\int \frac{1}{x+1} \mathrm{e}^x \, \mathrm{d}x = \frac{P(x)}{Q(x)} \mathrm{e}^x + \text{constant}.$$

You need consider only the case when $P(x)$ and $Q(x)$ have no common factors.

[STEP 3, 2016Q4]

(i) By considering $\frac{1}{1+x^r} - \frac{1}{1+x^{r+1}}$ for $|x| \neq 1$, simplify

$$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})}.$$

Show that, for $|x| < 1$,

$$\sum_{r=1}^{\infty} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{x}{1-x^2}.$$

(ii) Deduce that

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}\big((r+1)y\big) = 2\mathrm{e}^{-y} \operatorname{cosech}(2y)$$

for $y > 0$.

Hence simplify

$$\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry) \operatorname{sech}\big((r+1)y\big),$$

for $y > 0$.

[STEP 3, 2016Q5]

(i) By considering the binomial expansion of $(1+x)^{2m+1}$, prove that

$$\binom{2m+1}{m} < 2^{2m},$$

for any positive integer $m$.

(ii) For any positive integers $r$ and $s$ with $r < s$, $P_{r,s}$ is defined as follows: $P_{r,s}$ is the product of all the prime numbers greater than $r$ and less than or equal to $s$, if there are any such primes numbers; if there are no such primes numbers, then $P_{r,s} = 1$.

For example, $P_{3,7} = 35$, $P_{7,10} = 1$ and $P_{14,18} = 17$.

Show that, for any positive integer $m$, $P_{m+1,2m+1}$ divides $\binom{2m+1}{m}$, and deduce that

$$P_{m+1,2m+1} < 2^{2m}.$$

(iii) Show that, if $P_{1,k} < 4^k$ for $k = 2, 3, \dots, 2m$, then $P_{1,2m+1} < 4^{2m+1}$.

(iv) Prove that $P_{1,n} < 4^n$ for $n \geq 2$.

[STEP 3, 2016Q6]

Show, by finding $R$ and $\gamma$, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x+\gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of $B$.

Two curves have equations $y = \operatorname{sech} x$ and $y = a\tanh x + b$, where $a > 0$.

(i) In the case $b > a$, show that if the curves intersect then the $x$-coordinates of the points of intersection can be written in the form

$$\pm\operatorname{arcosh}\left(\frac{1}{\sqrt{b^2-a^2}}\right) - \operatorname{artanh}\frac{a}{b}.$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on $a$ and $b$ for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on $a$ and $b$ for the curves to touch and, given that they touch, express the $y$-coordinate of the point of contact in terms of $a$.

[STEP 3, 2016Q7]

Let $\omega = e^{\frac{2\pi i}{n}}$, where $n$ is a positive integer. Show that, for any complex number $z$,

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1.$$

The points $X_0, X_1, \ldots, X_{n-1}$ lie on a circle with centre $O$ and radius 1, and are the vertices of a regular polygon.

(i) The point $P$ is equidistant from $X_0$ and $X_1$. Show that, if $n$ is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1,$$

where $|PX_k|$ denotes the distance from $P$ to $X_k$.

Give the corresponding result when $n$ is odd. (There are two cases to consider.)

(ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n.$$

[STEP 3, 2016Q8]

(i) The function $f$ satisfies, for all $x$, the equation

$$f(x) + (1-x)f(-x) = x^2.$$

Show that $f(-x) + (1+x)f(x) = x^2$. Hence find $f(x)$ in terms of $x$. You should verify that your function satisfies the original equation.

(ii) The function $K$ is defined, for $x \neq 1$, by

$$K(x) = \frac{x+1}{x-1}.$$

Show that, for $x \neq 1$, $K\big(K(x)\big) = x$.

The function $g$ satisfies the equation

$$g(x) + xg\left(\frac{x+1}{x-1}\right) = x \qquad (x \neq 1).$$

Show that, for $x \neq 1$, $g(x) = \frac{2x}{x^2+1}$.

(iii) Find $h(x)$, for $x \neq 0, x \neq 1$, given that

$$h(x) + h\left(\frac{1}{1-x}\right) = 1 - x - \frac{1}{1-x} \qquad (x \neq 0, x \neq 1).$$

## Section B: Mechanics

### [STEP 3, 2016Q9]

Three pegs $P$, $Q$ and $R$ are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side $2a$. A particle $X$ of mass $m$ lies on the table. It is attached to the pegs by three springs, $PX$, $QX$ and $RX$, each of modulus of elasticity $\lambda$ and natural length $l$, where $l < \frac{2}{\sqrt{3}}a$. Initially the particle is in equilibrium. Show that the extension in each spring is $\frac{2}{\sqrt{3}}a - l$.

The particle is then pulled a small distance directly towards $P$ and released. Show that the tension $T$ in the spring $RX$ is given by

$$T = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l\right),$$

where $x$ is the displacement of $X$ from its equilibrium position.

Show further that the particle performs approximate simple harmonic motion with period

$$2\pi\sqrt{\frac{4mla}{3(4a - \sqrt{3}l)\lambda}}.$$

### [STEP 3, 2016Q10]



A smooth plane is inclined at an angle $\alpha$ to the horizontal. A particle $P$ of mass $m$ is attached to a fixed point $A$ above the plane by a light inextensible string of length $a$. The particle rests in equilibrium on the plane, and the string makes an angle $\beta$ with the plane.

The particle is given a horizontal impulse parallel to the plane so that it has an initial speed of $u$. Show that the particle will not immediately leave the plane if $ag\cos(\alpha + \beta) > u^2 \tan\beta$.

Show further that a necessary condition for the particle to perform a complete circle whilst in contact with the plane is $6\tan\alpha\tan\beta < 1$.

[STEP 3, 2016Q11]

A car of mass $m$ travels along a straight horizontal road with its engine working at a constant rate $P$. The resistance to its motion is such that the acceleration of the car is zero when it is moving with speed $4U$.

(i) Given that the resistance is proportional to the car's speed, show that the distance $X_1$ travelled by the car while it accelerates from speed $U$ to speed $2U$, is given by

$$\lambda X_1 = 2\ln\frac{9}{5} - 1,$$

where $\lambda = \frac{P}{16mU^3}$.

(ii) Given instead that the resistance is proportional to the square of the car's speed, show that the distance $X_2$ travelled by the car while it accelerates from speed $U$ to speed $2U$ is given by

$$\lambda X_2 = \frac{4}{3}\ln\frac{9}{8}.$$

(iii) Given that $3.17 < \ln 24 < 3.18$ and $1.60 < \ln 5 < 1.61$, determine which is the larger of $X_1$ and $X_2$.

## Section C: Probability and Statistics

### [STEP 3, 2016Q12]

Let $X$ be a random variable with mean $\mu$ and standard deviation $\sigma$. *Chebyshev's inequality*, which you may use without proof, is

$$\mathrm{P}(|X-\mu| > k\sigma) \leq \frac{1}{k^2},$$

where $k$ is any positive number.

(i) The probability of a biased coin landing heads up is 0.2. It is thrown $100n$ times, where $n$ is an integer greater than 1. Let $\alpha$ be the probability that the coin lands heads up $N$ times, where $16n \leq N \leq 24n$.

Use Chebyshev's inequality to show that

$$\alpha \geq 1 - \frac{1}{n}.$$

(ii) Use Chebyshev's inequality to show that

$$1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{(2n)!} \geq \left(1 - \frac{1}{n}\right)\mathrm{e}^n.$$

### [STEP 3, 2016Q13]

Given a random variable $X$ with mean $\mu$ and standard deviation $\sigma$, we define the *kurtosis*, $\kappa$, of $X$ by

$$\kappa = \frac{\mathrm{E}((X-\mu)^4)}{\sigma^4} - 3.$$

Show that the random variable $X - a$, where $a$ is a constant, has the same kurtosis as $X$.

(i) Show by integration that a random variable which is Normally distributed with mean 0 has kurtosis 0.

(ii) Let $Y_1, Y_2, \dots, Y_n$ be $n$ independent, identically distributed, random variables with mean 0, and let $T = \sum_{r=1}^{n} Y_r$. Show that

$$\mathrm{E}(T^4) = \sum_{r=1}^{n} \mathrm{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mathrm{E}(Y_s^2)\mathrm{E}(Y_r^2).$$

(iii) Let $X_1, X_2, \dots, X_n$ be $n$ independent, identically distributed, random variables each with kurtosis $\kappa$. Show that the kurtosis of their sum is $\frac{\kappa}{n}$.

# STEP 3 2017

STEP 3 2017
On-line Exam

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Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

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## INFORMATION FOR CANDIDATES

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## Section A: Pure Mathematics

### [STEP 3, 2017Q1]

(i) Prove that, for any positive $n$ and $r$,

$$\frac{1}{^{n+r}C_{r+1}} = \frac{r+1}{r}\left(\frac{1}{^{n+r-1}C_r} - \frac{1}{^{n+r}C_r}\right).$$

Hence determine

$$\sum_{n=1}^{\infty} \frac{1}{^{n+r}C_{r+1}},$$

and deduce that $\sum_{n=2}^{\infty} \frac{1}{^{n+2}C_3} = \frac{1}{2}$.

(ii) Show that, for $n \geq 3$,

$$\frac{3!}{n^3} < \frac{1}{^{n+1}C_3} \quad \text{and} \quad \frac{20}{^{n+1}C_3} - \frac{1}{^{n+2}C_5} < \frac{5!}{n^3}.$$

By summing these inequalities for $n \geq 3$, show that

$$\frac{115}{96} < \sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{116}{96}.$$

**Note:** $^nC_r$ is another notation for $\binom{n}{r}$.



### [STEP 3, 2017Q2]

The transformation $R$ in the complex plane is a rotation (anticlockwise) by an angle $\theta$ about the point represented by the complex number $a$. The transformation $S$ in the complex plane is a rotation (anticlockwise) by an angle $\phi$ about the point represented by the complex number $b$.

(i) The point $P$ is represented by the complex number $z$. Show that the image of $P$ under $R$ is represented by

$$e^{i\theta} z + a(1 - e^{i\theta}).$$

(ii) Show that the transformation $SR$ (equivalent to $R$ followed by $S$) is a rotation about the point represented by $c$, where

$$c \sin\frac{1}{2}(\theta+\phi) = a e^{\frac{i\phi}{2}} \sin\frac{1}{2}\theta + b e^{-\frac{i\theta}{2}} \sin\frac{1}{2}\phi,$$

provided $\theta + \phi \neq 2n\pi$ for any integer $n$.

What is the transformation $SR$ if $\theta + \phi = 2\pi$?

(iii) Under what circumstances is $RS = SR$?

[STEP 3, 2017Q3]

Let $\alpha, \beta, \gamma$ and $\delta$ be the roots of the quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

You are given that, for any such equation, $\alpha\beta + \gamma\delta$, $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ satisfy a cubic equation of the form

$$y^3 + Ay^2 + (pr - 4s)y + (4qs - p^2s - r^2) = 0.$$

Determine $A$.

Now consider the quartic equation given by $p = 0$, $q = 3$, $r = -6$ and $s = 10$.

(i) Find the value of $\alpha\beta + \gamma\delta$, given that it is the largest root of the corresponding cubic equation.

(ii) Hence, using the values of $q$ and $s$, find the value of $(\alpha + \beta)(\gamma + \delta)$ and the value of $\alpha\beta$ given that $\alpha\beta > \gamma\delta$.

(iii) Using these results, and the values of $p$ and $r$, solve the quartic equation.

[STEP 3, 2017Q4]

For any function $f$ satisfying $f(x) > 0$, we define the *geometric mean*, $F$, by

$$F(y) = e^{\frac{1}{y}\int_0^y \ln f(x)\,dx} \qquad (y > 0).$$



(i) The function $f$ satisfies $f(x) > 0$ and $a$ is a positive number with $a \neq 1$. Prove that

$$F(y) = a^{\frac{1}{y}\int_0^y \log_a f(x)\,dx}.$$

(ii) The functions $f$ and $g$ satisfy $f(x) > 0$ and $g(x) > 0$, and the function $h$ is defined by $h(x) = f(x)g(x)$. Their geometric means are $F$, $G$ and $H$, respectively. Show that $H(y) = F(y)G(y)$.

(iii) Prove that, for any positive number $b$, the geometric mean of $b^x$ is $\sqrt{b^y}$.

(iv) Prove that, if $f(x) > 0$ and the geometric mean of $f(x)$ is $\sqrt{f(y)}$, then $f(x) = b^x$ for some positive number $b$.

[STEP 3, 2017Q5]

The point with cartesian coordinates $(x, y)$ lies on a curve with polar equation $r = f(\theta)$. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $f(\theta)$, $f'(\theta)$ and $\tan\theta$.

Two curves, with polar equations $r = f(\theta)$ and $r = g(\theta)$, meet at right angles. Show that where they meet

$$f'(\theta)g'(\theta) + f(\theta)g(\theta) = 0.$$

The curve $C$ has polar equation $r = f(\theta)$ and passes through the point given by $r = 4$, $\theta = -\frac{1}{2}\pi$. For each positive value of $a$, the curve with polar equation $r = a(1 + \sin\theta)$ meets $C$ at right angles. Find $f(\theta)$.

Sketch on a single diagram the three curves with polar equations $r = 1 + \sin\theta$, $r = 4(1 + \sin\theta)$ and $r = f(\theta)$.

[STEP 3, 2017Q6]

In this question, you are not permitted to use any properties of trigonometric functions or inverse trigonometric functions.

The function $T$ is defined for $x > 0$ by

$$T(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u,$$

and $T_\infty = \int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u$ (which has a finite value).

(i) By making an appropriate substitution in the integral for $T(x)$, show that

$$T(x) = T_\infty - T(x^{-1}).$$

(ii) Let $v = \frac{u+a}{1-au}$, where $a$ is a constant. Verify that, for $u \neq a^{-1}$,

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1+v^2}{1+u^2}.$$

Hence show that, for $a > 0$ and $x < \frac{1}{a}$,

$$T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a).$$

Deduce that

$$T(x^{-1}) = 2T_\infty - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1})$$

and hence that, for $b > 0$ and $y > \frac{1}{b}$,

$$T(y) = 2T_\infty - T\left(\frac{y+b}{by-1}\right) - T(b).$$

(iii) Use the above results to show that $T(\sqrt{3}) = \frac{2}{3}T_\infty$ and $T(\sqrt{2}-1) = \frac{1}{4}T_\infty$.

[STEP 3, 2017Q7]

Show that the point $T$ with coordinates

$$\left(\frac{a(1-t^2)}{1+t^2}, \frac{2bt}{1+t^2}\right) \qquad (*)$$

(where $a$ and $b$ are non-zero) lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(i) The line $L$ is the tangent to the ellipse at $T$. The point $(X, Y)$ lies on $L$, and $X^2 \neq a^2$.

Show that

$$(a + X)bt^2 - 2aYt + b(a - X) = 0.$$

Deduce that if $a^2Y^2 > (a^2 - X^2)b^2$, then there are two distinct lines through $(X, Y)$ that are tangents to the ellipse. Interpret this result geometrically. Show, by means of a sketch, that the result holds also if $X^2 = a^2$.

(ii) The distinct points $P$ and $Q$ are given by $(*)$, with $t = p$ and $t = q$, respectively. The tangents to the ellipse at $P$ and $Q$ meet at the point with coordinates $(X, Y)$, where $X^2 \neq a^2$. Show that

$$(a + X)pq = a - X$$

and find an expression for $p + q$ in terms of $a$, $b$, $X$ and $Y$.

Given that the tangents meet the $y$-axis at points $(0, y_1)$ and $(0, y_2)$, where $y_1 + y_2 = 2b$, show that

$$\frac{X^2}{a^2} + \frac{Y}{b} = 1.$$

[STEP 3, 2017Q8]

Prove that, for any numbers $a_1, a_2, \ldots$, and $b_1, b_2, \ldots$, and for $n \geq 1$,

$$\sum_{m=1}^{n} a_m(b_{m+1} - b_m) = a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m).$$

(i) By setting $b_m = \sin mx$, show that

$$\sum_{m=1}^{n} \cos\left(m + \frac{1}{2}\right)x = \frac{1}{2}(\sin(n+1)x - \sin x)\operatorname{cosec}\frac{1}{2}x.$$

**Note:** $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

(ii) Show that

$$\sum_{m=1}^{n} m \sin mx = (p\sin(n+1)x + q\sin nx)\operatorname{cosec}^2\frac{1}{2}x,$$

where $p$ and $q$ are to be determined in terms of $n$.

**Note:** $2\sin A \sin B = \cos(A - B) - \cos(A + B)$.

$2\cos A \sin B = \sin(A + B) - \sin(A - B)$.

## Section B: Mechanics

### [STEP 3, 2017Q9]

Two particles $A$ and $B$ of masses $m$ and $2m$, respectively, are connected by a light spring of natural length $a$ and modulus of elasticity $\lambda$. They are placed on a smooth horizontal table with $AB$ perpendicular to the edge of the table, and $A$ is held on the edge of the table. Initially the spring is at its natural length.

Particle $A$ is released. At a time $t$ later, particle $A$ has dropped a distance $y$ and particle $B$ has moved a distance $x$ from its initial position (where $x < a$). Show that $y + 2x = \frac{1}{2}gt^2$.

The value of $\lambda$ is such that particle $B$ reaches the edge of the table at a time $T$ given by $T = \sqrt{\frac{6a}{g}}$. By considering the total energy of the system (without solving any differential equations), show that the speed of particle $B$ at this time is $\sqrt{\frac{2ag}{3}}$.

### [STEP 3, 2017Q10]

A uniform rod $PQ$ of mass $m$ and length $3a$ is freely hinged at $P$.

The rod is held horizontally and a particle of mass $m$ is placed on top of the rod at a distance $l$ from $P$, where $l < 2a$. The coefficient of friction between the rod and the particle is $\mu$.

The rod is then released. Show that, while the particle does not slip along the rod,

$$(3a^2 + l^2)\dot{\theta}^2 = g(3a + 2l)\sin\theta,$$

where $\theta$ is the angle through which the rod has turned, and the dot denotes the time derivative.

Hence, or otherwise, find an expression for $\ddot{\theta}$ and show that the normal reaction of the rod on the particle is non-zero when $\theta$ is acute.

Show further that, when the particle is on the point of slipping,

$$\tan\theta = \frac{\mu a(2a - l)}{2(l^2 + al + a^2)}.$$

What happens at the moment the rod is released if, instead, $l > 2a$?

[STEP 3, 2017Q11]

A railway truck, initially at rest, can move forwards without friction on a long straight horizontal track. On the truck, $n$ guns are mounted parallel to the track and facing backwards, where $n > 1$. Each of the guns is loaded with a single projectile of mass $m$. The mass of the truck and guns (but not including the projectiles) is $M$.

When a gun is fired, the projectile leaves its muzzle horizontally with a speed $v - V$ relative to the ground, where $V$ is the speed of the truck immediately before the gun is fired.

(i) All $n$ guns are fired simultaneously. Find the speed, $u$, with which the truck moves, and show that the kinetic energy, $K$, which is gained by the system (truck, guns and projectiles) is given by

$$K = \frac{1}{2}nmv^2\left(1 + \frac{nm}{M}\right).$$

(ii) Instead, the guns are fired one at a time. Let $u_r$ be the speed of the truck when $r$ guns have been fired, so that $u_0 = 0$. Show that, for $1 \le r \le n$,

$$u_r - u_{r-1} = \frac{mv}{M + (n-r)m} \qquad (*)$$

and hence that $u_n < u$.

(iii) Let $K_r$ be the total kinetic energy of the system when $r$ guns have been fired (one at a time), so that $K_0 = 0$. Using$(*)$, show that, for $1 \le r \le n$,

$$K_r - K_{r-1} = \frac{1}{2}mv^2 + \frac{1}{2}mv(u_r - u_{r-1})$$

and hence show that

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n.$$

Deduce that $K_n < K$.

# Section C: Probability and Statistics

## [STEP 3, 2017Q12]

The discrete random variables $X$ and $Y$ can each take the values $1, \dots, n$ (where $n \geq 2$).

Their joint probability distribution is given by

$$P(X = x, Y = y) = k(x + y),$$

where $k$ is a constant.

(i) Show that

$$P(X = x) = \frac{n + 1 + 2x}{2n(n + 1)}.$$

Hence determine whether $X$ and $Y$ are independent.

(ii) Show that the covariance of $X$ and $Y$ is negative.

## [STEP 3, 2017Q13]

The random variable $X$ has mean $\mu$ and variance $\sigma^2$, and the function $V$ is defined, for $-\infty < x < \infty$, by

$$V(x) = E((X - x)^2).$$

Express $V(x)$ in terms of $x$, $\mu$ and $\sigma$.

The random variable $Y$ is defined by $Y = V(X)$. Show that

$$E(Y) = 2\sigma^2. \qquad (*)$$

Now suppose that $X$ is uniformly distributed on the interval $0 \leq x \leq 1$. Find $V(x)$. Find also the probability density function of $Y$ and use it to verify that $(*)$ holds in this case.

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# Section A: Pure Mathematics

## [STEP 3, 2018Q1]

(i) The function $f$ is given by

$$f(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0).$$

Find the stationary point of the curve $y = f(\beta)$ and sketch the curve.

Sketch also the curve $y = g(\beta)$, where

$$g(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0).$$

(ii) Let $u$ and $v$ be the roots of the equation

$$x^2 + \alpha x + \beta = 0,$$

where $\beta \neq 0$. Obtain expressions in terms of $\alpha$ and $\beta$ for $u + v + \frac{1}{uv}$ and $\frac{1}{u} + \frac{1}{v} + uv$.

(iii) Given that $u + v + \frac{1}{uv} = -1$, and that $u$ and $v$ are real, show that $\frac{1}{u} + \frac{1}{v} + uv \leq -1$.

(iv) Given instead that $u + v + \frac{1}{uv} = 3$, and that $u$ and $v$ are real, find the greatest value $\frac{1}{u} + \frac{1}{v} + uv$.



## [STEP 3, 2018Q2]

The sequence of functions $y_0, y_1, y_2, \ldots$ is defined by $y_0 = 1$ and, for $n \geq 1$,

$$y_n = (-1)^n \frac{1}{z} \frac{\mathrm{d}^n z}{\mathrm{d}x^n},$$

where $z = \mathrm{e}^{-x^2}$.

(i) Show that $\frac{\mathrm{d}y_n}{\mathrm{d}x} = 2xy_n - y_{n+1}$ for $n \geq 1$.

(ii) Prove by induction that, for $n \geq 1$,

$$y_{n+1} = 2xy_n - 2ny_{n-1}.$$

Deduce that, for $n \geq 1$,

$$y_{n+1}^2 - y_n y_{n+2} = 2n(y_n^2 - y_{n-1}y_{n+1}) + 2y_n^2.$$

(iii) Hence show that $y_n^2 - y_{n-1}y_{n+1} > 0$ for $n \geq 1$.

[STEP 3, 2018Q3]

Show that the second-order differential equation

$$x^2y'' + (1-2p)xy' + (p^2-q^2)y = f(x),$$

where $p$ and $q$ are constants, can be written in the form

$$x^a\big(x^b(x^cy)'\big)' = f(x), \qquad (*)$$

where $a$, $b$ and $c$ are constants.

(i) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + (p^2-q^2)y = 0$$

in the different cases that arise according to the values of $p$ and $q$.

(ii) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1-2p)xy' + p^2y = x^n$$

in the different cases that arise according to the values of $p$ and $n$.

[STEP 3, 2018Q4]

The point $P(a\sec\theta\,, b\tan\theta)$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $a > b > 0$. Show that the equation of the tangent to the hyperbola at $P$ can be written as

$$bx - ay\sin\theta = ab\cos\theta.$$

(i) This tangent meets the lines $\frac{x}{a} = \frac{y}{b}$ and $\frac{x}{a} = -\frac{y}{b}$ at $S$ and $T$, respectively.

How is the mid-point of $ST$ related to $P$?

(ii) The point $Q(a\sec\phi\,, b\tan\phi)$ also lies on the hyperbola and the tangents to the hyperbola at $P$ and $Q$ are perpendicular. These two tangents intersect at $(x, y)$.

Obtain expressions for $x^2$ and $y^2$ in terms of $a$, $\theta$ and $\phi$.

Hence, or otherwise, show that $x^2 + y^2 = a^2 - b^2$.

[STEP 3, 2018Q5]

The real numbers $a_1, a_2, a_3, \ldots$ are all positive. For each positive integer $n$, $A_n$ and $G_n$ are defined by

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \quad \text{and} \quad G_n = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}.$$

(i) Show that, for any given positive integer $k$,

$$(k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k)$$

if and only if

$$\lambda_k^{k+1} - (k+1)\lambda_k + k \geq 0,$$

where $\lambda_k = \left(\frac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}$.

(ii) Let

$$f(x) = x^{k+1} - (k+1)x + k,$$

where $x > 0$ and $k$ is a positive integer. Show that $f(x) \geq 0$ and that $f(x) = 0$ if and only if $x = 1$.

(iii) Deduce that:

(a) $A_n \geq G_n$ for all $n$.

(b) if $A_n = G_n$ for some $n$, then $a_1 = a_2 = \cdots = a_n$.



[STEP 3, 2018Q6]

(i) The distinct points $A$, $Q$ and $C$ lie on a straight line in the Argand diagram, and represent the distinct complex numbers $a$, $q$ and $c$, respectively. Show that $\frac{q-a}{c-a}$ is real and hence that $(c-a)(q^* - a^*) = (c^* - a^*)(q-a)$.

Given that $aa^* = cc^* = 1$, show further that

$$q + acq^* = a + c.$$

(ii) The distinct points $A, B, C$ and $D$ lie, in anticlockwise order, on the circle of unit radius with centre at the origin (so that, for example, $aa^* = 1$). The lines $AC$ and $BD$ meet at $Q$. Show that

$$(ac - bd)q^* = (a + c) - (b + d),$$

where $b$ and $d$ are complex numbers represented by the points $B$ and $D$ respectively, and show further that

$$(ac - bd)(q + q^*) = (a - b)(1 + cd) + (c - d)(1 + ab).$$

(iii) The lines $AB$ and $CD$ meet at $P$, which represents the complex number $p$. Given that $p$ is real, show that $p(1 + ab) = a + b$. Given further that $ac - bd \neq 0$, show that

$$p(q + q^*) = 2.$$

[STEP 3, 2018Q7]

(i) Use De Moivre's theorem to show that, if $\sin\theta \neq 0$, then

$$\frac{(\cot\theta + \mathbf{i})^{2n+1} - (\cot\theta - \mathbf{i})^{2n+1}}{2\mathbf{i}} = \frac{\sin(2n+1)\,\theta}{\sin^{2n+1}\theta},$$

for any positive integer $n$.

Deduce that the solutions of the equation

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \cdots + (-1)^n = 0$$

are

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right)$$

where $m = 1, 2, \ldots, n$.

(ii) Hence show that

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n(2n-1)}{3}.$$

(iii) Given that $0 < \sin\theta < \theta < \tan\theta$ for $0 < \theta < \frac{1}{2}\pi$, show that

$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta.$$

Hence show that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

[STEP 3, 2018Q8]

In this question, you should ignore issues of convergence.

(i) Let

$$I = \int_0^1 \frac{f(x^{-1})}{1+x}\,\mathrm{d}x,$$

where $f(x)$ is a function for which the integral exists.

Show that

$$I = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{f(y)}{y(1+y)}\,\mathrm{d}y$$

and deduce that, if $f(x) = f(x+1)$ for all $x$, then

$$I = \int_0^1 \frac{f(x)}{1+x}\,\mathrm{d}x.$$

(ii) The *fractional part*, $\{x\}$, of a real number $x$ is defined to be $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$. For example $\{3.2\} = 0.2$ and $\{3\} = 0$.

Use the result of part (i) to evaluate

$$\int_0^1 \frac{\{x^{-1}\}}{1+x}\,\mathrm{d}x \quad \text{and} \quad \int_0^1 \frac{\{2x^{-1}\}}{1+x}\,\mathrm{d}x.$$

## Section B: Mechanics

### [STEP 3, 2018Q9]

A particle $P$ of mass $m$ is projected with speed $u_0$ along a smooth horizontal floor directly towards a wall. It collides with a particle $Q$ of mass $km$ which is moving directly away from the wall with speed $v_0$. In the subsequent motion, $Q$ collides alternately with the wall and with $P$. The coefficient of restitution between $Q$ and $P$ is $e$, and the coefficient of restitution between $Q$ and the wall is 1.

Let $u_n$ and $v_n$ be the velocities of $P$ and $Q$, respectively, towards the wall after the $n$th collision between $P$ and $Q$.

(i) Show that, for $n \geq 2$,

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0. \qquad (*)$$

(ii) You are now given that $e = \frac{1}{2}$ and $k = \frac{1}{34}$, and that the solution of $(*)$ is of the form

$$u_n = A\left(\frac{7}{10}\right)^n + B\left(\frac{5}{7}\right)^n \qquad (n \geq 0),$$

where $A$ and $B$ are independent of $n$. Find expressions for $A$ and $B$ in terms of $u_0$ and $v_0$.

Show that, if $0 < 6u_0 < v_0$, then $u_n$ will be negative for large $n$.

### [STEP 3, 2018Q10]

A uniform disc with centre $O$ and radius $a$ is suspended from a point $A$ on its circumference, so that it can swing freely about a horizontal axis $L$ through $A$. The plane of the disc is perpendicular to $L$. A particle $P$ is attached to a point on the circumference of the disc. The mass of the disc is $M$ and the mass of the particle is $m$.

In equilibrium, the disc hangs with $OP$ horizontal, and the angle between $AO$ and the downward vertical through $A$ is $\beta$. Find $\sin\beta$ in terms of $M$ and $m$ and show that

$$\frac{AP}{a} = \sqrt{\frac{2M}{M+m}}.$$

The disc is rotated about $L$ and then released. At later time $t$, the angle between $OP$ and the horizontal is $\theta$; when $P$ is higher than $O$, $\theta$ is positive and when $P$ is lower than $O$, $\theta$ is negative. Show that

$$\frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (m+M)ga\cos\beta\,(1-\cos\theta)$$

is constant during the motion, where $I$ is the moment of inertia of the disc about $L$.

Given that $m = \frac{3}{2}M$ and that $I = \frac{3}{2}Ma^2$, show that the period of small oscillations is

$$3\pi\sqrt{\frac{3a}{5g}}.$$

[STEP 3, 2018Q11]

A particle is attached to one end of a light inextensible string of length $b$. The other end of the string is attached to a fixed point $O$. Initially the particle hangs vertically below $O$. The particle then receives a horizontal impulse.

The particle moves in a circular arc with the string taut until the acute angle between the string and the upward vertical is $\alpha$, at which time it becomes slack. Express $V$, the speed of the particle when the string becomes slack, in terms of $b$, $g$ and $\alpha$.

Show that the string becomes taut again a time $T$ later, where

$$gT = 4V \sin\alpha,$$

and that just before this time the trajectory of the particle makes an angle $\beta$ with the horizontal where $\tan\beta = 3\tan\alpha$.

When the string becomes taut, the momentum of the particle in the direction of the string is destroyed. Show that the particle comes instantaneously to rest at this time if and only if

$$\sin^2\alpha = \frac{1+\sqrt{3}}{4}.$$

## Section C: Probability and Statistics

### [STEP 3, 2018Q12]

A random process generates, independently, $n$ numbers each of which is drawn from a uniform (rectangular) distribution on the interval 0 to 1. The random variable $Y_k$ is defined to be the $k$th smallest number (so there are $k-1$ smaller numbers).

(i) Show that, for $0 \le y \le 1$,

$$\mathrm{P}(Y_k \le y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}. \qquad (*)$$

(ii) Show that

$$m\binom{n}{m} = n\binom{n-1}{m-1}$$

and obtain a similar expression for $(n-m)\binom{n}{m}$.

Starting from $(*)$, show that the probability density function of $Y_k$ is

$$n\binom{n-1}{k-1} y^{k-1}(1-y)^{n-k}.$$

Deduce an expression for $\int_0^1 y^{k-1}(1-y)^{n-k}\,\mathrm{d}y$.

(iii) Find $\mathrm{E}(Y_k)$ in terms of $n$ and $k$.

[STEP 3, 2018Q13]

The random variable $X$ takes only non-negative integer values and has probability generating function $G(t)$. Show that

$$\mathrm{P}(X = 0 \text{ or } 2 \text{ or } 4 \text{ or } 6 \ldots) = \frac{1}{2}\big(G(1) + G(-1)\big).$$

You are now given that $X$ has a Poisson distribution with mean $\lambda$. Show that

$$G(t) = \mathrm{e}^{-\lambda(1-t)}.$$

(i) The random variable $Y$ is defined by

$$\mathrm{P}(Y = r) = \begin{cases} k\mathrm{P}(X = r), & \text{if } r = 0, 2, 4, 6, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$

where $k$ is an appropriate constant.

Show that the probability generating function of $Y$ is $\frac{\cosh \lambda t}{\cosh \lambda}$.

Deduce that $\mathrm{E}(Y) < \lambda$ for $\lambda > 0$.

(ii) The random variable $Z$ is defined by

$$\mathrm{P}(Z = r) = \begin{cases} c\mathrm{P}(X = r), & \text{if } r = 0, 4, 8, 12, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$

where $c$ is an appropriate constant.

Is $\mathrm{E}(Z) < \lambda$ for all positive values of $\lambda$?

# STEP 3 2019

STEP 3 2019

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# Section A: Pure Mathematics

## [STEP 3, 2019Q1]

The coordinates of a particle at time $t$ are $x$ and $y$. For $t \geq 0$, they satisfy the pair of coupled differential equations

$$\dot{x} = -x - ky$$
$$\dot{y} = x - y$$

where $k$ is a constant. When $t = 0$, $x = 1$ and $y = 0$.

(i) Let $k = 1$. Find $x$ and $y$ in terms of $t$ and sketch $y$ as a function of $t$.

Sketch the path of the particle in the $x$-$y$ plane, giving the coordinates of the point at which $y$ is greatest and the coordinates of the point at which $x$ is least.

(ii) Instead, let $k = 0$. Find $x$ and $y$ in terms of $t$ and sketch the path of the particle in the $x$-$y$ plane.

## [STEP 3, 2019Q2]

The definition of the derivative $f'$ of a (differentiable) function $f$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \qquad (*)$$

(i) The function $f$ has derivative $f'$ and satisfies

$$f(x+y) = f(x)f(y)$$

for all $x$ and $y$, and $f'(0) = k$ where $k \neq 0$. Show that $f(0) = 1$.

Using $(*)$, show that $f'(x) = kf(x)$ and find $f(x)$ in terms of $x$ and $k$.

(ii) The function $g$ has derivative $g'$ and satisfies

$$g(x+y) = \frac{g(x) + g(y)}{1 + g(x)g(y)}$$

for all $x$ and $y$, $|g(x)| < 1$ for all $x$, and $g'(0) = k$ where $k \neq 0$.

Find $g'(x)$ in terms of $g(x)$ and $k$, and hence find $g(x)$ in terms of $x$ and $k$.

[STEP 3, 2019Q3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(i) You are given that the transformation represented by $\mathbf{A}$ has a line $L_1$ of invariant points (so that each point on $L_1$ is transformed to itself). Let $(x, y)$ be a point on $L_1$. Show that $\big((a-1)(d-1) - bc\big)xy = 0$.

Show further that $(a-1)(d-1) = bc$.

What can be said about $\mathbf{A}$ if $L_1$ does not pass through the origin?

(ii) By considering the cases $b \neq 0$ and $b = 0$ separately, show that if $(a-1)(d-1) = bc$ then the transformation represented by $\mathbf{A}$ has a line of invariant points. You should identify the line in the different cases that arise.

(iii) You are given instead that the transformation represented by $\mathbf{A}$ has an invariant line $L_2$ (so that each point on $L_2$ is transformed to a point on $L_2$) and that $L_2$ does not pass through the origin. If $L_2$ has the form $y = mx + k$, show that $(a-1)(d-1) = bc$.

[STEP 3, 2019Q4]

The $n$th degree polynomial $P(x)$ is said to be *reflexive* if:

(a) $P(x)$ is of the form $x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$ where $n \geq 1$.

(b) $a_1, a_2, \ldots, a_n$ are real.

(c) the $n$ (not necessarily distinct) roots of the equation $P(x) = 0$ are $a_1, a_2, \ldots, a_n$.



(i) Find all reflexive polynomials of degree less than or equal to 3.

(ii) For a reflexive polynomial with $n > 3$, show that

$$2a_2 = -a_2^2 - a_3^2 - \cdots - a_n^2.$$

Deduce that, if all the coefficients of a reflexive polynomial of degree $n$ are integers and $a_n \neq 0$, then $n \leq 3$.



(iii) Determine all reflexive polynomials with integer coefficients.

[STEP 3, 2019Q5]

(i) Let

$$f(x) = \frac{x}{\sqrt{x^2 + p}},$$

where $p$ is a non-zero constant. Sketch the curve $y = f(x)$ for $x \geq 0$ in the case $p > 0$.

(ii) Let

$$I = \int \frac{1}{(b^2 - y^2)\sqrt{c^2 - y^2}} \mathrm{d}y,$$

where $b$ and $c$ are positive constant. Use the substitution $y = \frac{cx}{\sqrt{x^2+p}}$, where $p$ is a suitably chosen constant, to show that

$$I = \int \frac{1}{b^2 + (b^2 - c^2)x^2} \mathrm{d}x.$$

Evaluate

$$\int_1^{\sqrt{2}} \frac{1}{(3 - y^2)\sqrt{2 - y^2}} \mathrm{d}y.$$

[**Note:** $\int \frac{1}{a^2+x^2} \mathrm{d}x = \frac{1}{a} \tan^{-1} \frac{x}{a} + \text{constant.}$]

Hence evaluate

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2 - 1)\sqrt{2y^2 - 1}} \mathrm{d}y.$$

(iii) By means of a suitable substitution, evaluate

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{(3y^2 - 1)\sqrt{2y^2 - 1}} \mathrm{d}y.$$

[STEP 3, 2019Q6]

The point $P$ in the Argand diagram is represented by the the complex number $z$, which satisfies

$$zz^* - az^* - a^*z + aa^* - r^2 = 0.$$

Here, $r$ is a positive real number and $r^2 \neq a^*a$. By writing $|z-a|^2$ as $(z-a)(z-a)^*$, show that the locus of $P$ is a circle, $C$, the radius and the centre of which you should give.

(i) The point $Q$ is repesnted by $\omega$ and is related to $P$ by $\omega = \frac{1}{z}$. Let $C'$ be the locus of $Q$. Show that $C'$ is also a circle, and give its radius and centre.

If $C$ and $C'$ are the same circle, show that

$$(|a|^2 - r^2)^2 = 1$$

and that either $a$ is real or $a$ is imaginary. Give sketches to indicate the position of $C$ in these two cases.

(ii) Suppose instead that the point $Q$ is represented by $\omega$, where $\omega = \frac{1}{z^*}$. If the locus of $Q$ is $C$, is it the case that either $a$ is real or $a$ is imaginary?

[STEP 3, 2019Q7]

The *Devil's Curve* is given by

$$y^2(y^2 - b^2) = x^2(x^2 - a^2),$$

where $a$ and $b$ are positive constants.

(i) In the case $a = b$, sketch the Devil's Curve.

(ii) Now consider the case $a = 2$ and $b = \sqrt{5}$, and $x \geq 0$, $y \geq 0$.

(a) Show by considering a quadratic equation in $x^2$ that either $0 \leq y \leq 1$ or $y \geq 2$.

(b) Describe the curve very close to and very far from the origin.

(c) Find the points at which the tangent to the curve is parallel to the $x$-axis and the point at which the tangent to the curve is parallel to the $y$-axis.

Sketch the Devil's Curve in this case.

(iii) Sketch the Devil's Curve in the case $a = 2$ and $b = \sqrt{5}$ again, but with $-\infty < x < \infty$ and $-\infty < y < \infty$.

[STEP 3, 2019Q8]

A pyramid has a horizontal rectangular base $ABCD$ and its vertex $V$ is vertically above the centre of the base. The acute angle between the face $AVB$ and the base is $\alpha$, the acute angle between the face $BVC$ and the base is $\beta$ and the obtuse angle between the faces $AVB$ and $BVC$ is $\pi - \theta$.

(i) The edges $AB$ and $BC$ are parallel to the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, respectively, and the unit vector $\hat{\mathbf{k}}$ is vertical. Find a unit vector that is perpendicular to the face $AVB$.

Show that

$$\cos\theta = \cos\alpha\cos\beta.$$

(ii) The edge $BV$ makes an angle $\phi$ with the base. Show that

$$\cot^2\phi = \cot^2\alpha + \cot^2\beta.$$

Show also that

$$\cos^2\phi = \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1-\cos^2\theta} \geq \frac{2\cos\theta - 2\cos^2\theta}{1-\cos^2\theta}$$

and deduce that $\phi < \theta$.

## Section B: Mechanics

### [STEP 3, 2019Q9]

In this question, $\mathbf{\hat{i}}$ and $\mathbf{\hat{j}}$ are perpendicular unit vectors and $\mathbf{\hat{j}}$ is vertically upwards.

A smooth hemisphere of mass $M$ and radius $a$ rests on a smooth horizontal table with its plane face in contact with the table. The point $A$ is at the top of the hemisphere and the point $O$ is at the centre of its plane face.

Initially, a particle $P$ of mass $m$ rests at $A$. It is then given a small displacement in the positive $\mathbf{\hat{i}}$ direction. At a later time $t$, when the particle is still in contact with the hemisphere, the hemisphere has been displaced by $-s\mathbf{\hat{i}}$ and $\angle AOP = \theta$.

(i) Let $\mathbf{r}$ be the position vector of the particle at time $t$ with respect to the initial position of $O$. Write down an expression for $\mathbf{r}$ in terms of $a$, $\theta$ and $s$ and show that

$$\dot{\mathbf{r}} = \left(a\dot{\theta}\cos\theta - \dot{s}\right)\mathbf{\hat{i}} - a\dot{\theta}\sin\theta\,\mathbf{\hat{j}}.$$

Show also that

$$\dot{s} = (1-k)a\dot{\theta}\cos\theta,$$

where $k = \frac{M}{m+M}$, and deduce that

$$\dot{\mathbf{r}} = a\dot{\theta}(k\cos\theta\,\mathbf{\hat{i}} - \sin\theta\,\mathbf{\hat{j}}).$$

(ii) Show that

$$a\dot{\theta}^2(k\cos^2\theta + \sin^2\theta) = 2g(1-\cos\theta).$$

(iii) At time $T$, when $\theta = \alpha$, the particle leaves the hemisphere. By considering the component of $\ddot{\mathbf{r}}$ parallel to the vector $\sin\theta\,\mathbf{\hat{i}} + k\cos\theta\,\mathbf{\hat{j}}$, or otherwise, show that at time $T$

$$a\dot{\theta}^2 = g\cos\alpha.$$

Find a cubic equation for $\cos\alpha$ and deduce that $\cos\alpha > \frac{2}{3}$.

[STEP 3, 2019Q10]

Two identical smooth spheres $P$ and $Q$ can move on a smooth horizontal table. Initially, $P$ moves with speed $u$ and $Q$ is at rest. Then $P$ collides with $Q$. The direction of travel of $P$ before the collision makes an acute angle $\alpha$ with the line joining the centres of $P$ and $Q$ at the moment of the collision. The coefficient of restitution between $P$ and $Q$ is $e$ where $e < 1$.

As a result of the collision, $P$ has speed $v$ and $Q$ has speed $w$, and $P$ is deflected through an angle $\theta$.

(i) Show that

$$u \sin \alpha = v \sin(\alpha + \theta)$$

and find an expression for $w$ in terms of $v$, $\theta$ and $\alpha$.

(ii) Show further that

$$\sin \theta = \cos(\theta + \alpha) \sin \alpha + e \sin(\theta + \alpha) \cos \alpha$$

and find an expression for $\tan \theta$ in terms of $\tan \alpha$ and $e$.

Find, in terms of $e$, the maximum value of $\tan \theta$ as $\alpha$ varies.

# Section C: Probability and Statistics

## [STEP 3, 2019Q11]

The number of customers arriving at a builders' merchants each day follows a Poisson distribution with mean $\lambda$. Each customer is offered some free sand. The probability of any given customer taking the free sand is $p$.

(i) Show that the number of customers each day who take sand follows a Poisson distribution with mean $p\lambda$.

(ii) The merchant has a mass $S$ of sand at the beginning of the day. Each customer who takes the free sand gets a proportion $k$ of the remaining sand, where $0 \leq k < 1$. Show that by the end of the day the expected mass of sand taken is

$$(1 - \mathrm{e}^{-kp\lambda})S.$$

(iii) At the beginning of the day, the merchant's bag of sand contains a large number of grains, exactly one of which is made from solid gold. At the end of the day, the merchant's assistant takes a proportion $k$ of the remaining sand. Find the probability that the assistant takes the golden grain. Comment on the case $k = 0$ and on the limit $k \to 1$.

In the case $p\lambda > 1$ find the value of $k$ which maximises the probability that the assistant takes the golden grain.

## [STEP 3, 2019Q12]

The set $S$ is the set of all integers from 1 to $n$. The set $T$ is the set of all distinct subsets of $S$, including the empty set $\phi$ and $S$ itself. Show that $T$ contains exactly $2^n$ sets.

The sets $A_1, A_2, \ldots, A_m$, which are not necessarily distinct, are chosen randomly and independently from $T$, and for each $k$ $(1 \leq k \leq m)$, the set $A_k$ is equally likely to be any of the sets in $T$.

(i) Write down the value of $\mathrm{P}(1 \in A_1)$.

(ii) By considering each integer separately, show that $\mathrm{P}(A_1 \cap A_2 = \emptyset) = \left(\frac{3}{4}\right)^n$.

Find $\mathrm{P}(A_1 \cap A_2 \cap A_3 = \emptyset)$ and $\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_m = \emptyset)$.

(iii) Find $\mathrm{P}(A_1 \subseteq A_2)$, $\mathrm{P}(A_1 \subseteq A_2 \subseteq A_3)$ and $\mathrm{P}(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m)$.

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# Section A: Pure Mathematics

## [STEP 3, 2020Q1]

For non-negative integers $a$ and $b$, let

$$I(a,b) = \int_0^{\frac{\pi}{2}} \cos^a x \cos bx \, \mathrm{d}x.$$

(i) Show that for positive integers $a$ and $b$,

$$I(a,b) = \frac{a}{a+b} I(a-1, b-1).$$

(ii) Prove by induction on $n$ that for non-negative integers $n$ and $m$,

$$\int_0^{\frac{\pi}{2}} \cos^n x \cos(n+2m+1)x \, \mathrm{d}x = (-1)^m \frac{2^n n! \, (2m)! \, (n+m)!}{m! \, (2n+2m+1)!}.$$

## [STEP 3, 2020Q2]

The curve $C$ has equation $\sinh x + \sinh y = 2k$, where $k$ is a positive constant.

(i) Show that the curve $C$ has no stationary points and that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$ at the point $(x, y)$ on the curve if and only if

$$1 + \sinh x \sinh y = 0.$$

Find the co-ordinates of the points of inflection on the curve $C$, leaving your answers in terms of inverse hyperbolic functions.

(ii) Show that if $(x, y)$ lies on the curve $C$ and on the line $x + y = a$, then

$$\mathrm{e}^{2x}(1 - \mathrm{e}^{-a}) - 4k\mathrm{e}^{x} + (\mathrm{e}^{a} - 1) = 0$$

and deduce that $1 < \cosh a \leq 2k^2 + 1$.

(iii) Sketch the curve $C$.

## [STEP 3, 2020Q3]

Given distinct points $A$ and $B$ in the complex plane, the point $G_{AB}$ is defined to be the centroid of the triangle $ABK$, where the point $K$ is the image of $B$ under rotation about $A$ through a clockwise angle of $\frac{1}{3}\pi$.

**Note:** if the points $P$, $Q$ and $R$ are represented in the complex plane by $p$, $q$ and $r$, the centroid of triangle $PQR$ is defined to be the point represented by $\frac{1}{3}(p+q+r)$.

(i) If $A$, $B$ and $G_{AB}$ are represented in the complex plane by $a$, $b$ and $g_{ab}$, show that

$$g_{ab} = \frac{1}{\sqrt{3}}(\omega a + \omega^* b),$$

where $\omega = \mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$.

(ii) The quadrilateral $Q_1$ has vertices $A$, $B$, $C$ and $D$, in that order, and the quadrilateral $Q_2$ has vertices $G_{AB}$, $G_{BC}$, $G_{CD}$ and $G_{DA}$, in that order. Using the result in part(i), show that $Q_1$ is a parallelogram if and only if $Q_2$ is a parallelogram.

(iii) The triangle $T_1$ has vertices $A$, $B$ and $C$ and the triangle $T_2$ has vertices $G_{AB}$, $G_{BC}$, and $G_{CA}$. Using the result in part (i), show that $T_2$ is always an equilateral triangle.

## [STEP 3, 2020Q4]

The plane $\Pi$ has equation $\mathbf{r}\cdot\hat{\mathbf{n}} = 0$ where $\hat{\mathbf{n}}$ is a unit vector. Let $P$ be a point with position vector $\mathbf{x}$ which does not lie on the plane $\Pi$. Show that the point $Q$ with position vector $\mathbf{x} - (\mathbf{x}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$ lies on $\Pi$ and that $PQ$ is perpendicular to $\Pi$.

(i) Let transformation $T$ be a reflection in the plane $ax + by + cz = 0$, where $a^2 + b^2 + c^2 = 1$.

Show that the image of $\hat{\mathbf{i}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ under $T$ is $\begin{pmatrix}b^2+c^2-a^2\\-2ab\\-2ac\end{pmatrix}$, and find the images of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ under $T$.

Write down the matrix $\mathbf{M}$ which represents transformation $T$.

(ii) The matrix

$$\begin{pmatrix}0.64 & 0.48 & 0.6\\ 0.48 & 0.36 & -0.8\\ 0.6 & -0.8 & 0\end{pmatrix}$$

represents a reflection in a plane. Find the cartesian equation of the plane.

(iii) The matrix $\mathbf{N}$ represents a rotation through angle $\pi$ about the line through the origin parallel to $\begin{pmatrix}a\\b\\c\end{pmatrix}$, where $a^2 + b^2 + c^2 = 1$. Find the matrix $\mathbf{N}$.

(iv) Identify the single transformation which is represented by the matrix $\mathbf{NM}$.

[STEP 3, 2020Q5]

Show that for positive integer $n$, $x^n - y^n = (x-y)\sum_{r=1}^{n} x^{n-r}y^{r-1}$.

(i) Let $F$ be defined by

$$F(x) = \frac{1}{x^n(x-k)} \qquad \text{for } x \neq 0, k$$

when $n$ is a positive integer and $k \neq 0$.

(a) Given that

$$F(x) = \frac{A}{x-k} + \frac{f(x)}{x^n},$$

where $A$ is a constant and $f(x)$ is a polynomial, show that

$$f(x) = \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right).$$

Deduce that

$$F(x) = \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n} \frac{1}{k^{n-r}x^r}.$$

(b) By differentiating $x^n F(x)$, prove that

$$\frac{1}{x^n(x-k)^2} = \frac{1}{k^n(x-k)^2} - \frac{n}{xk^n(x-k)} + \sum_{r=1}^{n} \frac{n-r}{k^{n+1-r}x^{r+1}}.$$

(ii) Hence evaluate the limit of

$$\int_2^N \frac{1}{x^3(x-1)^2}\,\mathrm{d}x$$

as $N \to \infty$, justifying your answer.

[STEP 3, 2020Q6]

(i) Sketch the curve $y = \cos x + \sqrt{\cos 2x}$ for $-\frac{1}{4}\pi \le x \le \frac{1}{4}\pi$.

(ii) The equation of curve $C_1$ in polar co-ordinates is

$$r = \cos\theta + \sqrt{\cos 2\theta} \qquad -\frac{1}{4}\pi \le \theta \le \frac{1}{4}\pi.$$

Sketch the curve $C_1$.

(iii) The equation of curve $C_2$ in polar co-ordinates is

$$r^2 - 2r\cos\theta + \sin^2\theta = 0 \qquad -\frac{1}{4}\pi \le \theta \le \frac{1}{4}\pi.$$

Find the value of $r$ when $\theta = \pm\frac{1}{4}\pi$.

Show that, when $r$ is small, $r \approx \frac{1}{2}\theta^2$.

Sketch the curve $C_2$, indicating clearly the behaviour of the curve near $r = 0$ and near $\theta = \pm\frac{1}{4}\pi$.

Show that the area enclosed by curve $C_2$ and above the line $\theta = 0$ is $\frac{\pi}{2\sqrt{2}}$.

[STEP 3, 2020Q7]



(i) Given that the variables $x$, $y$ and $u$ are connected by the differential equations

$$\frac{du}{dx} + f(x)u = h(x) \qquad \text{and} \qquad \frac{dy}{dx} + g(x)y = u\,,$$

show that

$$\frac{d^2y}{dx^2} + \big(g(x) + f(x)\big)\frac{dy}{dx} + \big(g'(x) + f(x)g(x)\big)y = h(x). \tag{1}$$

(ii) Given that the differential equation

$$\frac{d^2y}{dx^2} + \left(1 + \frac{4}{x}\right)\frac{dy}{dx} + \left(\frac{2}{x} + \frac{2}{x^2}\right)y = 4x + 12 \tag{2}$$

can be written in the same form as (1), find a first order differential equation which is satisfied by $g(x)$.

If $g(x) = kx^n$, find a possible value of $n$ and the corresponding value of $k$.

Hence find a solution of (2) with $y = 5$ and $\frac{dy}{dx} = -3$ at $x = 1$.

[STEP 3, 2020Q8]

A sequence $u_k$, for integer $k \geq 1$, is defined as follows.

$$u_1 = 1$$
$$u_{2k} = u_k \text{ for } k \geq 1$$
$$u_{2k+1} = u_k + u_{k+1} \text{ for } k \geq 1$$

(i) Show that, for every pair of consecutive terms of this sequence, except the first pair, the term with odd subscript is larger than the term with even subscript.

(ii) Suppose that two consecutive terms in this sequence have a common factor greater than one. Show that there are then two consecutive terms earlier in the sequence which have the same common factor. Deduce that any two consecutive terms in this sequence are co-prime (do not have a common factor greater than one).

(iii) Prove that it is not possible for two positive integers to appear consecutively in the same order in two different places in the sequence.

(iv) Suppose that $a$ and $b$ are two co-prime positive integers which do not occur consecutively in the sequence with $b$ following $a$. If $a > b$, show that $a - b$ and $b$ are two co-prime positive integers which do not occur consecutively in the sequence with $b$ following $a - b$, and whose sum is smaller than $a + b$. Find a similar result for $a < b$.

(v) For each integer $n \geq 1$, define the function $f$ from the positive integers to the positive rational numbers by $f(n) = \frac{u_n}{u_{n+1}}$. Show that the range of $f$ is all the positive rational numbers, and that $f$ has an inverse.

## Section B: Mechanics

### [STEP 3, 2020Q9]

Two inclined planes $\Pi_1$ and $\Pi_2$ meet in a horizontal line at the lowest points of both planes and lie on either side of this line. $\Pi_1$ and $\Pi_2$ make angles of $\alpha$ and $\beta$, respectively, to the horizontal, where $0 < \beta < \alpha < \frac{1}{2}\pi$.

A uniform rigid rod $PQ$ of mass $m$ rests with $P$ lying on $\Pi_1$ and $Q$ lying on $\Pi_2$ so that the rod lies in a vertical plane perpendicular to $\Pi_1$ and $\Pi_2$ with $P$ higher than $Q$.

(i) It is given that both planes are smooth and that the rod makes an angle $\theta$ with the horizontal. Show that $2\tan\theta = \cot\beta - \cot\alpha$.

(ii) It is given instead that $\Pi_1$ is smooth, that $\Pi_2$ is rough with coefficient of friction $\mu$ and that the rod makes an angle $\phi$ with the horizontal. Given that the rod is in limiting equilibrium, with $P$ about to slip down the plane $\Pi_1$, show that

$$\tan\theta - \tan\phi = \frac{\mu}{(\mu + \tan\beta)\sin 2\beta}$$

where $\theta$ is the angle satisfying $2\tan\theta = \cot\beta - \cot\alpha$.

[STEP 3, 2020Q10]

A light elastic spring $AB$, of natural length $a$ and modulus of elasticity $kmg$, hangs vertically with one end $A$ attached to a fixed point. A particle of mass $m$ is attached to the other end $B$. The particle is held at rest so that $AB > a$ and is released.

Find the equation of motion of the particle and deduce that the particle oscillates vertically.

If the period of oscillation is $\frac{2\pi}{\Omega}$, show that $kg = a\Omega^2$.

Suppose instead that the particle, still attached to $B$, lies on a horizontal platform which performs simple harmonic motion vertically with amplitude $b$ and period $\frac{2\pi}{\omega}$.

At the lowest point of its oscillation, the platform is a distance $h$ below $A$.

Let $x$ be the distance of the particle above the lowest point of the oscillation of the platform. When the particle is in contact with the platform, show that the upward force on the particle from the platform is

$$mg + m\Omega^2(a + x - h) + m\omega^2(b - x).$$

Given that $\omega < \Omega$, show that, if the particle remains in contact with the platform throughout its motion,

$$h \leq a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}.$$

Find the corresponding inequality if $\omega > \Omega$.

Hence show that, if the particle remains in contact with the platform throughout its motion, it is necessary that

$$h \leq a\left(1 + \frac{1}{k}\right) + b\,,$$

whatever the value of $\omega$.

# Section C: Probability and Statistics

## [STEP 3, 2020Q11]

The continuous random variable $X$ is uniformly distributed on $[a,b]$ where $0 < a < b$.

(i) Let $f$ be a function defined for all $x \in [a,b]$

- with $f(a) = b$ and $f(b) = a$,
- which is strictly decreasing on $[a,b]$,
- for which $f(x) = f^{-1}(x)$ for all $x \in [a,b]$.

The random variable $Y$ is defined by $Y = f(X)$. Show that

$$\mathrm{P}(Y \leq y) = \frac{b - f(y)}{b - a} \quad \text{for} \quad y \in [a,b].$$

Find the probability density function for $Y$ and hence show that

$$\mathrm{E}(Y^2) = -ab + \int_a^b \frac{2xf(x)}{b-a}\,\mathrm{d}x.$$

(ii) The random variable $Z$ is defined by $\frac{1}{Z} + \frac{1}{X} = \frac{1}{c}$ where $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$. By finding the variance of $Z$, show that

$$\ln\left(\frac{b-c}{a-c}\right) < \frac{b-a}{c}.$$



## [STEP 3, 2020Q12]

$A$ and $B$ both toss the same biased coin. The probability that the coin shows heads is $p$, where $0 < p < 1$, and the probability that it shows tails is $q = 1 - p$.

Let $X$ be the number of times $A$ tosses the coin until it shows heads. Let $Y$ be the number of times $B$ tosses the coin until it shows heads.

(i) The random variable $S$ is defined by $S = X + Y$ and the random variable $T$ is the maximum of $X$ and $Y$. Find an expression for $\mathrm{P}(S = s)$ and show that



$$\mathrm{P}(T = t) = pq^{t-1}(2 - q^{t-1} - q^t).$$

(ii) The random variable $U$ is defined by $U = |X - Y|$, and the random variable $W$ is the minimum of $X$ and $Y$. Find expressions for $\mathrm{P}(U = u)$ and $\mathrm{P}(W = w)$.

(iii) Show that $\mathrm{P}(S = 2 \text{ and } T = 3) \neq \mathrm{P}(S = 2) \times \mathrm{P}(T = 3)$.

(iv) Show that $U$ and $W$ are independent, and show that no other pair of the four variables $S$, $T$, $U$ and $W$ are independent.

# STEP 3 2021

STEP 3 2021
On-line Exam

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**TIME ALLOWED: 180 MINUTES**

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**Calculators are not permitted.**

# Section A: Pure Mathematics

## [STEP 3, 2021Q1]

(i) A curve has parametric equations

$$x = -4\cos^3 t\,, y = 12\sin t - 4\sin^3 t.$$

Find the equation of the normal to this curve at the point

$$(-4\cos^3\phi\,, 12\sin\phi - 4\sin^3\phi),$$

where $0 < \phi < \frac{1}{2}\pi$.

Verify that this normal is a tangent to the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

at the point $(8\cos^3\phi\,, 8\sin^3\phi)$.

(ii) A curve has parametric equations

$$x = \cos t + t\sin t\,, y = \sin t - t\cos t.$$

Find the equation of the normal to this curve at the point

$$(\cos\phi + \phi\sin\phi\,, \sin\phi - \phi\cos\phi),$$

where $0 < \phi < \frac{1}{2}\pi$.

Determine the perpendicular distance from the origin to this normal, and hence find he equation of a curve, independent of $\phi$, to which this normal is a tangent.

[STEP 3, 2021Q2]

(i) Let

$$x = \frac{a}{b-c}, y = \frac{b}{c-a} \text{ and } a = \frac{c}{a-b},$$

where $a$, $b$ and $c$ are distinct real numbers.

Show that

$$\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and use this result to deduce that $yz + zx + xy = -1$.

Hence show that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geq 2.$$

(ii) Let

$$x = \frac{2a}{b+c}, y = \frac{2b}{c+a} \text{ and } z = \frac{2c}{a+b},$$

where $a$, $b$ and $c$ are positive real numbers.

Using a suitable matrix, show that $xyz + yz + zx + xy = 4$.

Hence show that

$$(2a+b+c)(a+2b+c)(a+b+2c) > 5(b+c)(c+a)(a+b).$$

Show further that

$$(2a+b+c)(a+2b+c)(a+b+2c) > 7(b+c)(c+a)(a+b).$$



[STEP 3, 2021Q3]



(i) Let $I_n = \int_0^\beta (\sec x + \tan x)^n \, \mathrm{d}x$, where $n$ is a non-negative integer and $0 < \beta < \frac{\pi}{2}$.

For $n \geq 1$, show that

$$\frac{1}{2}(I_{n+1} + I_{n-1}) = \frac{1}{n}(\sec\beta + \tan\beta)^n - 1.$$

Show also that

$$I_n < \frac{1}{n}((\sec\beta + \tan\beta)^n - 1).$$

(ii) Let $J_n = \int_0^\beta (\sec x \cos\beta + \tan x)^n \, \mathrm{d}x$, where $n$ is a non-negative integer and $0 < \beta < \frac{\pi}{2}$.

For $n \geq 1$, show that

$$J_n < \frac{1}{n}((1 + \tan\beta)^n - \cos^n\beta).$$

[STEP 3, 2021Q4]

Let $\hat{\mathbf{n}}$ be a vector of unit length and $\Pi$ be the plane through the origin perpendicular to $\hat{\mathbf{n}}$. For any vector $\mathbf{x}$, the *projection* of $\mathbf{x}$ onto the plane $\Pi$ is defined to be the vector $\mathbf{x} - (\mathbf{x}.\hat{\mathbf{n}})\hat{\mathbf{n}}$ .

The vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ each have unit length and the angle between them is $\theta$, which satisfies $0 < \theta < \pi$. The vector $\mathbf{m}$ is given by $\mathbf{m} = \frac{1}{2}(\hat{\mathbf{a}} + \hat{\mathbf{b}})$.

(i) Show that $\mathbf{m}$ bisects the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

(ii) The vector $\hat{\mathbf{c}}$ also has unit length. The angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ is $\alpha$, and the angle between $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ is $\beta$. Both angles are acute and non-zero.

Let $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{b}}_1$ be the projections of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, respectively, onto the plane through the origin perpendicular to $\hat{\mathbf{c}}$. Show that $\hat{\mathbf{a}}_1.\hat{\mathbf{c}} = 0$ and, by considering $|\hat{\mathbf{a}}_1|^2 = \hat{\mathbf{a}}_1.\hat{\mathbf{a}}_1$, show that $|\hat{\mathbf{a}}_1| = \sin\alpha$.

Show also that the angle $\phi$ between $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{b}}_1$ satisfies

$$\cos\phi = \frac{\cos\theta - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

(iii) Let $\mathbf{m}_1$ be the projection of $\mathbf{m}$ onto the plane through the origin perpendicular to $\hat{\mathbf{c}}$.

Show that $\mathbf{m}_1$ bisects the angle between $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{b}}_1$ if and only if

$$\alpha = \beta \text{ or } \cos\theta = \cos(\alpha - \beta).$$



[STEP 3, 2021Q5]

Two curves have polar equations $r = a + 2\cos\theta$ and $r = 2 + \cos 2\theta$, where $r \geq 0$ and $a$ is a constant.

(i) Show that these curves meet when

$$2\cos^2\theta - 2\cos\theta + 1 - a = 0.$$

Hence show that these curves touch if $a = \frac{1}{2}$ and find the other two values of $a$ for which the curves touch.

(ii) Sketch the curves $r = a + 2\cos\theta$ and $r = 2 + \cos 2\theta$ on the same diagram in the case $a = \frac{1}{2}$. Give the values of $r$ and $\theta$ at the points at which the curves touch and justify the other features you show on your sketch.

(iii) On two further diagrams, one for each of the other two values of $a$, sketch both the curves $r = a + 2\cos\theta$ and $r = 2 + \cos 2\theta$. Give the values of $r$ and $\theta$ at the points at which the curves touch and justify the other features you show on your sketch.

[STEP 3, 2021Q6]

(i) For $x \neq \tan\alpha$, the function $f_\alpha$ is defined by

$$f_\alpha(x) = \tan^{-1}\left(\frac{x\tan\alpha + 1}{\tan\alpha - x}\right)$$

where $0 < \alpha < \frac{1}{2}\pi$.

Show that $f_\alpha'(x) = \frac{1}{1+x^2}$.

Hence sketch $y = f_\alpha(x)$.

On a separate diagram, sketch $y = f_\alpha(x) - f_\beta(x)$ where $0 < \alpha < \beta < \frac{1}{2}\pi$.

(ii) For $0 \leq x \leq 2\pi$ and $x \neq \frac{1}{2}\pi, \frac{3}{2}\pi$, the function $g(x)$ is defined by

$$g(x) = \tanh^{-1}(\sin x) - \sinh^{-1}(\tan x).$$

For $\frac{1}{2}\pi < x < \frac{3}{2}\pi$, show that $g'(x) = 2\sec x$.

Use this result to sketch $y = g(x)$ for $0 \leq x \leq 2\pi$.

[STEP 3, 2021Q7]

(i) Let

$$z = \frac{e^{i\theta} + e^{i\phi}}{e^{i\theta} - e^{i\phi}},$$

where $\theta$ and $\phi$ are real, and $\theta - \phi \neq 2n\pi$ for any integer $n$. Show that

$$z = \mathbf{i}\cot\left(\frac{1}{2}(\phi - \theta)\right)$$

and give expressions for the modulus and argument of $z$.

(ii) The distinct points $A$ and $B$ lie on a circle with radius 1 and centre $O$. In the complex plane, $A$ and $B$ are represented by the complex numbers $a$ and $b$, and $O$ is at the origin. The point $X$ is represented by the complex number $x$, where $x = a + b$ and $a + b \neq 0$. Show that $OX$ is perpendicular to $AB$.

If the distinct points $A$, $B$ and $C$ in the complex plane, which are represented by the complex numbers $a$, $b$ and $c$, lie on a circle with radius 1 and centre $O$, and $h = a + b + c$ represents the point $H$, then $H$ is said to be the *orthocentre* of the triangle $ABC$.

(iii) The distinct points $A$, $B$ and $C$ lie on a circle with radius 1 and centre $O$. In the complex plane, $A$, $B$ and $C$ are represented by the complex numbers $a$, $b$ and $c$, and $O$ is at the origin.

Show that, if the point $H$, represented by the complex number $h$, is the orthocentre of the triangle $ABC$, then either $h = a$ or $AH$ is perpendicular to $BC$.

(iv) The distinct points $A$, $B$, $C$ and $D$ (in that order, anticlockwise) all lie on a circle with radius 1 and centre $O$. The points $P$, $Q$, $R$ and $S$ are the orthocentres of the triangles $ABC$, $BCD$, $CDA$ and $DAB$, respectively. By considering the midpoint of $AQ$, show that there is a single transformation which maps the quadrilateral $ABCD$ on to the quadrilateral $QRSP$ and describe this transformation fully.

[STEP 3, 2021Q8]

A sequence $x_1, x_2, \ldots$ of real numbers is defined by $x_{n+1} = x_n^2 - 2$ for $n \geq 1$ and $x_1 = a$.

(i) Show that if $a > 2$ then $x_n \geq 2 + 4^{n-1}(a-2)$.

(ii) Show also that $x_n \longrightarrow \infty$ as $n \to \infty$ if and only if $|a| > 2$.

(iii) When $a > 2$, a second sequence $y_1, y_2, \ldots$ is defined by

$$y_n = \frac{Ax_1x_2 \cdots x_n}{x_{n+1}},$$

where $A$ is a positive constant and $n \geq 1$.

Prove that, for a certain value of $a$, with $a > 2$, which you should find in terms of $A$,

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for all $n \geq 1$.

Determine whether, for this value of $a$, the second sequence converges.

## Section B: Mechanics

### [STEP 3, 2021Q9]

An equilateral triangle $ABC$ has sides of length $a$. The points $P$, $Q$ and $R$ lie on the sides $BC$, $CA$ and $AB$, respectively, such that the length $BP$ is $x$ and $QR$ is parallel to $CB$. Show that

$$(\sqrt{3}\cot\phi + 1)(\sqrt{3}\cot\theta + 1)x = 4(a - x),$$

where $\theta = \angle CPQ$ and $\phi = \angle BRP$.

A horizontal triangular frame with sides of length $a$ and vertices $A$, $B$ and $C$ is fixed on a smooth horizontal table. A small ball is placed at a point $P$ inside the frame, in contact with side $BC$ at a distance $x$ from $B$. It is struck so that it moves round the triangle $PQR$ described above, bouncing off the frame at $Q$ and then $R$ before returning to point $P$. The frame is smooth and the coefficient of restitution between the ball and the frame is $e$.

Show that

$$x = \frac{ae}{1+e}.$$

Show further that if the ball continues to move round $PQR$ after returning to $P$, then $e = 1$.

### [STEP 3, 2021Q10]

The origin $O$ of coordinates lies on a smooth horizontal table and the $x$- and $y$-axes lie in the plane of the table. A cylinder of radius a is fixed to the table with its axis perpendicular to the $x$-$y$ plane and passing through $O$, and with its lower circular end lying on the table. One end, $P$, of a light inextensible string $PQ$ of length $b$ is attached to the bottom edge of the cylinder at $(a, 0)$. The other end, $Q$, is attached to a particle of mass $m$, which rests on the table.

Initially $PQ$ is straight and perpendicular to the radius of the cylinder at $P$, so that $Q$ is at $(a, b)$. The particle is then given a horizontal impulse parallel to the $x$-axis so that the string immediately begins to wrap around the cylinder. At time $t$, the part of the string that is still straight has rotated through an angle $\theta$, where $a\theta < b$.

(i) Obtain the Cartesian coordinates of the particle at this time.

Find also an expression for the speed of the particle in terms of $\theta$, $\dot{\theta}$, $a$ and $b$.

(ii) Show that

$$\dot{\theta}(b - a\theta) = u,$$

where $u$ is the initial speed of the particle.

(iii) Show further that the tension in the string at time $t$ is

$$\frac{mu^2}{\sqrt{b^2 - 2aut}}.$$

# Section C: Probability and Statistics

## [STEP 3, 2021Q11]

The continuous random variable $X$ has probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda$ is a positive constant.

The random variable $Y$ is the greatest integer less than or equal to $X$, and $Z = X - Y$.

(i) Show that, for any non-negative integer $n$,

$$\mathrm{P}(Y = n) = \left(1 - e^{-\lambda}\right)e^{-n\lambda}.$$

(ii) Show that

$$\mathrm{P}(Z < z) = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}} \text{ for } 0 \leq z \leq 1.$$

(iii) Evaluate $\mathrm{E}(Z)$.

(iv) Obtain an expression for

$$\mathrm{P}(Y = n \text{ and } z_1 < Z < z_2),$$

where $0 \leq z_1 < z_2 \leq 1$ and $n$ is a non-negativeinteger.

Determine whether $Y$ and $Z$ are independent.

[STEP 3, 2021Q12]

(i) In a game, each member of a team of $n$ players rolls a fair six-sided die.

The total score of the team is the number of pairs of players rolling the same number. For example, if 7 players roll 3, 3, 3, 3, 6, 6, 2 the total score is 7, as six different pairs of players both score 3 and one pair of players both score 6.

Let $X_{ij}$, for $1 \leq i < j \leq n$, be the random variable that takes the value 1 if players $i$ and $j$ roll the same number and the value 0 otherwise.

Show that $X_{12}$ is independent of $X_{23}$.

Hence find the mean and variance of the team's total score.

(ii) Show that, if $Y_i$, for $1 \leq i \leq m$, are random variables with mean zero, then

$$\operatorname{Var}(Y_1 + Y_2 + \cdots + Y_m) = \sum_{i=1}^{m} \mathrm{E}\big(Y_i^2\big) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \mathrm{E}\big(Y_iY_j\big).$$

(iii) In a different game, each member of a team of $n$ players rolls a fair six-sided die.

The total score of the team is the number of pairs of players rolling the same even number minus the number of pairs of players rolling the same odd number. For example, if 7 players roll 3, 3, 3, 3, 6, 6,2 the total score is $-5$.

Let $Z_{ij}$, for $1 \leq i < j \leq n$, be the random variable that takes the value 1 if players $i$ and $j$ roll the same even number, the value $-1$ if players $i$ and $j$ roll the same odd number and the value 0 otherwise.

Show that $Z_{12}$ is not independent of $Z_{23}$.

Find the mean of the team's total score and show that the variance of the team's total score is $\frac{1}{36}n(n^2 - 1)$.

# STEP 3 2022

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# Section A: Pure Mathematics

## [STEP 3, 2022Q1]

Let $C_1$ be the curve given by the parametric equations

$$x = ct, \qquad y = \frac{c}{t},$$

where $c > 0$ and $t \neq 0$, and let $C_2$ be the circle

$$(x-a)^2 + (y-b)^2 = r^2.$$

$C_1$ and $C_2$ intersect at the four points $P_i$ $(i = 1, 2, 3, 4)$, and the corresponding values of the parameter $t$ at these points are $t_i$.

(i) Show that $t_i$ are the roots of the equation

$$c^2t^4 - 2act^3 + (a^2 + b^2 - r^2)t^2 - 2bct + c^2 = 0. \qquad (*)$$

(ii) Show that

$$\sum_{i=1}^{4} t_i^2 = \frac{2}{c^2}(a^2 - b^2 + r^2)$$

and find a similar expression for $\sum_{i=1}^{4} \frac{1}{t_i^2}$.

(iii) Hence show that $\sum_{i=1}^{4} OP_i^2 = 4r^2$, where $OP_i$ denotes the distance of the point $P_i$ from the origin.

(iv) Suppose that the curves $C_1$ and $C_2$ touch at two distinct points.

By considering the product of the roots of $(*)$, or otherwise, show that the centre of circle $C_2$ must lie on either the line $y = x$ or $y = -x$.

## [STEP 3, 2022Q2]

(i) Suppose that there are three non-zero integers $a$, $b$ and $c$ for which $a^3 + 2b^3 + 4c^3 = 0$. Explain why there must exist an integer $p$, with $|p| < |a|$, such that $4p^3 + b^3 + 2c^3 = 0$, and show further that there must exist integers $p$, $q$ and $r$, with $|p| < |a|$, $|q| < |b|$ and $|r| < |c|$, such that $p^3 + 2q^3 + 4r^3 = 0$. Deduce that no such integers $a$, $b$ and $c$ can exist.

(ii) Prove that there are no non-zero integers $a$, $b$ and $c$ for which $9a^3 + 10b^3 + 6c^3 = 0$.

(iii) By considering the expression $(3n \pm 1)^2$, prove that, unless an integer is a multiple of three, its square is one more than a multiple of 3. Deduce that the sum of the squares of two integers can only be a multiple of three if each of the integers is a multiple of three.

Hence prove that there are no non-zero integers $a$, $b$ and $c$ for which $a^2 + b^2 = 3c^2$.

(iv) Prove that there are no non-zero integers $a$, $b$ and $c$ for which $a^2 + b^2 + c^2 = 4abc$.

[STEP 3, 2022Q3]

(i) The curve $C_1$ has equation

$$ax^2 + bxy + cy^2 = 1$$

where $abc \neq 0$ and $a > 0$.

Show that, if the curve has two stationary points, then $b^2 < 4ac$.

(ii) The curve $C_2$ has equation

$$ay^3 + bx^2y + cx = 1$$

where $abc \neq 0$ and $b > 0$.

Show that the $x$-coordinates of stationary points on this curve satisfy

$$4cb^3x^4 - 8b^3x^3 - ac^3 = 0.$$

Show that, if the curve has two stationary points, then $4ac^6 + 27b^3 > 0$.

(iii) Consider the simultaneous equations

$$\begin{aligned} ay^3 + bx^2y + cx &= 1 \\ 2bxy + c &= 0 \\ 3ay^2 + bx^2 &= 0 \end{aligned}$$

where $abc \neq 0$ and $b > 0$.

Show that, if these simultaneous equations have a solution, then $4ac^6 + 27b^3 = 0$.

[STEP 3, 2022Q4]

*You may assume that all infinite sums and products in this question converge.*

(i) Prove by induction that for all positive integers $n$,

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\sinh\left(\frac{x}{2^n}\right)$$

and deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x}\,\frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right).$$

(ii) You are given that the Maclaurin series for $\sinh x$ is

$$\sinh x = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}.$$

Use this result to show that, as $y$ tends to 0, $\frac{y}{\sinh y}$ tends to 1.

Deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\cdots.$$

(iii) Let $x = \ln 2$. Evaluate $\cosh\left(\frac{x}{2}\right)$ and show that

$$\cosh\left(\frac{x}{4}\right) = \frac{1 + 2^{\frac{1}{2}}}{2 \times 2^{\frac{1}{4}}}.$$

Use part (ii) to show that

$$\frac{1}{\ln 2} = \frac{1 + 2^{\frac{1}{2}}}{2} \times \frac{1 + 2^{\frac{1}{4}}}{2} \times \frac{1 + 2^{\frac{1}{8}}}{2} \cdots.$$

(iv) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2 + \sqrt{2}}}{2} \times \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots.$$

[STEP 3, 2022Q5]

(i) Show that

$$\int_{-a}^{a} \frac{1}{1+e^{x}}\,\mathrm{d}x = a \quad \text{for all } a \geq 0.$$

(ii) Explain why, if $g$ is a continuous function and

$$\int_{0}^{a} g(x)\,\mathrm{d}x = 0 \quad \text{for all } a \geq 0.$$

then $g(x) = 0$ for all $x \geq 0$.

Let $f$ be a continuous function with $f(x) \geq 0$ for all $x$. Show that

$$\int_{-a}^{a} \frac{1}{1+f(x)}\,\mathrm{d}x = a \quad \text{for all } a \geq 0$$

if and only if

$$\frac{1}{1+f(x)} + \frac{1}{1+f(-x)} - 1 = 0 \quad \text{for all } a \geq 0.$$

and hence if and only if $f(x)f(-x) = 1$ for all $x$.

(iii) Let $f$ be a continuous function such that, for all $x$, $f(x) \geq 0$ and $f(x)f(-x) = 1$. Show that, if $h$ is a continuous function with $h(x) = h(-x)$ for all $x$, then

$$\int_{-a}^{a} \frac{h(x)}{1+f(x)}\,\mathrm{d}x = \int_{0}^{a} h(x)\,\mathrm{d}x.$$



(iv) Hence find the exact value of

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\mathrm{e}^{-x}\cos x}{\cosh x}\,\mathrm{d}x.$$

[STEP 3, 2022Q6]

(i) Show that when $\alpha$ is small, $\cos(\theta + \alpha) - \cos\theta \approx -\alpha\sin\theta - \frac{1}{2}\alpha^2\cos\theta$.

Find the limit as $\alpha \to 0$ of

$$\frac{\sin(\theta + \alpha) - \sin\theta}{\cos(\theta + \alpha) - \cos\theta} \qquad (*)$$

in the case $\sin\theta \neq 0$.

In the case $\sin\theta = 0$, what happens to the value of expression $(*)$ when $\alpha \to 0$?

(ii) A circle $C_1$ of radius $a$ rolls without slipping in an anti-clockwise direction on a fixed circle $C_2$ with centre at the origin $O$ and radius $(n-1)a$, where $n$ is an integer greater than 2. The point $P$ is fixed on $C_1$. Initially the centre of $C_1$ is at $(na, 0)$ and $P$ is at $\big((n+1)a, 0\big)$.

(a) Let $Q$ be the point of contact of $C_1$ and $C_2$ at any time in the rolling motion. Show that when $OQ$ makes an angle $\theta$, measured anticlockwise, with the positive $x$-axis, the $x$-coordinate of $P$ is $x(\theta) = a(n\cos\theta + \cos n\theta)$, and find the corresponding expression for the $y$-coordinate, $y(\theta)$, of $P$.

(b) Find the values of $\theta$ for which the distance $OP$ is $(n-1)a$.

(c) Let $\theta_0 = \frac{1}{n-1}\pi$. Find the limit as $\alpha \to 0$ of

$$\frac{y(\theta_0 + \alpha) - y(\theta_0)}{x(\theta_0 + \alpha) - x(\theta_0)}.$$

Hence show that, at the point $\big(x(\theta_0), y(\theta_0)\big)$, the tangent to the curve traced out by $P$ is parallel to $OP$.

[STEP 3, 2022Q7]

Let $\mathbf{n}$ be a vector of unit length in three dimensions. For each vector $\mathbf{r}$, $f(\mathbf{r})$ is defined by $f(\mathbf{r}) = \mathbf{n} \times \mathbf{r}$.

(i) Given that

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

show that the $x$-component of $f\big(f(\mathbf{r})\big)$ is $-x(b^2 + c^2) + aby + acz$. Show further that

$$f\big(f(\mathbf{r})\big) = (\mathbf{n}.\mathbf{r})\mathbf{n} - \mathbf{r}.$$

Explain, by means of a diagram, how $f\big(f(\mathbf{r})\big)$ is related to $\mathbf{n}$ and $\mathbf{r}$.

(ii) Let $R$ be the point with position vector $\mathbf{r}$ and $P$ be the point with position vector $g(\mathbf{r})$, where $g$ is defined by

$$g(\mathbf{s}) = \mathbf{s} + \sin\theta\, f(\mathbf{s}) + (1 - \cos\theta) f\big(f(\mathbf{s})\big).$$

By considering $g(\mathbf{n})$ and $g(\mathbf{r})$ when $\mathbf{r}$ is perpendicular to $\mathbf{n}$, state, with justification, the geometric transformation which maps $R$ onto $P$.

(iii) Let $R$ be the point with position vector $\mathbf{r}$ and $Q$ be the point with position vector $h(\mathbf{r})$, where $h$ is defined by

$$h(\mathbf{s}) = -\mathbf{s} - 2f\big(f(\mathbf{s})\big).$$

State, with justification, the geometric transformation which maps $R$ onto $Q$.



[STEP 3, 2022Q8]

(i) Use De Moivre's theorem to prove that for any positive integer $k > 1$,

$$\sin(k\theta) = \sin\theta \cos^{k-1}\theta \left(k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right)$$

and find a similar expression for $\cos(k\theta)$.

(ii) Let $\theta = \cos^{-1}\left(\frac{1}{a}\right)$, where $\theta$ is measured in degrees, and $a$ is an odd integer greater than 1.

Suppose that there is a positive integer $k$ such that $\sin(k\theta) = 0$ and $\sin(m\theta) \neq 0$ for all integers $m$ with $0 < m < k$.

Show that it would be necessary to have $k$ even and $\cos\left(\frac{1}{2}k\theta\right) = 0$.

Deduce that $\theta$ is irrational.

(iii) Show that if $\phi = \cot^{-1}\left(\frac{1}{b}\right)$, where $\phi$ is measured in degrees, and $b$ is an even integer greater than 1, then $\phi$ is irrational.

## Section B: Mechanics

### [STEP 3, 2022Q9]

(i) Two particles $A$ and $B$, of masses $m$ and $km$ respectively, lie at rest on a smooth horizontal surface. The coefficient of restitution between the particles is $e$, where $0 < e < 1$. Particle $A$ is then projected directly towards particle $B$ with speed $u$.

Let $v_1$ and $v_2$ be the velocities of particles $A$ and $B$, respectively, after the collision, in the direction of the initial velocity of $A$.

Show that $v_1 = \alpha u$ and $v_2 = \beta u$, where $\alpha = \frac{1-ke}{k+1}$ and $\beta = \frac{1+e}{k+1}$.

Particle $B$ strikes a vertical wall which is perpendicular to its direction of motion and a distance $D$ from the point of collision with $A$, and rebounds. The coefficient of restitution between particle $B$ and the wall is also $e$.

Show that, if $A$ and $B$ collide for a second time at a point $\frac{1}{2}D$ from the wall, then $k = \frac{1+e-e^2}{e(2e+1)}$.

(ii) Three particles $A$, $B$ and $C$, of masses $m$, $km$ and $k^2m$ respectively, lie at rest on a smooth horizontal surface in a straight line, with $B$ between $A$ and $C$. A vertical wall is perpendicular to this line and lies on the side of $C$ away from $A$ and $B$. The distance between $B$ and $C$ is equal to $d$ and the distance between $C$ and the wall is equal to $3d$. The coefficient of restitution between each pair of particles, and between particle $C$ and the wall, is $e$, where $0 < e < 1$. Particle $A$ is then projected directly towards particle $B$ with speed $u$.

Show that, if all three particles collide simultaneously at a point $\frac{3}{2}d$ from the wall, then $e = \frac{1}{2}$.

### [STEP 3, 2022Q10]

Two light elastic springs each have natural length $a$. One end of each spring is attached to a particle $P$ of weight $W$. The other ends of the springs are attached to the end-points, $B$ and $C$, of a fixed horizontal bar $BC$ of length $2a$. The moduli of elasticity of the springs $PB$ and $PC$ are $s_1W$ and $s_2W$ respectively; these values are such that the particle $P$ hangs in equilibrium with angle $BPC$ equal to $90°$.

(i) Let angle $PBC = \theta$. Show that $s_1 = \frac{\sin\theta}{2\cos\theta - 1}$ and find $s_2$ in terms of $\theta$.

(ii) Take the zero level of gravitational potential energy to be the horizontal bar $BC$ and let the total potential energy of the system be $-paW$. Show that $p$ satisfies

$$\frac{1}{2}\sqrt{2} \geq p > \frac{1}{4}(1+\sqrt{3})$$

and hence that $p = 0.7$, correct to one significant figure.

## Section C: Probability and Statistics

### [STEP 3, 2022Q11]

A fair coin is tossed $N$ times and the random variable $X$ records the number of heads. The mean deviation, $\delta$, of $X$ is defined by

$$\delta = \mathrm{E}(|X - \mu|)$$

where $\mu$ is the mean of $X$.

(i) Let $N = 2n$ where $n$ is a positive integer.

(a) Show that $\mathrm{P}(X \le n-1) = \frac{1}{2}\big(1 - \mathrm{P}(X = n)\big)$.

(b) Show that

$$\delta = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\frac{1}{2^{2n-1}}.$$

(c) Show that for $r > 0$,

$$r\binom{2n}{r} = 2n\binom{2n-1}{r-1}.$$

Hence show that

$$\delta = \frac{n}{2^{2n}}\binom{2n}{n}.$$

(ii) Find a similar expression for $\delta$ in the case $N = 2n+1$.

[STEP 3, 2022Q12]

(i) The point $A$ lies on the circumference of a circle of radius $a$ and centre $O$. The point $B$ is chosen at random on the circumference, so that the angle $AOB$ has a uniform distribution on $[0, 2\pi]$. Find the expected length of the chord $AB$.

(ii) The point $C$ is chosen at random in the interior of a circle of radius $a$ and centre $O$, so that the probability that it lies in any given region is proportional to the area of the region. The random variable $R$ is defined as the distance between $C$ and $O$.

Find the probability density function of $R$.

Obtain a formula in terms of $a$, $R$ and $t$ for the length of a chord through $C$ that makes an acute angle of $t$ with $OC$.

Show that as $C$ varies (with $t$ fixed), the expected length $L(t)$ of such chords is given by

$$L(t) = \frac{4a(1-\cos^3 t)}{3\sin^2 t}.$$

Show further that

$$L(t) = \frac{4a}{3}\left(\cos t + \frac{1}{2}\sec^2\left(\frac{1}{2}t\right)\right).$$

(iii) The random variable $T$ is uniformly distributed on $\left[0, \frac{1}{2}\pi\right]$. Find the expected value of $L(T)$.

# STEP 3 2023

STEP 3 2023
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

**TIME ALLOWED: 180 MINUTES**

## INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

## INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

| | |
|---|---|
| **Section A** | Pure Mathematics |
| **Section B** | Mechanics |
| **Section C** | Probability and Statistics |

There are 12 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

**There is NO Mathematical Formulae booklet.**

**Calculators are not permitted.**

## Section A: Pure Mathematics

### [STEP 3, 2023Q1]

The distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the curve $x^2 = 4ay$, where $a > 0$.

(i) Given that

$$(p+q)^2 = p^2q^2 + 6pq + 5, \qquad (*)$$

show that the line through $P$ and $Q$ is a tangent to the circle with centre $(0, 3a)$ and radius $2a$.

(ii) Show that, for any given value of $p$ with $p^2 \neq 1$, there are two distinct real values of $q$ that satisfy equation $(*)$.

Let these values be $q_1$ and $q_2$. Find expressions, in terms of $p$, for $q_1 + q_2$ and $q_1q_2$.

(iii) Show that, for any given value of $p$ with $p^2 \neq 1$, there is a triangle with one vertex at $P$ such that all three vertices lie on the curve $x^2 = 4ay$ and all three sides are tangents to the circle with centre $(0, 3a)$ and radius $2a$.

[STEP 3, 2023Q2]

The polar curves $C_1$ and $C_2$ are defined for $0 \leq \theta \leq \pi$ by

$$r = k(1 + \sin\theta)$$
$$r = k + \cos\theta$$

respectively, where $k$ is a constant greater than 1.

(i) Sketch the curves on the same diagram. Show that if $\theta = \alpha$ at the point where the curves intersect, $\tan\alpha = \frac{1}{k}$.

(ii) The region A is defined by the inequalities

$$0 \leq \theta \leq \alpha \quad \text{and} \quad r \leq k(1 + \sin\theta).$$

Show that the area of A can be written as

$$\frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1 - \cos\alpha).$$

(iii) The region B is defined by the inequalities

$$\alpha \leq \theta \leq \pi \quad \text{and} \quad r \leq k + \cos\theta\,.$$

Find an expression in terms of $k$ and $\alpha$ for the area of B.

(iv) The total area of regions A and B is denoted by $R$. The area of the region enclosed by $C_1$ and the lines $\theta = 0$ and $\theta = \pi$ is denoted by $S$. The area of the region enclosed by $C_2$ and the lines $\theta = 0$ and $\theta = \pi$ is denoted by $T$.

Show that as $k \to \infty$,

$$\frac{R}{T} \to 1$$

and find the limit of

$$\frac{R}{S}$$

as $k \to \infty$.

[STEP 3, 2023Q3]

(i) Show that, if $a$ and $b$ are complex numbers, with $b \neq 0$, and $s$ is a positive real number, then the points in the Argand diagram representing the complex numbers $a + sb\text{i}$, $a - sb\text{i}$ and $a + b$ form an isosceles triangle.

Given three points which form an isosceles triangle in the Argand diagram, explain with the aid of a diagram how to determine the values of $a$, $b$ and $s$ so that the vertices of the triangle represent complex numbers $a + sb\text{i}$, $a - sb\text{i}$ and $a + b$.

(ii) Show that, if the roots of the equation $z^3 + pz + q = 0$, where $p$ and $q$ are complex numbers, are represented in the Argand diagram by the vertices of an isosceles triangle, then there is a non-zero real number $s$ such that

$$\frac{p^3}{q^2} = \frac{27(3s^2 - 1)^3}{4(9s^2 + 1)^2}.$$

(iii) Sketch the graph $y = \frac{(3x-1)^3}{(9x+1)^2}$, identifying any stationary points.

(iv) Show that if the roots of the equation $z^3 + pz + q = 0$ are represented in the Argand diagram by the vertices of an isosceles triangle then $\frac{p^3}{q^2}$ is a real number and $\frac{p^3}{q^2} > -\frac{27}{4}$.

[STEP 3, 2023Q4]

Let $n$ be a positive integer. The polynomial $p$ is defined by the identity

$$p(\cos\theta) \equiv \cos\big((2n+1)\theta\big) + 1.$$



(i) Show that

$$\cos\big((2n+1)\theta\big) = \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta\,(\cos^2\theta - 1)^r.$$

(ii) By considering the expansion of $(1+t)^{2n+1}$ for suitable values of $t$, show that the coefficient of $x^{2n+1}$ in the polynomial $p(x)$ is $2^{2n}$.

(iii) Show that the coefficient of $x^{2n-1}$ in the polynomial $p(x)$ is $-(2n+1)2^{2n-2}$.

(iv) It is given that there exists a polynomial $q$ such that

$$p(x) = (x+1)[q(x)]^2$$

and the coefficient of $x^n$ in $q(x)$ is greater than 0.

Write down the coefficient of $x^n$ in the polynomial $q(x)$ and, for $n \geq 2$, show that the coefficient of $x^{n-2}$ in the polynomial $q(x)$ is

$$2^{n-2}(1-n).$$

[STEP 3, 2023Q5]

(i) Show that if

$$\frac{1}{x}+\frac{2}{y}=\frac{2}{7},$$

then $(2x-7)(y-7)=49$.

By considering the factors of 49, find all the pairs of positive integers $x$ and $y$ such that

$$\frac{1}{x}+\frac{2}{y}=\frac{2}{7}.$$

(ii) Let $p$ and $q$ be prime numbers such that

$$p^2+pq+q^2=n^2$$

where $n$ is a positive integer. Show that

$$(p+q+n)(p+q-n)=pq$$

and hence explain why $p+q=n+1$.

Hence find the possible values of $p$ and $q$.

(iii) Let $p$ and $q$ be positive and

$$p^3+q^3+3pq^2=n^3.$$

Show that $p+q-n<p$ and $p+q-n<q$.

Show that there are no prime numbers $p$ and $q$ such that $p^3+q^3+3pq^2$ is the cube of an integer.

[STEP 3, 2023Q6]

(i) By considering the Maclaurin series for $\mathrm{e}^x$, show that for all real $x$,

$$\cosh^2 x \geq 1+x^2.$$

Hence show that the function $f$, defined for all real $x$ by $f(x)=\tan^{-1}x-\tanh x$, is an increasing function.

Sketch the graph $y=f(x)$.

(ii) Function $g$ is defined for all real $x$ by $g(x)=\tan^{-1}x-\frac{1}{2}\pi\tanh x$.

(a) Show that $g$ has at least two stationary points.

(b) Show, by considering its derivative, that $(1+x^2)\sinh x - x\cosh x$ is non-negative for $x\geq 0$.

(c) Show that $\frac{\cosh^2 x}{1+x^2}$ is an increasing function for $x \geq 0$.

(d) Hence or otherwise show that $g$ has exactly two stationary points.

(e) Sketch the graph $y=g(x)$.

[STEP 3, 2023Q7]

(i) Let $f$ be a continuous function defined for $0 \leq x \leq 1$. Show that

$$\int_0^1 f(\sqrt{x})\,\mathrm{d}x = 2\int_0^1 xf(x)\,\mathrm{d}x.$$

(ii) Let $g$ be a continuous function defined for $0 \leq x \leq 1$ such that

$$\int_0^1 \big(g(x)\big)^2\,\mathrm{d}x = \int_0^1 g(\sqrt{x})\,\mathrm{d}x - \frac{1}{3}.$$

Show that $\int_0^1 (g(x) - x)^2\,\mathrm{d}x = 0$ and explain why $g(x) = x$ for $0 \leq x \leq 1$.

(iii) Let $h$ be a continuous function defined for $0 \leq x \leq 1$ with derivative $h'$ such that

$$\int_0^1 \big(h'(x)\big)^2\,\mathrm{d}x = 2h(1) - 2\int_0^1 h(x)\,\mathrm{d}x - \frac{1}{3}.$$

Given that $h(0) = 0$, find $h$.

(iv) Let $k$ be a continuous function defined for $0 \leq x \leq 1$ and $a$ be a real number, such that

$$\int_0^1 \mathrm{e}^{ax}\big(k(x)\big)^2\,\mathrm{d}x = 2\int_0^1 k(x)\,\mathrm{d}x + \frac{\mathrm{e}^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}.$$

Show that $a$ must be equal to 2 and find $k$.

[STEP 3, 2023Q8]

If

$$y = \begin{cases} k_1(x) & x \le b \\ k_2(x) & x \ge b \end{cases}$$

with $k_1(b) = k_2(b)$, then $y$ is said to be continuously differentiable at $x = b$ if $k_1'(b) = k_2'(b)$.

(i) Let $f(x) = x\mathrm{e}^{-x}$. Verify that, for all real $x$, $y = f(x)$ is a solution to the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

and that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Show that $f'(x) \ge 0$ for $x \le 1$.

(ii) You are given the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + y = 0$$

where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$. Let

$$y = \begin{cases} g_1(x) & x \le 1 \\ g_2(x) & x \ge 1 \end{cases}$$

be a solution of the differential equation which is continuously differentiable at $x = 1$.

Write down an expression for $g_1(x)$ and find an expression for $g_2(x)$.

(iii) State the geometrical relationship between the curves $y = g_1(x)$ and $y = g_2(x)$.

(iv) Prove that if $y = k(x)$ is a solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0$$

in the interval $r \le x \le s$, where $p$ and $q$ are constants, then, in a suitable interval which you should state, $y = k(c - x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0.$$

(v) You are given the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0$$

where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Let $h(x) = \mathrm{e}^{-x}\sin x$. Show that $h'\left(\frac{1}{4}\pi\right) = 0$.

It is given that $y = h(x)$ satisfies the differential equation in the interval $-\frac{3}{4}\pi \le x \le \frac{1}{4}\pi$ and that $h'(x) \ge 0$ in this interval.

In a solution to the differential equation which is continuously differentiable at $\left(n + \frac{1}{4}\right)\pi$ for all $n \in \mathbb{Z}$, find $y$ in terms of $x$ in the intervals

(a) $\frac{1}{4}\pi \leq x \leq \frac{5}{4}\pi$,

(b) $\frac{5}{4}\pi \leq x \leq \frac{9}{4}\pi$.

## Section B: Mechanics

### [STEP 3, 2023Q9]

Two particles, $A$ of mass $m$ and $B$ of mass $M$, are fixed to the ends of a light inextensible string $AB$ of length $r$ and lie on a smooth horizontal plane. The origin of coordinates and the $x$- and $y$-axes are in the plane.

Initially, $A$ is at $(0,0)$ and $B$ is $(r,0)$. $B$ is at rest and $A$ is given an instantaneous velocity of magnitude $u$ in the positive $y$ direction.

At a time $t$ after this, $A$ has position $(x,y)$ and $B$ has position $(X,Y)$. You may assume that, in the subsequent motion, the string remains taut.

(i) Explain by means of a diagram why

$$X = x + r\cos\theta$$
$$Y = y - r\sin\theta$$

where $\theta$ is the angle clockwise from the positive $x$-axis of the vector $\overrightarrow{AB}$.

(ii) Find expressions for $\dot{X}, \dot{Y}, \ddot{X}$ and $\ddot{Y}$ in terms of $\ddot{x}, \ddot{y}, \dot{x}, \dot{y}, r, \ddot{\theta}, \dot{\theta}$ and $\theta$, as appropriate.

Assume that the tension $T$ in the string is the only force acting on either particle.

(iii) Show that

$$\ddot{x}\sin\theta + \ddot{y}\cos\theta = 0$$
$$\ddot{X}\sin\theta + \ddot{Y}\cos\theta = 0$$

and hence that $\theta = \frac{ut}{r}$.

(iv) Show that

$$m\ddot{x} + M\ddot{X} = 0$$
$$m\ddot{y} + M\ddot{Y} = 0$$

and find $my + MY$ in terms of $t$ and $m, M, u, r$ as appropriate.

(v) Show that

$$y = \frac{1}{m+M}\left(mut + Mr\sin\left(\frac{ut}{r}\right)\right).$$

(vi) Show that, if $M > m$, then the $y$ component of the velocity of particle $A$ will be negative at some time in the subsequent motion.

[STEP 3, 2023Q10]

A thin uniform beam $AB$ has mass $3m$ and length $2h$. End $A$ rests on rough horizontal ground and the beam makes an angle of $2\beta$ to the vertical, supported by a light inextensible string attached to end $B$. The coefficient of friction between the beam and the ground at $A$ is $\mu$.

The string passes over a small frictionless pulley fixed to a point $C$ which is a distance $2h$ vertically above $A$. A particle of mass $km$, where $k < 3$, is attached to the other end of the string and hangs freely.

(i) Given that the system is in equilibrium, find an expression for $k$ in terms of $\beta$ and show that $k^2 \leq \frac{9\mu^2}{\mu^2+1}$.

(ii) A particle of mass $m$ is now fixed to the beam at a distance $xh$ from $A$, where $0 \leq x \leq 2$. Given that $k = 2$, and that the system is in equilibrium, show that

$$\frac{F^2}{N^2} = \frac{x^2 + 6x + 5}{4(x + 2)^2},$$

where $F$ is the frictional force and $N$ is the normal reaction on the beam at $A$.

By considering $\frac{1}{3} - \frac{F^2}{N^2}$, or otherwise, find the minimum value of $\mu$ for which the beam can be in equilibrium whatever the value of $x$.

## Section C: Probability and Statistics

### [STEP 3, 2023Q11]

Show that

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k = (x+1)\mathrm{e}^x - 1.$$

In the remainder of this question, $n$ is a fixed positive integer.

(i) Random variable $Y$ has a Poisson distribution with mean $n$. One observation of $Y$ is taken. Random variable $D$ is defined as follows. If the observed value of $Y$ is zero then $D = 0$. If the observed value of $Y$ is $k$, where $k \geq 1$, then a fair $k$-sided die (with sides numbered 1 to $k$ ) is rolled once and $D$ is the number shown on the die.

(a) Write down $\mathrm{P}(D = 0)$.

(b) Show, from the definition of the expectation of a random variable, that

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} \left[ d \sum_{k=d}^{\infty} \left( \frac{1}{k} \cdot \frac{n^k}{k!} \mathrm{e}^{-n} \right) \right].$$

Show further that

$$\mathrm{E}(D) = \sum_{k=1}^{\infty} \left( \frac{1}{k} \cdot \frac{n^k}{k!} \mathrm{e}^{-n} \sum_{d=1}^{k} d \right).$$

(c) Show that $\mathrm{E}(D) = \frac{1}{2}(n + 1 - \mathrm{e}^{-n})$.

(ii) Random variables $X_1, X_2, \ldots, X_n$ all have Poisson distributions. For each $k \in \{1,2,\ldots,n\}$, the mean of $X_k$ is $k$.

A fair $n$-sided die, with sides numbered 1 to $n$, is rolled. When $k$ is the number shown, one observation of $X_k$ is recorded. Let $Z$ be the number recorded.

(a) Find $\mathrm{P}(Z = 0)$.

(b) Show that $\mathrm{E}(Z) > \mathrm{E}(D)$.

[STEP 3, 2023Q12]

A drawer contains $n$ pairs of socks. The two socks in each pair are indistinguishable, but each pair of socks is a different colour from all the others. A set of $2k$ socks, where $k$ is an integer with $2k \leq n$, is selected at random from this drawer: that is, every possible set of $2k$ socks is equally likely to be selected.

(i) Find the probability that, among the socks selected, there is no pair of socks.

(ii) Let $X_{n,k}$ be the random variable whose value is the number of pairs of socks found amongst those selected. Show that

$$\mathrm{P}(X_{n,k} = r) = \frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}$$

for $0 \leq r \leq k$.

(iii) Show that

$$r\mathrm{P}(X_{n,k} = r) = \frac{k(2k-1)}{2n-1}\mathrm{P}(X_{n-1,k-1} = r-1),$$

for $1 \leq r \leq k$, and hence find $\mathrm{E}(X_{n,k})$.