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CAMBRIDGE SIXTH TERM EXAM PAPER

STEP 2

PAST PAPERS

1987-2023

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STEP 2 1987

STEP 2 1987
[On-line Exam]

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TIME ALLOWED: 180 MINUTES

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Calculators are not permitted.

[STEP 2, 1987Q1]

Prove that:

(i) if $a + 2b + 3c = 7x$, then

$$a^2 + b^2 + c^2 = (x - a)^2 + (2x - b)^2 + (3x - c)^2.$$

(ii) if $2a + 3b + 3c = 11x$, then

$$a^2 + b^2 + c^2 = (2x - a)^2 + (3x - b)^2 + (3x - c)^2.$$

Give a general result of which (i) and (ii) are special cases.

[STEP 2, 1987Q2]

Show that if at least one of the four angles $A \pm B \pm C$ is a multiple of π, then

$$\sin^4 A + \sin^4 B + \sin^4 C - 2\sin^2 B \sin^2 C - 2\sin^2 C \sin^2 A - 2\sin^2 A \sin^2 B + 4\sin^2 A \sin^2 B \sin^2 C = 0.$$

[STEP 2, 1987Q3]

Let a and b be positive integers such that $b < 2a - 1$. For any given positive integer n, the integers N and M are defined by

$$\left[a + \sqrt{a^2 - b}\right]^n = N - r,$$

$$\left[a - \sqrt{a^2 - b}\right]^n = M + s,$$

where $0 \le r < 1$ and $0 \le s < 1$. Prove that

(i) $M = 0$,

(ii) $r = s$,

(iii) $r^2 - Nr + b^n = 0$.

Show that for large n, $\left(8 + 3\sqrt{7}\right)^n$ differs from an integer by about 2^{-4n}.

[STEP 2, 1987Q4]

Explain the geometrical relationship between the points in the Argand diagram represented by the complex numbers z and $z\mathrm{e}^{\mathrm{i}\theta}$.

Write down necessary and sufficient conditions that the distinct complex numbers α, β and γ represent the vertices of an equilateral triangle taken in anticlockwise order.

Show that α, β and γ represent the vertices of an equilateral triangle (taken in any order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.$$

Find necessary and sufficient conditions on the complex coefficients a. b and c for the roots of the equation

$$z^3 + az^2 + bz + c = 0$$

to lie at the vertices of an equilateral triangle in the Argand diagram.

[STEP 2, 1987Q5]

If $y = f(x)$, then the inverse of f (when it exists) can be obtained from Lagrange's identity. This identity, which you may use without proof, is

$$f^{-1}(y) = y + \sum_{1}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} [y - f(y)]^n,$$

provided the series converges.

(i) Verify Lagrange's identity when $f(x) = \alpha x$, $(0 < \alpha < 2)$.

(ii) Show that one root of the equation

$$\frac{1}{2} = x - \frac{1}{4}x^3$$

is

$$x = \sum_{0}^{\infty} \frac{(3n)!}{n!\,(2n+1)!\,2^{4n+1}}.$$

(iii) Find a solution for x, as a series in λ, of the equation

$$x = \mathrm{e}^{\lambda x}.$$

[You may assume that the series in part (ii) converges, and that the series in part (iii) converges for suitable λ.]

[STEP 2, 1987Q6]

Let

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1-\sin\theta\sin 2\alpha}\,\mathrm{d}\theta,$$

where $0 < \alpha < \frac{1}{4}\pi$. Show that

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1+\sin\theta\sin 2\alpha}\,\mathrm{d}\theta,$$

and hence that

$$I = \frac{\pi}{\sin^2 2\alpha} - \cot^2 2\alpha \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2\theta}{1+\cos^2 2\alpha\tan^2\theta}\,\mathrm{d}\theta.$$

Show that $I = \frac{1}{2}\pi\sec^2\alpha$, and state the value of I if $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$.

[STEP 2, 1987Q7]

A definite integral can be evaluated approximately by means of the Trapezium rule:

$$\int_{x_0}^{x_N} f(x)\,\mathrm{d}x \approx \frac{1}{2}h\{f(x_0)+2f(x_1)+\cdots+\ 2f(x_{N-1})+f(x_N)\},$$

where the interval length h is given by $Nh = x_N - x_0$, and $x_r = x_0 + rh$. Justify briefly this approximation.

Use the Trapezium rule with intervals of unit length to evaluate approximately the integral

$$\int_1^n \ln x\,\mathrm{d}x,$$

where $n\ (> 2)$ is an integer. Deduce that $n! \approx g(n)$, where

$$g(n) = n^{n+\frac{1}{2}}\mathrm{e}^{1-n},$$

and show by means of a sketch, or otherwise, that

$$n! < g(n).$$

By using the Trapezium rule on the above integral with intervals of width k^{-1}, where k is a positive integer, show that

$$(kn)! \approx k!\,n^{kn+\frac{1}{2}}\left(\frac{\mathrm{e}}{k}\right)^{k(1-n)}.$$

Determine whether this approximation or $g(kn)$ is closer to $(kn)!$.

[STEP 2, 1987Q8]

Let $\mathbf{r}$ be the position vector of a point in three-dimensional space. Describe fully the locus of the point whose position vector is $\mathbf{r}$ in each of the following four cases:

(i) $(\mathbf{a}-\mathbf{b}).\mathbf{r} = \frac{1}{2}(|\mathbf{a}|^2 - |\mathbf{b}|^2)$.

(ii) $(\mathbf{a}-\mathbf{r}).(\mathbf{b}-\mathbf{r}) = 0$.

(iii) $|\mathbf{r}-\mathbf{a}|^2 = \frac{1}{2}|\mathbf{a}-\mathbf{b}|^2$.

(iv) $|\mathbf{r}-\mathbf{b}|^2 = \frac{1}{2}|\mathbf{a}-\mathbf{b}|^2$.

Prove algebraically that the equations (i) and (ii) together are equivalent to (iii) and (iv) together. Explain carefully the geometric meaning of this equivalence.

[STEP 2, 1987Q9]

For any square matrix $\mathbf{A}$ such that $\mathbf{I}-\mathbf{A}$ is non-singular (where $\mathbf{I}$ is the unit matrix), the matrix $\mathbf{B}$ is defined by

$$\mathbf{B} = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}.$$

Prove that $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$ if and only if $\mathbf{A}+\mathbf{A}^{\mathrm{T}} = \mathbf{O}$ (where $\mathbf{O}$ is the zero matrix), explaining clearly each step of your proof.

[You may quote standard results about matrices without proof.]

[STEP 2, 1987Q10]

The set S consists of N (>2) elements $a_1, a_2, \ldots, a_N$. S is acted upon by a binary operation $\diamond$, defined by

$$a_j \diamond a_k = a_m\,,$$

where m is equal to the greater of j and k.

Determine, giving reasons, which of the four group axioms hold for S under $\diamond$, and which do not.

Determine also, giving reasons, which of the group axioms hold for S under $*$, where $*$ is defined by

$$a_j * a_k = a_n\,,$$

where $n = |j-k|+1$.

[STEP 2, 1987Q11]

A rough ring of radius a is fixed so that it lies in a plane inclined at an angle α to the horizontal. A uniform heavy rod of length b $(> a)$ has one end smoothly pivoted at the centre of the ring, so that the rod is free to move in any direction. It rests on the circumference of the ring, making an angle θ with the radius to the highest point on the circumference. Find the relation between α, θ and the coefficient of friction, μ, which must hold when the rod is in limiting equilibrium.

[STEP 2, 1987Q12]

A long, light, inextensible string passes through a small fixed ring. One end of the string is attached to a particle of mass m, which hangs freely. The other end is attached to a bead also of mass m which is threaded on a smooth rigid wire fixed in the same vertical plane as the ring. The curve of the wire is such that the system can be in static equilibrium for all positions of the bead. The shortest distance between the wire and the ring is d (> 0). Using plane polar coordinates centred on the ring, find the equation of the curve.

The bead is set in motion. Assuming that the string remains taut, show that the speed of the bead when it is a distance r from the ring is $\left(\frac{r}{2r-d}\right)^{\frac{1}{2}} v$, where v is the speed of the bead when $r = d$.

[STEP 2, 1987Q13]

Ice snooker is played on a rectangular horizontal table, of length L and width B, on which a small disc (the *puck*) slides without friction. The table is bounded by smooth vertical walls (the *cushions*) and the coefficient of restitution between the puck and any cushion is e. If the puck is hit so that it bounces off two adjacent cushions, show that its final path (after two bounces) is parallel to its original path.

The puck rests against the cushion at a point which divides the side of length L in the ratio z : 1. Show that it is possible, whatever z, to hit the puck so that it bounces off the three other cushions in succession clockwise and returns to the spot at which it started.

By considering these paths as z varies, explain briefly why there are two different ways in which, starting at any point away from the cushions, it is possible to perform a shot in which the puck bounces off all four cushions in succession clockwise and returns to its starting point.

[STEP 2, 1987Q14]

A thin uniform elastic band of mass m, length l and modulus of elasticity λ is pushed on to a smooth circular cone of vertex angle 2α, in such a way that all elements of the band are the same distance from the vertex. It is then released from rest. Let $x(t)$ be the length of the band at time t after release, and let t_0 be the time at which the band becomes slack.

Assuming that a small element of the band which subtends an angle $\delta\theta$ at the axis of the cone experiences a force, due to the tension T in the band, of magnitude $T\delta\theta$ directed towards the axis, and ignoring the effects of gravity, show that

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{4\pi^2\lambda}{ml}(x - l)\sin^2\alpha = 0, \qquad (0 < t < t_0).$$

Find the value of t_0.

[STEP 2, 1987Q15]

A train of length l_1 and a lorry of length l_2 are heading for a level crossing at speeds u_1 and u_2 respectively. Initially the front of the train and the front of the lorry are at distances d_1, and d_2 from the crossing. Find conditions on u_1 and u_2 under which a collision will occur. On a diagram with u_1 and u_2 measured along the x and y axes respectively, shade in the region which represents collision.

Hence show that if u_1 and u_2 are two independent random variables, both uniformly distributed on $(0, V)$, then the probability of a collision in the case when initially the back of the train is nearer to the crossing than the front of the lorry is

$$\frac{l_1 l_2 + l_2 d_1 + l_1 d_2}{2d_2(l_2 + d_2)}.$$

Find the probability of a collision in each of the other two possible cases.

[STEP 2, 1987Q16]

My two friends, who shall remain nameless, but whom I shall refer to as P and Q, both told me this afternoon that there is a body in my fridge. I'm not sure what to make of this, because P tells the truth with a probability of only p, while Q (independently) tells the truth with probability q. I haven't looked in the fridge for some time, so if you had asked me this morning, I would have said that there was just as likely to be a body in it as not. Clearly, in view of what P and Q told me, I must revise this estimate. Explain carefully why my new estimate of the probability of there being a body in the fridge should be

$$\frac{pq}{1-p-q+2pq}.$$

I have now been to look in the fridge, and there is indeed a body in it; perhaps more than one. It seems to me that only my enemy A, or my enemy B, or (with a bit of luck) both A and B could be in my fridge, and this evening I would have judged these three possibilities to be equally likely. But tonight I asked P and Q separately whether or not A was in the fridge, and they each said that he was. What should be my new estimate of the probability that both A and B are in my fridge?

Of course, I tell the truth always.

STEP 2 1988

STEP 2 1988
On-line Exam

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[STEP 2, 1988Q1]

The function f is defined, for $x \neq 1$ and $x \neq 2$, by

$$f(x) = \frac{1}{(x-1)(x-2)}.$$

Show that for $|x| < 1$

$$f(x) = \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n,$$

and that for $1 < |x| < 2$

$$f(x) = -\sum_{n=1}^{\infty} x^{-n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n.$$

Find an expansion for $f(x)$ which is valid for $|x| > 2$.

[STEP 2, 1988Q2]

The numbers x, y and z are non-zero, and satisfy

$$2a - 3y = \frac{(z-x)^2}{y}$$

and

$$2a - 3z = \frac{(x-y)^2}{z},$$

for some number a. If $y \neq z$, prove that

$$x + y + z = a,$$

and that

$$2a - 3x = \frac{(y-z)^2}{x}.$$

Determine whether this last equation holds *only* if $y \neq z$.

[STEP 2, 1988Q3]

The quadratic equation $x^2 + bx + c = 0$, where b and c are real, has the property that if k is a (possibly complex) root, then k^{-1} is a root. Determine carefully the restrictions that this property places on b and c. If, in addition to this property, the equation has the further property that if k is a root, then $1 - k$ is a root, find b and c.

Show that

$$x^3 - \frac{3}{2}x^2 - \frac{3}{2}x + 1 = 0$$

is the only cubic equation of the form $x^3 + px^2 + qx + r = 0$, where p, q and r are real, which has both the above properties.

[STEP 2, 1988Q4]

The complex number ω is such that $\omega^2 - 2\omega$ is real.

(i) Sketch the locus of ω in the Argand diagram.

(ii) If $\omega^2 = x + \mathrm{i}y$, describe fully and sketch the locus of points $(x,\ y)$ in the x-y plane.

The complex number t is such that $t^2 - 2t$ is imaginary. If $t^2 = p + \mathrm{i}q$, sketch the locus of points (p, q) in the p-q plane.

[STEP 2, 1988Q5]

By considering the imaginary part of the equation $z^7 = 1$, or otherwise, find all the roots of the equation

$$t^6 - 21t^4 + 35t^2 - 7 = 0.$$

You should justify each step carefully.

Hence, or otherwise, prove that

$$\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7} = \sqrt{7}.$$

Find the corresponding result for

$$\tan\frac{2\pi}{n}\tan\frac{4\pi}{n}\cdots\tan\frac{(n-1)\pi}{n}$$

in the two cases $n = 9$ and $n = 11$.

[STEP 2, 1988Q6]

Show that the following functions are positive when x is positive:

(i) $x - \tanh x$

(ii) $x \sinh x - 2\cosh x + 2$

(iii) $2x \cosh 2x - 3\sinh 2x + 4x$.

The function f is defined for $x > 0$ by

$$f(x) = \frac{x(\cosh x)^{\frac{1}{3}}}{\sinh x}.$$

Show that $f(x)$ has no turning points when $x > 0$, and sketch $f(x)$ for $x > 0$.

[STEP 2, 1988Q7]

The integral I is defined by

$$I = \int_1^2 \frac{(2 - 2x + x^2)^k}{x^{k+1}} \mathrm{d}x,$$

where k is a constant. Show that

$$I = \int_0^1 \frac{(1+x^2)^k}{(1+x)^{k+1}} \mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{1}{\left[\sqrt{2}\cos\theta\cos\left(\frac{\pi}{4} - \theta\right)\right]^{k+1}} \mathrm{d}\theta$$
$$= 2\int_0^{\frac{\pi}{8}} \frac{1}{\left[\sqrt{2}\cos\theta\cos\left(\frac{\pi}{4} - \theta\right)\right]^{k+1}} \mathrm{d}\theta.$$

Hence show that

$$I = 2\int_0^{\sqrt{2}-1} \frac{(1+x^2)^k}{(1+x)^{k+1}} \mathrm{d}x.$$

Deduce that

$$\int_1^{\sqrt{2}} \left(\frac{2 - 2x^2 + x^4}{x^2}\right)^k \frac{1}{x} \mathrm{d}x = \int_1^{\sqrt{2}} \left(\frac{2 - 2x + x^2}{x}\right)^k \frac{1}{x} \mathrm{d}x.$$

[STEP 2, 1988Q8]

In a crude model of the population dynamics of a community of aardvarks and buffaloes, it is assumed that, if the numbers of aardvarks and buffaloes in any year are A and B respectively, then the numbers in the following year are $\frac{1}{4}A + \frac{3}{4}B$ and $\frac{3}{2}B - \frac{1}{2}A$ respectively. It does not matter if the model predicts fractions of animals, but a non-positive number of buffaloes means that the species has become extinct, and the model ceases to apply. Using matrices or otherwise, show that the ratio of the number of aardvarks to the number of buffaloes can remain the same each year, provided it takes one of two possible values.

Let these two possible values be x and y, and let the numbers of aardvarks and buffaloes in a given year be a and b respectively. By writing the vector (a, b) as a linear combination of the vectors $(x, 1)$ and $(y, 1)$, or otherwise, show how the numbers of aardvarks and buffaloes in subsequent years may be found. On a sketch of the a - b plane, mark the regions which correspond to the following situations:

(i) an equilibrium population is reached as time $t \to \infty$.

(ii) buffaloes become extinct after a finite time.

(iii) buffaloes approach extinction as $t \to \infty$.

[STEP 2, 1988Q9]

Give a careful argument to show that, if G_1 and G_2 are subgroups of a finite group G such that every element of G is either in G_1 or in G_2, then either $G_1 = G$ or $G_2 = G$.

Give an example of a group H which has three subgroups H_1, H_2, and H_3 such that every element of H is either in H_1, H_2 or H_3 and $H_1 \neq H, H_2 \neq H, H_3 \neq H$.

[STEP 2, 1988Q10]

The surface S in three dimensional space is described by the equation

$$\mathbf{a}.\mathbf{r} + ar = a^2,$$

where $\mathbf{r}$ is the position vector with respect to the origin O, $\mathbf{a}(\neq \mathbf{0})$ is the position vector of a fixed point, $r = |\mathbf{r}|$ and $a = |\mathbf{a}|$. Show, with the aid of a diagram, that S is the locus of points which are equidistant from the origin O and the plane $\mathbf{r}.\mathbf{a} = a^2$.

The point P, with position vector $\mathbf{p}$, lies in S, and the line joining P to O meets S again at Q. Find the position vector of Q.

The line through O orthogonal to $\mathbf{p}$ and $\mathbf{a}$ meets S at T and T'. Show that the position vectors of T and T' are

$$\pm\frac{1}{\sqrt{2ap - a^2}}\,\mathbf{a} \times \mathbf{p},$$

where $p = |\mathbf{p}|$.

Show that the area of the triangle PQT is

$$\frac{ap^2}{(2p - a)}.$$

[STEP 2, 1988Q11]

A heavy particle lies on a smooth horizontal table, and is attached to one end of a light inextensible string of length L. The other end of the string is attached to a point P on the circumference of the base of a vertical post which is fixed into the table. The base of the post is a circle of radius a with its centre at a point O on the table. Initially, at time $t = 0$, the string is taut and perpendicular to the line OP. The particle is then struck in such a way that the string starts winding round the post and remains taut. At a later time t, a length $a\theta(t)$ $(< L)$ of the string is in contact with the post. Using cartesian axes with origin O, find the position and velocity vectors of the particle at time t in terms of a, L, θ, and $\dot{\theta}$, and hence show that the speed of the particle is $(L - a\theta)\dot{\theta}$.

If the initial speed of the particle is v, show that the particle hits the post at a time $\frac{L^2}{2av}$.

[STEP 2, 1988Q12]

One end of a thin uniform inextensible, but perfectly flexible, string of length l and uniform mass per unit length is held at a point on a smooth table a distance d $(< l)$ away from a small vertical hole in the surface of the table. The string passes through the hole so that a length $l - d$ of the string hangs vertically. The string is released from rest. Assuming that the height of the table is greater than l, find the time taken for the end of the string to reach the top of the hole.

[STEP 2, 1988Q13]

A librarian wishes to pick up a row of identical books from a shelf, by pressing her hands on the outer covers of the two outermost books and lifting the whole row together. The covers of the books are all in parallel vertical planes, and the weight of each book is W. With each arm, the librarian can exert a maximum force of P in the vertical direction, and, independently, a maximum force of Q in the horizontal direction. The coefficient of friction between each pair of books and also between each hand and a book is μ. Derive an expression for the maximum number of books that can be picked up without slipping, using this method.

[You may assume that the books are thin enough for the rotational effect of the couple on each book to be ignored.]

[STEP 2, 1988Q14]

Two particles of masses M and m $(M > m)$ are attached to the ends of a light rod of length $2l$. The rod is fixed at its midpoint to a point O on a horizontal axle so that the rod can swing freely about O in a vertical plane normal to the axle. The axle rotates about a *vertical* axis through O at a constant angular speed ω such that the rod makes a constant angle α $(0 < \alpha < \frac{\pi}{2})$ with the vertical. Show that

$$\omega^2 = \left(\frac{M-m}{M+m}\right)\frac{g}{l\cos\alpha}.$$

Show also that the force of reaction of the rod on the axle is inclined at an angle

$$\tan^{-1}\left[\left(\frac{M-m}{M+m}\right)^2 \tan\alpha\right]$$

with the downward vertical.

[STEP 2, 1988Q15]

An examination consists of several papers, which are marked independently. The mark given for each paper can be any integer from 0 to m inclusive, and the total mark for the examination is the sum of the marks on the individual papers. In order to make the examination completely fair, the examiners decide to allocate the mark for each paper at random, so that the probability that any given candidate will be allocated k marks $(0 \leq k \leq m)$ for a given paper is $(m+1)^{-1}$. If there are just two papers, show that the probability that a given candidate will receive a total of n marks is

$$\frac{2m-n+1}{(m+1)^2}$$

for $m < n \leq 2m$, and find the corresponding result for $0 \leq n \leq m$.

If the examination consists of three papers, show that the probability that a given candidate will receive a total of n marks is

$$\frac{6mn - 3m^2 - 2n^2 + 3m + 2}{2(m+1)^3}$$

in the case $m < n \leq 2m$. Find the corresponding result for $0 \leq n \leq m$, and deduce the result for $2m < n \leq 3m$.

[STEP 2, 1988Q16]

Find the probability that the quadratic equation

$$X^2 + 2BX + 1 = 0$$

has real roots when B is normally distributed with zero mean and unit variance.

Given that the two roots X_1 and X_2 are real, find:

(i) the probability that both X_1 and X_2 are greater than $\frac{1}{5}$;

(ii) the expected value of $|X_1 + X_2|$;

giving your answers to three significant figures.

STEP 2 1989

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Calculators are not permitted.

[STEP 2, 1989Q1]

Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Show how the cubic equation

$$24x^3 - 72x^2 + 66x - 19 = 0 \qquad (*)$$

can be reduced to the form

$$4z^3 - 3z = k$$

by means of the substitutions $y = x + a$ and $z = by$, for suitable values of the constants a and b. Hence find the three roots of the equation $(*)$, to three significant figures.

Show, by means of a counterexample, or otherwise, that not all cubic equations of the form $x^3 + \alpha x^2 + \beta x + \gamma = 0$ can be solved by this method.

[STEP 2, 1989Q2]

Let $\tan x = \sum_0^\infty a_n x^n$, for small x, and let $x\cot x = 1 + \sum_1^\infty b_n x^n$, for x small and not zero.

Using the relation

$$\cot x - \tan x = 2\cot 2x, \qquad (*)$$

or otherwise, prove that $a_{n-1} = (1 - 2^n)b_n$, for $n \geq 1$.

Let $x \operatorname{cosec} x = 1 + \sum_1^\infty c_n x^n$, for x small and not zero. Using a relation similar to $(*)$ involving $2\operatorname{cosec} 2x$, or otherwise, prove that

$$c_n = \frac{2^{n-1} - 1}{2^n - 1}\frac{1}{2^{n-1}}a_{n-1} \qquad (n \geq 1).$$

[STEP 2, 1989Q3]

The real numbers x and y are related to the real numbers u and v by

$$2(u+\mathrm{i}v)=\mathrm{e}^{x+\mathrm{i}y}-\mathrm{e}^{-x-\mathrm{i}y}.$$

Show that the line in the x-y plane given by $x=a$, where a is a positive constant, corresponds to the ellipse

$$\left(\frac{u}{\sinh a}\right)^2+\left(\frac{v}{\cosh a}\right)^2=1$$

in the u-v plane. Show also that the line given by $y=b$, where b is a constant and $0<\sin b<1$, corresponds to one branch of a hyperbola in the u-v plane. Write down the u and v coordinates of one point of intersection of the ellipse and hyperbola branch, and show that the curves intersect at right-angles at this point.

Make a sketch of the u-v plane showing the ellipse, the hyperbola branch and the line segments corresponding to:

(i) $x=0$.

(ii) $y=\frac{\pi}{2}, 0\le x\le a$.

[STEP 2, 1989Q4]

The function f is defined by

$$f(x)=\frac{(x-a)(x-b)}{(x-c)(x-d)}\qquad (x\neq c,\qquad x\neq d),$$

where a,b,c and d are real and distinct, and $a+b\neq c+d$. Show that

$$\frac{xf'(x)}{f(x)}=\left(1-\frac{a}{x}\right)^{-1}+\left(1-\frac{b}{x}\right)^{-1}-\left(1-\frac{c}{x}\right)^{-1}-\left(1-\frac{d}{x}\right)^{-1},$$

$(x\neq 0, x\neq a, x\neq b)$ and deduce that when $|x|$ is much larger than each of $|a|$, $|b|$, $|c|$ and $|d|$, the gradient of $f(x)$ has the same sign as $(a+b-c-d)$.

It is given that there is a real value of x for which $f(x)$ takes the real value z if and only if

$$[(c-d)^2z+(a-c)(b-d)+(a-d)(b-c)]^2 \ge 4(a-c)(b-d)(a-d)(b-c).$$

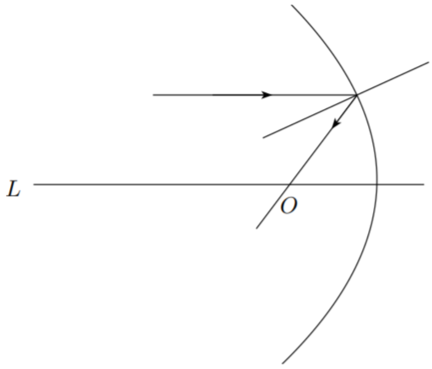
Describe briefly a method by which this result could be proved, but do not attempt to prove it.

Given that $a<b$ and $a<c<d$, make sketches of the graph of f in the four distinct cases which arise, indicating the cases for which the range of f is not the whole of $\mathbb{R}$.

[STEP 2, 1989Q5]

(i) Show that in polar coordinates, the gradient of any curve at the point (r, θ) is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r\right)\Big/\left(\frac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta\right).$$



(ii) A mirror is designed so that any ray of light which hits one side of the mirror and which is parallel to a certain fixed line L is reflected through a fixed point O on L. For any ray hitting the mirror, the normal to the mirror at the point of reflection bisects the angle between the incident ray and the reflected ray, as shown in the figure. Prove that the mirror intersects any plane containing L in a parabola.

[STEP 2, 1989Q6]

The function f satisfies the condition $f'(x) > 0$ for $a \le x \le b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b f(x)\,\mathrm{d}x = b\beta - a\alpha - \int_\alpha^\beta g(y)\,\mathrm{d}y,$$

where $\alpha = f(a)$ and $\beta = f(b)$. Interpret this formula geometrically, in the case where α and a are both positive.

Prove similarly and interpret (for $\alpha > 0$ and $a > 0$) the formula

$$2\pi \int_a^b x f(x)\,\mathrm{d}x = \pi(b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [g(y)]^2\,\mathrm{d}y.$$

[STEP 2, 1989Q7]

By means of the substitution $t = x^\alpha$, where α is a suitably chosen constant, find the general solution for $x > 0$ of the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - b\frac{\mathrm{d}y}{\mathrm{d}x} + x^{2b+1}y = 0,$$

where b is a constant and $b > -1$.

Show that, if $b > 0$, there exist solutions which satisfy $y \to 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} \to 0$ as $x \to 0$, but that these conditions do not determine a unique solution. For what values of b do these conditions determine a unique solution?

[STEP 2, 1989Q8]

Let $\Omega = \exp\left(\frac{\mathrm{i}\pi}{3}\right)$. Prove that $\Omega^2 - \Omega + 1 = 0$.

Two transformations, R and T, of the complex plane are defined by

$$R : z \longmapsto \Omega^2 z$$

and

$$T : z \longmapsto \frac{\Omega z + \Omega^2}{2\Omega^2 z + 1}.$$

Verify that each of R and T permute the four points $z_0 = 0$, $z_1 = 1$, $z_2 = \Omega^2$ and $z_3 = -\Omega$. Explain, without explicitly producing a group multiplication table, why the smallest group of transformations which contains elements R and T has order at least 12.

Are there any permutations of these points which cannot be produced by repeated combinations of R and T?

[STEP 2, 1989Q9]

The matrix $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n A^n,$$

where $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}$, and t is a variable scalar. Evaluate $\mathbf{A}^2$, and show that

$$\mathbf{F} = \mathbf{I} \cosh t + \mathbf{A} \sinh t.$$

Show also that $\mathbf{F}^{-1} = \mathbf{I} \cosh t - \mathbf{A} \sinh t$, and that $\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \mathbf{F}\mathbf{A}$.

The vector $\mathbf{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{A}\mathbf{r} = \mathbf{0},$$

with $x = \alpha$ and $y = \beta$ at $t = 0$. Solve this equation by means of a suitable matrix integrating factor, and hence show that

$$x(t) = \alpha \cosh t + (3\alpha + \beta) \sinh t$$
$$y(t) = \beta \cosh t - (8\alpha + 3\beta) \sinh t.$$

[STEP 2, 1989Q10]

State carefully the conditions which the fixed vectors **a, b, u** and **v** must satisfy in order to ensure that the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ intesects the line $\mathbf{r} = \mathbf{b} + \mu\mathbf{v}$ in exactly one point.

Find the two values of the fixed scalar b for which the planes with equations

$$\begin{cases} x + y + bz = b + 2 \\ bx + by + z = 2b + 1 \end{cases} \qquad (*)$$

do not intersect in a line. For other values of b, express the line of intersection of the two planes in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$, where $\mathbf{a}.\mathbf{u} = \mathbf{0}$.

Find conditions which b and the fixed scalars c and d must satisfy to ensure that there is exactly one point on the line

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} + \mu \begin{pmatrix} 1 \\ d \\ 0 \end{pmatrix}$$

whose coordinates satisfy both equations $(*)$.

[STEP 2, 1989Q11]

A lift of mass M and its counterweight of mass M are connected by a light inextensible cable which passes over a light frictionless pulley. The lift is constrained to move vertically between smooth guides. The distance between the floor and the ceiling of the lift is h. Initially, the lift is at rest, and the distance between the top of the lift and the pulley is greater than h. A small tile of mass m becomes detached from the ceiling of the lift. Show that the time taken for it to fall to the floor is

$$t = \sqrt{\frac{(2M - m)h}{Mg}}.$$

The collision between the tile and the lift floor is perfectly inelastic. Show that the lift is reduced to rest by the collision, and that the loss of energy of the system is mgh.

[STEP 2, 1989Q12]

A uniform rectangular lamina of sides $2a$ and $2b$ rests in a vertical plane. It is supported in equilibrium by two smooth pegs fixed in the same horizontal plane, a distance d apart, so that one corner of the lamina is below the level of the pegs. Show that if the distance between this (lowest) corner and the peg upon which the side of length $2a$ rests is less than a, then the distance between this corner and the other peg is less than b.

Show also that

$$b\cos\theta - a\sin\theta = d\cos 2\theta,$$

where θ is the acute angle which the sides of length $2b$ make with the horizontal.

[STEP 2, 1989Q13]

A body of mass m and centre of mass O is said to be *dynamically equivalent* to a system of particles of total mass m and centre of mass O if the moment of inertia of the system of particles is the same as the moment of inertia of the body, about any axis through O. Show this implies that the moment of inertia of the system of particles is the same as that of the body about *any* axis.

Show that a uniform rod of length $2a$ and mass m is dynamically equivalent to a suitable system of three particles, one at each end of the rod, and one at the midpoint.

Use this result to deduce that a uniform rectangular lamina of mass M is dynamically equivalent to a system consisting of particles each of mass $\frac{M}{36}$ at the corners, particles each of mass $\frac{M}{9}$ at the midpoint of each side, and a particle of mass $\frac{4M}{9}$ at the centre. Hence find the moment of inertia of a square lamina, of side $2a$ and mass M, about one of its diagonals:

The mass per unit length of a thin rod of mass m is proportional to the distance from one end of the rod, and a dynamically equivalent system consists of one particle at each end of the rod and one at the midpoint. Write down a set of equations which determines these masses, and show that, in fact, only two particles are required.

[STEP 2, 1989Q14]

One end of a light inextensible string of length l is fixed to a point on the upper surface of a thin, smooth, horizontal table-top, at a distance $(l-a)$ from one edge of the table-top. A particle of mass m is fixed to the other end of the string, and held a distance a away from this edge of the table-top, so that the string is horizontal and taut. The particle is then released. Find the tension in the string after the string has rotated through an angle θ, and show that the largest magnitude of the force on the edge of the table top is $\frac{8mg}{\sqrt{3}}$.

[STEP 2, 1989Q15]

Two points are chosen independently at random on the perimeter (including the diameter) of a semicircle of unit radius. What is the probability that exactly one of them lies on the diameter?

Let the area of the triangle formed by the two points and the midpoint of the diameter be denoted by the random variable A.

(i) Given that exactly one point lies on the diameter, show that the expected value of A is $(2\pi)^{-1}$.

(ii) Given that neither point lies on the diameter, show that the expected value of A is π^{-1}. [You may assume that if two points are chosen at random on a line of length π units, the probability density function for the distance X betwen the two points is $\frac{2(\pi-x)}{\pi^2}$ for $0 \le x \le \pi$.]

Using these results, or otherwise, show that the expected value of A is $(2+\pi)^{-1}$.

[STEP 2, 1989Q16]

Widgets are manufactured in batches of size $(n + N)$. Any widget has a probability p of being faulty, independent of faults in other widgets. The batches go through a quality control procedure in which a sample of size n, where $n \geq 2$, is taken from each batch and tested. If two or more widgets in the sample are found to be faulty, all widgets in the batch are tested and all faults corrected. If fewer than two widgets in the sample are found to be faulty, the sample is replaced in the batch, and no faults are corrected. Show that the probability that the batch contains exactly k, where $k \leq N$, faulty widgets after quality control is

$$\frac{[N + 1 + k(n-1)]N!}{(N-k+1)!\,k!}p^k(1-p)^{N+n-k},$$

and verify that this formula also gives the correct answer for $k = N + 1$.

Show that the expected number of faulty widgets in a batch after quality control is

$$[N + n + pN(n-1)]p(1-p)^{n-1}.$$

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[STEP 2, 1990Q1]

Prove that both $x^4 - 2x^3 + x^2$ and $x^2 - 8x + 17$ are non-negative for all real x. By considering the intervals $x \leq 0$, $0 < x \leq 2$ and $x > 2$ separately, or otherwise, prove that the equation

$$x^4 - 2x^3 + x^2 - 8x + 17 = 0$$

has no real roots.

Prove that the equation $x^4 - x^3 + x^2 - 4x + 4 = 0$ has no real roots.

[STEP 2, 1990Q2]

Prove that if $A + B + C + D = \pi$, then

$$\sin(A + B)\sin(A + D) - \sin B \sin D = \sin A \sin C.$$

The points P, Q, R and S lie, in that order, on a circle of centre O. Prove that

$$PQ \cdot RS + QR \cdot PS = PR \cdot QS.$$

[STEP 2, 1990Q3]

Sketch the curves given by

$$y = x^3 - 2bx^2 + c^2x,$$

where b and c are non-negative, in the cases:

(i) $2b < c\sqrt{3}$.

(ii) $2b = c\sqrt{3} \neq 0$.

(iii) $c\sqrt{3} < 2b < 2c$.

(iv) $b = c \neq 0$.

(v) $b > c > 0$.

(vi) $c = 0,\ b \neq 0$.

(vii) $c = b = 0$.

Sketch also the curves given by

$$y^2 = x^3 - 2bx^2 + c^2x$$

in the cases (i), (v) and (vii).

[STEP 2, 1990Q4]

A plane contains n distinct given lines, no two of which are parallel, and no three of which intersect at a point. By first considering the cases $n = 1, 2, 3$ and 4, provide and justify, by induction or otherwise, a formula for the number of line segments (including the infinite segments).

Prove also that the plane is divided into $\frac{1}{2}(n^2 + n + 2)$ regions (including those extending to infinity).

[STEP 2, 1990Q5]

The distinct points L, M, P and Q of the Argand diagram lie on a circle S centred on the origin and the corresponding complex numbers are l, m, p and q. By considering the perpendicular bisectors of the chords, or otherwise, prove that the chord LM is perpendicular to the chord PQ if and only if $lm + pq = 0$.

Let A_1, A_2 and A_3 be three given distinct points on S. For any given point A_1' on S, the points A_2', A_3' and A_1'' are chosen on S such that $A_1'A_2'$, $A_2'A_3'$ and $A_3'A_1''$ are perpendicular to A_1A_2, A_2A_3 and A_3A_1, respectively. Show that for exactly two positions of A_1', the points A_1' and A_1'' coincide.

If, instead, A_1, A_2, A_3 and A_4 are four given distinct points on S and, for any given point A_1', the points A_2', A_3', A_4' and A_1'' are chosen on S such that $A_1'A_2'$, $A_2'A_3'$, $A_3'A_4'$ and $A_4'A_1''$ are respectively perpendicular to A_1A_2, A_2A_3, A_3A_4 and A_4A_1, show that A_1' coincides with A_1''.

Give the corresponding result for n distinct points on S.

[STEP 2, 1990Q6]

Let a, b, c, d, p and q be positive integers. Prove that:

(i) if $b > 1$ and $c > 1$, then $bc \geq 2c \geq 2 + c$.

(ii) if $a < b$ and $d < c$, then $bc - ad \geq a + c$.

(iii) if $\frac{a}{b} < p < \frac{c}{d}$, then $(bc - ad)p \geq a + c$.

(iv) if $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$, then $p \geq \frac{a+c}{bc-ad}$ and $q \geq \frac{b+d}{bc-ad}$.

Hence find all fractions with denominators less than 20 which lie between $\frac{8}{9}$ and $\frac{9}{10}$.

[STEP 2, 1990Q7]

A damped system with feedback is modelled by the equation

$$f'(t) + f(t) - kf(t-1) = 0, \qquad (\dagger)$$

where k is a given non-zero constant. Show that (non-zero) solutions for f of the form $f(t) = Ae^{pt}$, where A and p are constants, are possible provided p satisfies

$$p + 1 = ke^{-p}. \qquad (*)$$

Show also, by means of a sketch, or otherwise, that equation $(*)$ can have 0, 1 or 2 real roots, depending on the value of k, and find the set of values of k for which such solutions of $(\dagger)$ exist. For what set of values of k do such solutions tend to zero as $t \to +\infty$?

[STEP 2, 1990Q8]

The functions x and y are related by

$$x(t) = \int_0^t y(u)\,\mathrm{d}u,$$

so that $x'(t) = y(t)$. Show that

$$\int_0^1 x(t)y(t)\,\mathrm{d}t = \frac{1}{2}[x(1)]^2.$$

In addition, it is given that $y(t)$ satisfies

$$y'' + (y^2 - 1)y' + y = 0, \qquad (*)$$

with $y(0) = y(1)$ and $y'(0) = y'(1)$. By integrating $(*)$, prove that $x(1) = 0$.

By multiplying $(*)$ by $x(t)$ and integrating by parts, prove the relation

$$\int_0^1 [y(t)]^2\,\mathrm{d}t = \frac{1}{3}\int_0^1 [y(t)]^4\,\mathrm{d}t.$$

Prove also the relation

$$\int_0^1 [y'(t)]^2\,\mathrm{d}t = \int_0^1 [y(t)]^2\,\mathrm{d}t.$$

[STEP 2, 1990Q9]

Show by means of a sketch that the parabola $r(1+\cos\theta) = 1$ cuts the interior of the cardioid $r = 4(1+\cos\theta)$ into two parts.

Show that the total length of the boundary of the part that includes the point $r = 1, \theta = 0$ is $18\sqrt{3} + \ln(2+\sqrt{3})$.

[STEP 2, 1990Q10]

Two square matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathbf{AB} = \mathbf{0}$. Show that either $\det \mathbf{A} = 0$ or $\det \mathbf{B} = 0$ or $\det \mathbf{A} = \det \mathbf{B} = 0$. If $\det \mathbf{B} \neq 0$, what must $\mathbf{A}$ be? Give an example to show that the condition $\det \mathbf{A} = \det \mathbf{B} = 0$ is not sufficient for the equation $\mathbf{AB} = \mathbf{0}$ to hold.

Find real numbers p, q and r such that

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = (\mathbf{M} + p\mathbf{I})(\mathbf{M} + q\mathbf{I})(\mathbf{M} + r\mathbf{I}),$$

where $\mathbf{M}$ is any square matrix and $\mathbf{I}$ is the appropriate identity matrix.

Hence, or otherwise, find all matrices $\mathbf{M}$ of the form $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ which satisfy the equation

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = \mathbf{0}.$$

[STEP 2, 1990Q11]

A disc is free to rotate in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is mk^2. Along one diameter is a narrow groove in which a particle of mass m slides freely. At time $t = 0$, the disc is rotating with angular speed Ω, and the particle is at a distance a from the axis and is moving towards the axis with speed V, where $k^2V^2 = \Omega^2 a^2 (k^2 + a^2)$. Show that, at a later time t, while the particle is still moving towards the axis, the angular speed ω of the disc and the distance r of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \quad \text{and} \quad \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\Omega r(k^2 + a^2)}{k(k^2 + r^2)^{\frac{1}{2}}}.$$

Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}},$$

where θ is the angle through which the disc has turned at time t. By making the substitution $u = \frac{1}{r}$, or otherwise, show that $r\sinh(\theta + \alpha) = k$, where $\sinh\alpha = \frac{k}{a}$. Hence, or otherwise, show that the particle never reaches the axis.

[STEP 2, 1990Q12]

A straight staircase consists of N smooth horizontal stairs each of height h. A particle slides over the top stair at speed U, with velocity perpendicular to the edge of the stair, and then falls down the staircase, bouncing once on every stair. The coefficient of restitution between the particle and each stair is e, where $e < 1$. Show that the horizontal distance d_n travelled between the nth and $(n+1)$th bounces is given by

$$d_n = U\left(\frac{2h}{g}\right)^{\frac{1}{2}}(e\alpha_n + \alpha_{n+1}),$$

where $\alpha_n = \left(\frac{1-e^{2n}}{1-e^2}\right)^{\frac{1}{2}}$.

If N is very large, show that U must satisfy

$$U = \left(\frac{L^2 g}{2h}\right)^{\frac{1}{2}}\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}},$$

where L is the horizontal distance between the edges of successive stairs.

[STEP 2, 1990Q13]

A thin non-uniform rod PQ of length $2a$ has its centre of gravity a distance $a+d$ from P. It hangs (not vertically) in equilibrium suspended from a small smooth peg O by means of a light inextensible string of length $2b$ which passes over the peg and is attached at its ends to P and Q. Express OP and OQ in terms of a, b and d. By considering the angle POQ, or otherwise, show that $d < \frac{a^2}{b}$.

[STEP 2, 1990Q14]

The identical uniform smooth spherical marbles $A_1, A_2, \dots, A_n$, where $n \geq 3$, each of mass m, lie in that order in a smooth straight trough, with each marble touching the next. The marble A_{n+1}, which is similar to A_n but has mass λ_m, is placed in the trough so that it touches A_n. Another marble A_0, identical to A_n, slides along the trough with speed u and hits A_1. It is given that kinetic energy is conserved throughout.

(i) Show that if $\lambda < 1$, there is a possible subsequent motion in which only A_n and A_{n+1} move (and A_0 is reduced to rest), but that if $\lambda > 1$, such a motion is not possible.

(ii) If $\lambda > 1$, show that a subsequent motion in which only A_{n-1}, A_n and A_{n+1} move is not possible.

(iii) If $\lambda > 1$, find a possible subsequent motion in which only two marbles move.

[STEP 2, 1990Q15]

A target consists of a disc of unit radius and centre O. A certain marksman never misses the target, and the probability of any given shot hitting the target within a distance t from O is t^2, where $0 \leq t \leq 1$. The marksman fires n shots independently. The random variable Y is the radius of the smallest circle, with centre O, which encloses all the shots. Show that the probability density function of Y is $2ny^{2n-1}$ and find the expected area of the circle.

The shot which is furthest from O is rejected. Show that the expected area of the smallest circle, with centre O, which encloses the remaining $(n-1)$ shots is $\left(\frac{n-1}{n+1}\right)\pi$.

[STEP 2, 1990Q16]

Each day, I choose at random between my brown trousers, my grey trousers and my expensive but fashionable designer jeans. Also in my wardrobe, I have a black silk tie, a rather smart brown and fawn polka-dot tie, my regimental tie and an elegant powder-blue cravat tie which I was given for Christmas. With the brown or grey trousers, I choose ties at random, except of course that I don't wear the cravat with the brown trousers nor the polka-dot tie with the grey trousers. With the jeans, the choice depends on whether it is Sunday or one of the six weekdays: on the weekdays, half the time I wear a cream-coloured sweat-shirt with $E = mc^2$ on the front and no tie; otherwise, and on Sundays (when I always wear a tie), I just pick at random from my four ties.

This morning, I received through the post a compromising photograph of me. I often receive such photographs and they are equally likely to have been taken on any day of the week. However, in this particular photograph I am wearing my black tie. Show that, on the basis of this information, the probability that it was taken on a Sunday is $\frac{11}{68}$.

I should have mentioned that on Mondays I lecture on calculus, and therefore always wear my jeans (so that the lectures are easier to understand). Find, in the light of this new information, the probability that the photograph was taken on a Sunday.

[The phrase 'at random' means 'with equal probability'.]

STEP 2 1991

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[STEP 2, 1991Q1]

Let $h(x) = ax^2 + bx + c$,where a, b and c are constants, and $a \neq 0$. Give a condition which a, b and c must satisfy in order that $h(x)$ can be written in the form

$$a(x+k)^2, \qquad (*)$$

where k is a constant.

If $f(x) = 3x^2 + 4x$ and $g(x) = x^2 - 2$, find the two constant values of λ such that $f(x) + \lambda g(x)$ can be written in the form $(*)$. Hence, or otherwise, find constants A, B, C, D, m and n such that

$$\begin{aligned} f(x) &= A(x+m)^2 + B(x+n)^2 \\ g(x) &= C(x+m)^2 + D(x+n)^2. \end{aligned}$$

If $f(x) = 3x^2 + 4x$ and $g(x) = x^2 + \alpha$, and it is given that there is only one value of λ for which $f(x) + \lambda g(x)$ can be written in the form $(*)$, find α.

[STEP 2, 1991Q2]

The equation of a hyperbola (with respect to axes which are displaced and rotated with respect to the standard axes) is

$$3y^2 - 10xy + 3x^2 + 16y - 16x + 15 = 0. \qquad (\dagger)$$

By differentiating $(\dagger)$, or otherwise, show that the equation of the tangent through the point (s, t) on the curve is

$$y = \left(\frac{5t - 3s + 8}{3t - 5s + 8}\right)x - \left(\frac{8t - 8s + 15}{3t - 5s + 8}\right).$$

Show that the equations of the asymptotes (the limiting tangents as $s \to \infty$) are

$$y = 3x - 4 \quad \text{and} \quad 3y = x - 4.$$

[**Hint**: you will need to find a relationship between s and t which is valid in the limit as $s \to \infty$.]

Show that the angle between one asymptote and the x-axis is the same as the angle between the other asymptote and the y-axis. Deduce the slopes of the lines that bisect the angles between the asymptotes and find the equations of the axes of the hyperbola.

[STEP 2, 1991Q3]

It is given that x, y, and z are distinct and non-zero, and that they satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Show that $x^2y^2z^2 = 1$ and that the value of $x + \frac{1}{y}$ is either $+1$ or -1.

[STEP 2, 1991Q4]

Let $y = \cos\varphi + \cos 2\varphi$, where $\varphi = \frac{2\pi}{5}$. Verify by direct substitution that y satisfies the quadratic equation $2y^2 = 3y + 2$ and deduce that the value of y is $-\frac{1}{2}$.

Let $\theta = \frac{2\pi}{17}$. Show that

$$\sum_{k=0}^{16} \cos k\theta = 0.$$

If $z = \cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta$, show that the value of z is $-\frac{1-\sqrt{17}}{4}$.

[STEP 2, 1991Q5]

Given a rough sketch of the function $\tan^k \theta$ for $0 \leq \theta \leq \frac{1}{4}\pi$ in the two cases $k = 1$ and $k \gg 1$ (i.e. k is much grester than 1).

Show that for any positive integer n

$$\int_0^{\frac{\pi}{4}} \tan^{2n+1}\theta \, \mathrm{d}\theta = (-1)^n\left(\frac{1}{2}\ln 2 + \sum_{m=1}^{n} \frac{(-1)^m}{2m}\right),$$

And deduce that

$$\sum_1^{\infty} \frac{(-1)^{m-1}}{2m} = \frac{1}{2}\ln 2.$$

Show similarly that

$$\sum_1^{\infty} \frac{(-1)^{m-1}}{2m-1} = \frac{\pi}{4}.$$

[STEP 2, 1991Q6]

Show by means of a sketch, or otherwise, that if $0 \leq f(y) \leq g(y)$ for $0 \leq y \leq x$ then

$$0 \leq \int_0^x f(y)\,\mathrm{d}y \leq \int_0^x g(y)\,\mathrm{d}y.$$

Starting from the inequality $0 \leq \cos y \leq 1$, or otherwise, prove that if $0 \leq x \leq \frac{1}{2}\pi$ then $0 \leq \sin x \leq x$ and $\cos x \geq 1 - \frac{1}{2}x^2$. Deduce that

$$\frac{1}{1800} \leq \int_0^{\frac{1}{10}} \frac{x}{(2+\cos x)^2}\,\mathrm{d}x \leq \frac{1}{1797}.$$

Show further that if $0 \leq x \leq \frac{1}{2}\pi$ then $\sin x \geq x - \frac{1}{6}x^3$. Hence prove that

$$\frac{1}{3000} \leq \int_0^{\frac{1}{10}} \frac{x^2}{(1-x+\sin x)^2}\,\mathrm{d}x \leq \frac{2}{5999}.$$

[STEP 2, 1991Q7]

The function g satisfies, for all positive x and y,

$$g(x) + g(y) = g(z), \qquad (*)$$

where $z = \frac{xy}{x+y+1}$. By treating y as a constant, show that

$$g'(x) = \frac{y^2 + y}{(x+y+1)^2} g'(z) = \frac{z(z+1)}{x(x+1)} g'(z),$$

and deduce that $2g'(1) = (u^2 + u)g'(u)$ for all u satifying $0 < u < 1$. Now by treating u as a variable, show that

$$g(u) = A \ln\left(\frac{u}{u+1}\right) + B,$$

where A and B are constants. Verify that g satisfies $(*)$ for a suitable of B. Can A be determined from $(*)$?

The function f satisfies, for all positive x and y,

$$f(x) + f(y) = f(z)$$

where $z = xy$. Show that $f(x) = C \ln x$ where C is a constant.

[STEP 2, 1991Q8]

Solve the quadratic equation $u^2 + 2u \sinh x - 1 = 0$, giving u in terms of x.

Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

which satisfies $y = 0$ and $y' > 0$ at $x = 0$.

Find the solution of the differential equation

$$\sinh x \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh x = 0$$

which satisfies $y = 0$ at $x = 0$.

[STEP 2, 1991Q9]

Let G be the set of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a, b and c are integers modulo 5, and $a \neq 0 \neq c$. Show that G forms a group under matrix multiplication (which may be assumed to be associative). What is the order of G? Determine whether or not G is commutative.

Determine whether or not the set consisting of all elements in G of order 1 or 2 is a subgroup of G.

[STEP 2, 1991Q10]

A straight stick of length h stands vertically. On a sunny day, the stick casts a shadow on flat horizontal ground. In cartesian axes based on the centre of the Earth, the position of the Sun may be taken to be $R(\cos\theta\,, \sin\theta\,, 0)$ where θ varies but R is constant. The positions of the base and tip of the stick are $a(0, \cos\phi\,, \sin\phi)$ and $b(0, \cos\phi\,, \sin\phi)$, respectively, where $b - a = h$. Show that the displacement vector from the base of the stick to the tip of the shadow is

$$Rh(R\cos\phi\sin\theta - b)^{-1}\begin{pmatrix} -\cos\theta \\ -\sin^2\phi\sin\theta \\ \cos\phi\sin\phi\sin\theta \end{pmatrix}.$$

['Stands vertically' means that the centre of the Earth, the base of the stick and the tip of the stick are collinear; 'horizontal' means perpendicular to the stick.]

[STEP 2, 1991Q11]

The Ruritanian army is supplied with shells which may explode at any time in flight but not before the shell reaches its maximum height. The effect of the explosion on any observer depends only on the distance between the exploding shell and the observer (and decreases with distance). Ruritanian guns fire the shells with fixed muzzle speed, and it is the policy of the gunners to fire the shells at an angle of elevation which minimises the possible damage to themselves (assuming the ground is level) - i.e. they aim so that the point on the descending trajectory that is nearest to them is as far away as possible. With that intention, they choose the angle of elevation that minimises the damage to themselves if the shell explodes at its maximum height. What angle do they choose?

Does the shell then get any nearer to the gunners during its descent?

[STEP 2, 1991Q12]

A particle is attached to one end B of a light elastic string of unstretched length a. Initially the other end A is at rest and the particle hangs at rest at a distance $a + c$ vertically below A. At time $t = 0$, the end A is forced to oscillate vertically, its downwards displacement at time t being $b\sin pt$. Let $x(t)$ be the downwards displacement of the particle at time t from its initial equilibrium position. Show that, while the string remains taut, $x(t)$ satisfies

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -n^2(x - b\sin pt),$$

where $n^2 = \frac{g}{c}$, and that if $0 < p < n$, $x(t)$ is given by

$$x(t) = \frac{bn}{n^2 - p^2}(n\sin pt - p\sin nt).$$

Write down a necessary and sufficient condition that the string remains taut throughout the subsequent motion, and show that it is satisfied if $pb < (n - p)c$.

[STEP 2, 1991Q13]

A non-uniform rod AB of mass m is pivoted at one end A so that it can swing freely in a vertical plane. Its centre of mass is a distance d from A and its moment of inertia about any axis perpendicular to the rod through A is mk^2. A small ring of mass αm is free to slide along the rod and the coefficient of friction between the ring and rod is μ. The rod is initially held in a horizontal position with the ring a distance x from A. If $k^2 > xd$, show that when the rod is released, the ring will start to slide when the rod makes an angle θ with the downward vertical, where

$$u \tan\theta = \frac{3\alpha x^2 + k^2 + 2xd}{k^2 - xd}.$$

Explain what will happen if (i) $k^2 = xd$ and (ii) $k^2 < xd$.

[STEP 2, 1991Q14]

The current in a straight river of constant width h flows at uniform speed av parallel to the river banks, where $0 < \alpha < 1$. A boat has to cross from a point A on one bank to a point B on the other bank directly opposite to A. The boat moves at constant speed v relative to the water. When the position of the boat is (x, y), where x is the perpendicular distance from the opposite bank and y is the distance downstream from AB, the boat is pointing in a direction which makes an angle θ with AB. Determine the velocity vector of the boat in terms of v, θ and α.

The pilot of the boat steers in such a way that the boat always points exactly towards B. Show that the velocity vector of the boat is

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \tan\theta \dfrac{\mathrm{d}x}{\mathrm{d}t} + x\sec^2\theta \dfrac{\mathrm{d}\theta}{\mathrm{d}t} \end{pmatrix}.$$

By comparing this with your previous expression deduce that

$$\alpha \frac{\mathrm{d}x}{\mathrm{d}\theta} = -x\sec\theta$$

and hence show that

$$\left(\frac{x}{h}\right)^{\alpha} = (\sec\theta + \tan\theta)^{-1}.$$

Let $s(t)$ be a new variable defined by $\tan\theta = \sinh(\alpha s)$. Show that $x = he^{-s}$, and that

$$he^{-s}\cosh(\alpha s)\frac{\mathrm{d}s}{\mathrm{d}t} = v.$$

Hence show that the time of crossing is $hv^{-1}(1 - a^2)^{-1}$.

[STEP 2, 1991Q15]

Integers $n_1, n_2, \dots, n_r$ (possibly the same) are chosen independently at random from the integers $1, 2, 3, \dots, m$. Show that the probability that $|n_1 - n_2| = k$, where $1 \le k \le m-1$, is $\frac{2(m-k)}{m^2}$ and show that the expectation of $|n_1 - n_2|$ is $\frac{m^2-1}{3m}$. Verify, for the case $m = 2$, the result that the expectation of $|n_1 - n_2| + |n_2 - n_3|$ is $\frac{2(m^2-1)}{3m}$. Write down the expectation, for general m, of

$$|n_1 - n_2| + |n_2 - n_3| + \cdots + |n_{r-1} - n_r|.$$

Desks in an examination hall are placed a distance d apart in straight lines. Each invigilator looks after one line of m desks. When called by a candidate, the invigilator walks to that candidate's desk, and stays there until called again. He or she is equally likely to be called by any of the m candidates in the line but candidates never call simultaneously or while the invigilator is attending to another call. At the beginning of the examination the invigilator stands by the first desk. Show that the expected distance walked by the invigilator in dealing with $N+1$ calls is

$$\frac{\mathrm{d}(m-1)}{6m}[2N(m+1)+3m].$$

[STEP 2, 1991Q16]

Each time it rains over the Cabbibo dam, a volume V of water is deposited, almost instantaneously, in the reservoir. Each day (midnight to midnight) water flows from the reservoir at a constant rate u units of volume per day. An engineer, if present, may choose to alter the value of u at any midnight.

(i) Suppose that it rains at most once in any day, that there is a probability p that it will rain on any given day and that, if it does, the rain is equally likely to fall at any time in the 24 hours (i.e. the time at which the rain falls is a random variable uniform on the interval $[0, 24]$). The engineer decides to take two days' holiday starting at midnight. If at this time the volume of water in the reservoir is V below the top of the dam, find an expression for u such that the probability of overflow in the two days is Q, where $Q < p^2$.

For the engineer's summer holidays, which last 18 days, the reservoir is drained to a volume kV below the top of the dam and the rate of outflow u is set to zero. The engineer wants to drain off as little as possible, consistent with the requirement that the probability that the dam will overflow is less than $\frac{1}{10}$. In the case $p = \frac{1}{3}$, find by means of a suitable approximation the required value of k.

(ii) Suppose instead that it may rain at most once before noon and at most once after noon each day, that the probability of rain in any given half-day is $\frac{1}{6}$ and that it is equally likely to rain at any time in each half day. Is the required value of k lower or higher?

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[STEP 2, 1992Q1]

Find the limit, as $n \to \infty$, of each of the following. You should explain your reasoning briefly.

(i) $\frac{n}{n+1}$

(ii) $\frac{5n+1}{n^2-3n+4}$

(iii) $\frac{\sin n}{n}$

(iv) $\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

(v) $(\tan^{-1} n)^{-1}$

(vi) $\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}-\sqrt{n}}$.

[STEP 2, 1992Q2]

Suppose that y satisfies the differential equation

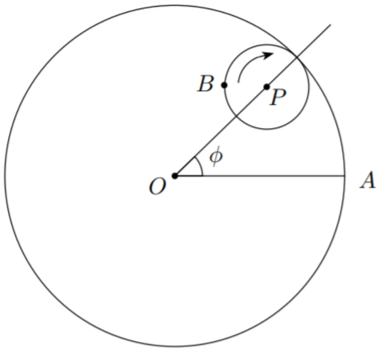
$$y = x\frac{\mathrm{d}y}{\mathrm{d}x} - \cosh\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right). \qquad (*)$$

By differentiating both sides of $(*)$ with respect to x, show that either

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \text{or} \quad x - \sinh\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0.$$

Find the general solutions of each of these two equations. Determine the solutions of $(*)$.

[STEP 2, 1992Q3]



In the figure, the large circle with centre O has radius 4 and the small circle with centre P has radius 1. The small circle rolls around the inside of the larger one. When P was on the line OA (before the small circle began to roll), the point B was in contact with the point A on the large circle. Sketch the curve C traced by B as the circle rolls.

Show that if we take O to be the origin of cartesian coordinates and the line OA to be the x-axis (so that A is the point $(4,0)$) then B is the point

$$(3\cos\phi + \cos 3\phi\,, 3\sin\phi - \sin 3\phi).$$

It is given that the area of the region enclosed by the curve C is

$$\int_0^{2\pi} x\frac{\mathrm{d}y}{\mathrm{d}\phi}\,\mathrm{d}\phi\,,$$

where B is the point (x,y). Calculate this area.

[STEP 2, 1992Q4]

$\lozenge$ is an operation which takes polynomials in x to polynomials in x: that is, given a polynomial $h(x)$ there is another polynomial called $\lozenge\, h(r)$. It is given that, if $f(x)$ and $g(x)$ are any two polynomials in x, the following are always true:

(a) $\lozenge\left(f(x)g(x)\right) = g(x)\lozenge f(x) + f(x)\lozenge g(x)$,

(b) $\lozenge\left(f(x)+g(x)\right) = \lozenge f(x) + \lozenge g(x)$,

(c) $\lozenge\, x = 1$,

(d) if λ is a constant then $\lozenge\left(\lambda f(x)\right) = \lambda \lozenge f(x)$. Show that, if $f(x)$ is a constant (i.e., a polynomial of degree zero), then $\lozenge f(x) = 0$.

Calculate $\lozenge\, x^2$ and $\lozenge\, x^3$. Prove that $\lozenge\, h(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)\right)$ for any polynomial $h(x)$.

[STEP 2, 1992Q5]

Explain what is meant by the order of an element g of a group G.

The set S consists of all 2×2 matrices whose determinant is 1. Find the inverse of the element $\mathbf{A}$ of S, where

$$\mathbf{A} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

Show that S is a group under matrix multiplication (you may assume that matrix multiplication is associative). For which elements $\mathbf{A}$ is $\mathbf{A}^{-1} = \mathbf{A}$? Which element or elements have order 2? Show that the element $\mathbf{A}$ of S has order 3 if, and only if, $w + z + 1 = 0$. Write down one such element.

[STEP 2, 1992Q6]

Sketch the graphs of $y = \sec x$ and $y = \ln(2 \sec x)$ for $0 \leq x < \frac{1}{2}\pi$. Show graphically that the equation

$$kx = \ln(2 \sec x)$$

has no solution with $0 \leq x < \frac{1}{2}\pi$ if k is a small positive number but two solutions if k is large.

Explain why there is a number k_0 such that

$$k_0 x = \ln(2 \sec x)$$

has exactly one solution with $0 \leq x < \frac{1}{2}\pi$. Let x_0 be this solution, so that $0 \leq x_0 < \frac{1}{2}\pi$ and $k_0 x_0 = \ln(2 \sec x)$. Show that

$$x_0 = \cot x_0 \ln(2 \sec x_0).$$

Use any appropriate method to find x_0 correct to two decimal places. Hence find an approximate value for k_0.

[STEP 2, 1992Q7]

The cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots a, b and c. Express p, q and r in terms of a, b and c.

(i) If $p = 0$ and two of the roots are equal to each other, show that

$$4q^3 + 27r^2 = 0.$$

(ii) Show that, if two of the roots of the original equation are equal to each other, then

$$4\left(q - \frac{p^2}{3}\right)^3 + 27\left(\frac{2p^3}{27} - \frac{pq}{3} + r\right)^2 = 0.$$

[STEP 2, 1992Q8]

Calculate the following integrals:

(i) $\int \frac{x}{(x-1)(x^2-1)} \mathrm{d}x$.

(ii) $\int \frac{1}{3\cos x + 4\sin x} \mathrm{d}x$.

(iii) $\int \frac{1}{\sinh x} \mathrm{d}x$.

[STEP 2, 1992Q9]

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C in three-dimensional space. Suppose that A, B, C and the origin O are not all in the same plane. Describe the locus of the point whose position vector $\mathbf{r}$ is given by

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where λ and μ are scalar parameters. By writing this equation in the form $\mathbf{r}.\mathbf{n} = p$ for a suitable vector $\mathbf{n}$ and scalar p, show that

$$-(\lambda + \mu)\mathbf{a}.(\mathbf{b} \times \mathbf{c}) + \lambda\mathbf{b}.(\mathbf{c} \times \mathbf{a}) + \mu\mathbf{c}.(\mathbf{a} \times \mathbf{b}) = 0.$$

for all scalars λ, μ.

Deduce that

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \mathbf{b}.(\mathbf{c} \times \mathbf{a}) = \mathbf{c}.(\mathbf{a} \times \mathbf{b}).$$

Say briefly what happens if A, B, C and O are all in the same plane.

[STEP 2, 1992Q10]

Let α be a fixed angle, $0 < \alpha \leq \frac{1}{2}\pi$. In each of the following cases, sketch the locus of z in the Argand diagram (the complex plane):

(i) $\arg\left(\frac{z-1}{z}\right) = \alpha$,

(ii) $\arg\left(\frac{z-1}{z}\right) = \alpha - \pi$,

(iii) $\left|\frac{z-1}{z}\right| = 1$.

Let z_1, z_2, z_3 and z_4 be four points lying (in that order) on a circle in the Argand diagram. If

$$w = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)}$$

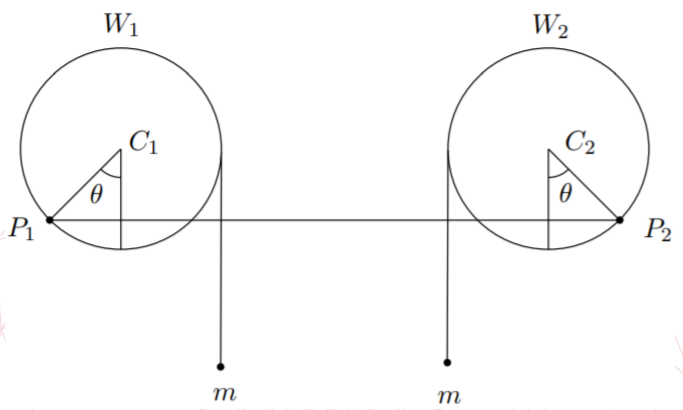
show, by considering $\arg(w)$, that w is real.

[STEP 2, 1992Q11]

I am standing next to an ice-cream van at a distance d from the top of a vertical cliff of height h. It is not safe for me to go any nearer to the top of the cliff. My niece Padma is on the broad level beach at the foot of the cliff. I have just discovered that I have left my wallet with her, so I cannot buy her an ice-cream unless she can throw the wallet up to me. She can throw it at speed V, at any angle she chooses and from anywhere on the beach. Air resistance is negligible; so is Padma's height compared to that of the cliff. Show that she can throw the wallet to me if and only if

$$V^2 \geq g(2h + d).$$

[STEP 2, 1992Q12]



In the figure, W_1 and W_2 are wheels, both of radius r. Their centres C_1 and C_2 are fixed at the same height, a distance d apart, and each wheel is free to rotate, without friction, about its centre. Both wheels are in the same vertical plane. Particles of mass m are suspended from W_1 and W_2 as shown, by light inextensible strings wound round the wheels. A light elastic string of natural length d and modulus of elasticity λ is fixed to the rims of the wheels at the points P_1 and P_2. The lines joining C_1 to P_1 and C_2 to P_2 both make an angle θ with the vertical. The system is in equilibrium. Show that

$$\sin 2\theta = \frac{mgd}{\lambda r}.$$

For what value or values of λ (in terms of m, d, r and g) are there

(i) no equilibrium positions,

(ii) just one equilibrium position,

(iii) exactly two equilibrium positions,

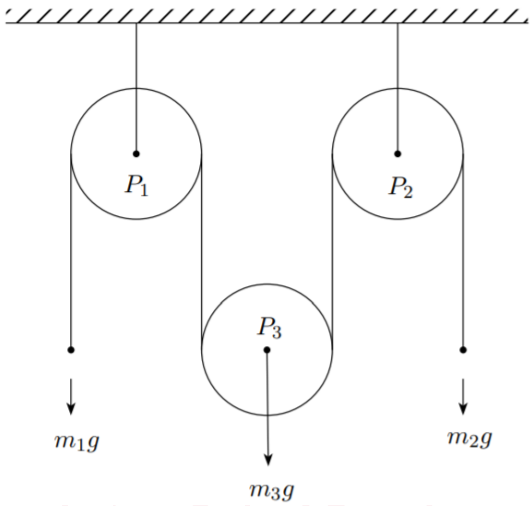
(iv) more than two equilibrium positions?

[STEP 2, 1992Q13]

Two particles P_1 and P_2, each of mass m, are joined by a light smooth inextensible string of length l. P_1 lies on a table top a distance d from the edge, and P_2 hangs over the edge of the table and is suspended a distance b above the ground. The coefficient of friction between P_1 and the table top is μ, and $\mu < 1$. The system is released from rest. Show that P_1 will fall off the edge of the table if and only if $\mu < \frac{b}{2d-b}$.

Suppose that $\mu > \frac{b}{2d-b}$, so that P_1 comes to rest on the table, and that the coefficient of restitution between P_2 and the floor is e. Show that, if $e > \frac{1}{2\mu}$, then P_1 comes to rest before P_2 bounces a second time.

[STEP 2, 1992Q14]



In the diagram P_1 and P_2 are smooth light pulleys fixed at the same height, and P_3 is a third smooth light pulley, freely suspended. A smooth light inextensible string runs over P_1, under P_3 and over P_2, as shown: the parts of the string not in contact with any pulley are vertical. A particle of mass m_3 is attached to P_3. There is a particle of mass m_1 attached to the end of the string below P_1 and a particle of mass m_2 attached to the other end, below P_2. The system is released from rest. Find the extension in the string, and show that the pulley P_3 will remain at rest if

$$4m_1m_2 = m_3(m_1 + m_2).$$

[STEP 2, 1992Q15]

A point moves in unit steps on the x-axis starting from the origin. At each step the point is equally likely to move in the positive or negative direction. The probability that after s steps it is at one of the points $x = 2, x = 3, x = 4$ or $x = 5$ is $\mathrm{P}(s)$. Show that $\mathrm{P}(5) = \frac{3}{16}$, $\mathrm{P}(6) = \frac{21}{64}$ and

$$\mathrm{P}(2k) = \binom{2k+1}{k-1}\left(\frac{1}{2}\right)^{2k}$$

where k is a positive integer. Find a similar expression for $\mathrm{P}(2k + 1)$. Determine the values of s for which $\mathrm{P}(s)$ has its greatest value.

[STEP 2, 1992Q16]

A taxi driver keeps a packet of toffees and a packet of mints in her taxi. From time to time she takes either a toffee (with probability p) or a mint (with probability $q = 1 - p$). At the beginning of the week she has n toffees and m mints in the packets. On the Nth occasion that she reaches for a sweet, she discovers (for the first time) that she has run out of that kind of sweet. What is the probability that she was reaching for a toffee?

STEP 2 1993

STEP 2 1993
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
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TIME ALLOWED: 180 MINUTES

INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

INFORMATION FOR CANDIDATES

There are 16 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

Calculators are not permitted.

[STEP 2, 1993Q1]

In the game of "Colonel Blotto" there are two players, Adam and Betty. First Adam chooses three non-negative integers a_1, a_2 and a_3, such that $a_1 + a_2 + a_3 = 9$, and then Betty chooses non-negative integers b_1, b_2 and b_3, such that $b_1 + b_2 + b_3 = 9$. If $a_1 > b_1$ then Adam scores one point; if $a_1 < b_1$ then Betty scores one point; and if $a_1 = b_1$ no points are scored. Similarly for a_2, b_2 and a_3, b_3.The winner is the player who scores the greater number of points: if the scores are equal then the game is drawn. Show that, if Betty knows the numbers a_1, a_2 and a_3, she can always choose her numbers so that she wins. Show that Adam can choose a_1, a_2 and a_3 in such a way that he will never win no matter what Betty does.

Now suppose that Adam is allowed to write down two triples of numbers and that Adam wins unless Betty can find one triple that beats both of Adam's choices (knowing what they are). Confirm that Adam wins by writing down $(5, 3, 1)$ and $(3, 1, 5)$.

[STEP 2, 1993Q2]

(i) Evaluate

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,\mathrm{d}x,$$

where m, n are integers, taking into account any special cases that arise.

(ii) Find $\int \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x$.

[STEP 2, 1993Q3]

(i) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y - 3y^2 = -2$$

by making the substitution

$$y = -\frac{1}{3u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

(ii) Solve the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy + x^2y^2 = 1$$

by making the substitution

$$y = \frac{1}{x} + \frac{1}{v},$$

where v is a function of x.

[STEP 2, 1993Q4]

Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\hat{\mathbf{m}}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\hat{\mathbf{m}}_2$ respectively, where $\hat{\mathbf{m}}_1$ and $\hat{\mathbf{m}}_2$ are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2)\cdot(\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2)|}{|\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2|}.$$

(i) Find the shortest distance between the lines in the case

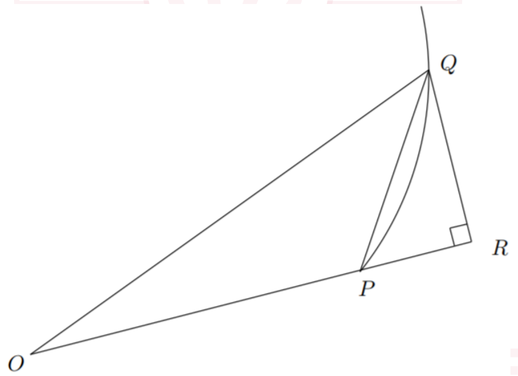
$$\mathbf{p}_1 = (2,1,-1) \qquad \mathbf{p}_2 = (1,0,-2)$$
$$\hat{\mathbf{m}}_1 = \frac{1}{5}(4,3,0) \qquad \hat{\mathbf{m}}_2 = \frac{1}{\sqrt{10}}(0,-3,1).$$

(ii) Two aircraft, A_1 and A_2, are flying in the directions given by the unit vectors $\hat{\mathbf{m}}_1$ and $\hat{\mathbf{m}}_2$ at constant speeds v_1 and v_2. At time $t = 0$ they pass the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2|v_1\hat{\mathbf{m}}_1 - v_2\hat{\mathbf{m}}_2|^2 - \big((\mathbf{p}_1 - \mathbf{p}_2).(v_1\hat{\mathbf{m}}_1 - v_2\hat{\mathbf{m}}_2)\big)^2}{|v_1\hat{\mathbf{m}}_1 - v_2\hat{\mathbf{m}}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1. How close is that, if $\mathbf{p}_1$, $\mathbf{p}_2$, $\hat{\mathbf{m}}_1$ and $\hat{\mathbf{m}}_2$ are as in (i)?

[STEP 2, 1993Q5]



In the diagram, O is the origin, P is a point of a curve $r = r(\theta)$ with coordinates (r,θ) and Q is another point of the curve, close to P, with coordinates $(r+\delta r, \theta+\delta\theta)$. The angle $\angle PRQ$ is a right angle. By calculating $\tan\angle QPR$, show that the angle at which the curve cuts OP is $\tan^{-1}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)$.

Let α be a constant angle, $0 < \alpha < \frac{\pi}{2}$. The curve with the equation

$$r = \mathrm{e}^{\theta\cot\alpha}$$

in polar coordinates is called an equiangular spiral. Show that it cuts every radius line at an angle α. Sketch the spiral.

Find the length of the complete turn of the spiral beginning at $r = 1$ and going outwards. What is the total length of the part of the spiral for which $r \leq 1$? [You may assume that the arc length s of the curve satisfies $\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$.]

[STEP 2, 1993Q6]

In this question, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ are non-zero 2×2 real matrices. Are the following assertions true or false? You must provide a proof or a counterexample in each case.

(i) If $\mathbf{AB} = \mathbf{0}$ then $\mathbf{BA} = \mathbf{0}$.

(ii) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

(iii) The equation $\mathbf{AX} = \mathbf{0}$ has a non-zero solution $\mathbf{X}$ if and only if $\det \mathbf{A} = 0$.

(iv) For any $\mathbf{A}$ and $\mathbf{B}$ there are at most two matrices $\mathbf{X}$ such that $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$.

[STEP 2, 1993Q7]

The integers a, b and c satisfy

$$2a^2 + b^2 = 5c^2.$$

By considering the possible values of $a \pmod 5$ and $b \pmod 5$, show that a and b must both be divisible by 5.

By considering how many times a, b and c can be divided by 5, show that the only solution is $a = b = c = 0$.

[STEP 2, 1993Q8]

Suppose that $a_i > 0$ for all $i > 0$. Show that

$$a_1 a_2 \leq \left(\frac{a_1 + a_2}{2}\right)^2.$$

Prove by induction that for all positive integers m

$$a_1 \dots a_{2^m} \leq \left(\frac{a_1 + \cdots + a_{2^m}}{2^m}\right)^{2^m}. \qquad (*)$$

If $n < 2^m$, put $b_1 = a_1, b_2 = a_2, \dots, b_n = a_n$ and $b_{n+1} = \cdots = b_{2^m} = A$, where

$$A = \frac{a_1 + \cdots + a_n}{n}.$$

By applying $(*)$ to the b_i, show that

$$a_1 \dots a_n A^{(2^m - n)} \leq A^{2^m}$$

(notice that $b_1 + \cdots + b_n = nA$). Deduce the (arithmetic mean)/(geometric mean) inequality

$$(a_1 \dots a_n)^{\frac{1}{n}} \leq \frac{a_1 + \cdots + a_n}{n}.$$

[STEP 2, 1993Q9]

In this question, the argument of a complex number is chosen to satisfy $0 \leq \arg z < 2\pi$. Let z be a complex number whose imaginary part is positive. What can you say about $\arg z$?

The complex numbers z_1, z_2, z_3 all have positive imaginary part and $\arg z_1 < \arg z_2 < \arg z_3$. Draw a diagram that shows why

$$\arg z_1 < \arg(z_1 + z_2 + z_3) < \arg z_3.$$

Prove that $\arg(z_1 z_2 z_3)$ is never equal to $\arg(z_1 + z_2 + z_3)$.

[STEP 2, 1993Q10]

Verify that if

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} -1 & 8 \\ 8 & 11 \end{pmatrix}$$

then $\mathbf{PAP}$ is a diagonal matrix.

Put $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. By writing

$$\mathbf{x} = \mathbf{P}\mathbf{x}_1 + \mathbf{a}$$

for a suitable vector $\mathbf{a}$, show that the equation

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} - 11 = 0,$$

where $\mathbf{b} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ and $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$, becomes

$$3x_1^2 - y_1^2 = c.$$

[STEP 2, 1993Q11]

[In this question, take the value of g to be 10 m s^{-2}.]

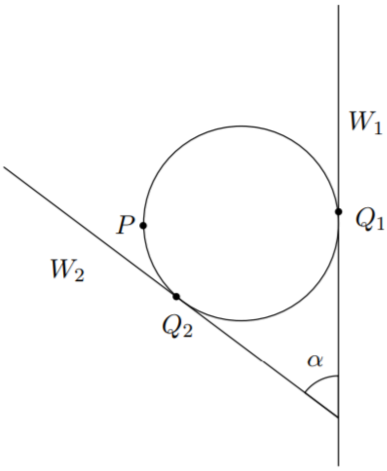
A body of mass m kg is dropped vertically into a deep pool of liquid. Once in the liquid, it is subject to gravity, an upward buoyancy force of $\frac{6}{5}$ times its weight, and a resistive force of $2mv^2$ N opposite to its direction of travel when it is travelling at speed v m s^{-1}. Show that the body stops sinking less than $\frac{\pi}{4}$ seconds after it enters the pool.

Suppose now that the body enters the liquid with speed 1 m s^{-1}. Show that the body descends to a depth of $\frac{1}{4}\ln 2$ metres and that it returns to the surface with speed $\frac{1}{\sqrt{2}}$ m s^{-1}, at a time

$$\frac{\pi}{8} + \frac{1}{4}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

seconds after entering the pool.

[STEP 2, 1993Q12]



A uniform sphere of mass M and radius r rests between a vertical wall W_1 and an inclined plane W_2 that meets W_1 at an angle α. Q_1 and Q_2 are the points of contact of the sphere with W_1 and W_2 respectively, as shown in the diagram. A particle of mass m is attached to the sphere at P, where PQ_1 is a diameter, and the system is released. The sphere is on the point of slipping at Q_1 and at Q_2. Show that if the coefficients of friction between the sphere and W_1 and W_2 are μ_1 and μ_2 respectively, then

$$m = \frac{\mu_2 + \mu_1 \cos\alpha - \mu_1\mu_2 \sin\alpha}{(2\mu_1\mu_2 + 1)\sin\alpha + (\mu_2 - 2\mu_1)\cos\alpha - \mu_2} M.$$

If the sphere is on the point of rolling about Q_2 instead of slipping, show that

$$m = \frac{M}{\sec\alpha - 1}.$$

[STEP 2, 1993Q13]

The force F of repulsion between two particles with positive charges Q and Q' is given by $F = \frac{kQQ'}{r^2}$, where k is a positive constant and r is the distance between the particles. Two small beads P_1 and P_2 are fixed to a straight horizontal smooth wire, a distance d apart. A third bead P_3 of mass m is free to move along the wire between P_1 and P_2. The beads carry positive electrical charges Q_1, Q_2 and Q_3. If P_3 is in equilibrium at a distance a from P_1, show that

$$a = \frac{d\sqrt{Q_1}}{\sqrt{Q_1} + \sqrt{Q_2}}.$$

Suppose that P_3 is displaced slightly from its equilibrium position and released from rest. Show that it performs approximate simple harmonic motion with period

$$\frac{\pi d}{\left(\sqrt{Q_1} + \sqrt{Q_2}\right)^2} \sqrt{\frac{2md\sqrt{Q_1 Q_2}}{kQ_3}}.$$

[You may use the fact that $\frac{1}{(a+y)^2} \approx \frac{1}{a^2} - \frac{2y}{a^3}$ for small y.]

[STEP 2, 1993Q14]

A ball of mass m is thrown vertically upwards from the floor of a room of height h with speed $\sqrt{2kgh}$, where $k > 1$. The coefficient of restitution between the ball and the ceiling or floor is a. Both the ceiling and floor are level. Show that the kinetic energy of the ball immediately before hitting the ceiling for the nth time is

$$mgh\left(a^{4n-4}(k-1)+\frac{a^{4n-4}-1}{a^2+1}\right).$$

Hence show that the number of times the ball hits the ceiling is at most

$$1-\frac{\ln(a^2(k-1)+k)}{4\ln a}.$$

[STEP 2, 1993Q15]

Two computers, LEP and VOZ, are programmed to add numbers after first approximating each number by an integer. LEP approximates the numbers by rounding: that is, it replaces each number by the nearest integer. VOZ approximates by truncation: that is, it replaces each number by the largest integer less than or equal to that number. The fractional parts of the numbers to be added are uniformly and independently distributed. (The fractional part of a number a is $a-\lfloor a\rfloor$, where $\lfloor a\rfloor$ is the largest integer less than or equal to a.) Both computers approximate and add 1500 numbers. For each computer, find the probability that the magnitude of the error in the answer will exceed 15.

How many additions can LEP perform before the probability that the magnitude of the error is less than 10 drops below 0.9?

[STEP 2, 1993Q16]

At the terminus of a bus route, passengers arrive at an average rate of 4 per minute according to a Poisson process. Each minute, on the minute, one bus arrives with probability $\frac{1}{4}$, independently of the arrival of passengers or previous buses. Just after eight o'clock there is no-one at the bus stop.

(i) What is the probability that the first bus arrives at n minutes past 8?

(ii) If the first bus arrives at 8:05, what is the probability that there are m people waiting for it?

(iii) Each bus can take 25 people and, since it is the terminus, the buses arrive empty. Explain carefully how you would calculate, to two significant figures, the probability that when the first bus arrives it is unable to pick up all the passengers. Your method should need the use of a calculator and standard tables only. There is no need to carry out the calculation.

STEP 2 1994

STEP 2 1994
On-line Exam

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TIME ALLOWED: 180 MINUTES

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INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

Section A	Pure Mathematics
Section B	Mechanics
Section C	Probability and Statistics

There are 14 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

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Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 1994Q1]

In this question we consider only positive, non-zero integers written out in the usual (decimal) way. We say, for example, that 207 ends in 7 and that 5310 ends in 1 followed by 0. Show that, if n does not end in 5 or an even number, then there exists m such that $n \times m$ ends in 1.

Show that, given any n, we can find m such that $n \times m$ ends either in 1 or in 1 followed by one or more zeros.

Show that, given any n which ends in 1 or in 1 followed by one or more zeros, we can find m such that $n \times m$ contains all the digits $0, 1, 2, \ldots, 9$.

[STEP 2, 1994Q2]

If Q is a polynomial, m is an integer, $m \geq 1$ and $P(x) = (x-a)^m Q(x)$, show that $P'(x) = (x-a)^{m-1}R(x)$ where R is a polynomial. Explain why $P^r(a) = 0$ whenever $1 \leq r \leq m-1$. (P^r is the rth derivative of P.)

If

$$P_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

for $n \geq 1$ show that P_n is a polynomial of degree n. By repeated integration by parts, or otherwise, show that, if $n-1 \geq m \geq 0$,

$$\int_{-1}^{1} x^m P_n(x)\mathrm{d}x = 0$$

and find the value of

$$\int_{-1}^{1} x^n P_n(x)\mathrm{d}x.$$

[You may use the formula $\int_0^{\frac{\pi}{2}} \cos^{2n+1} t \, \mathrm{d}t = \frac{(2^{2n})(n!)^2}{(2n+1)!}$ without proof if you need it. However some ways of doing this question do not use this formula.]

[STEP 2, 1994Q3]

The function f satisfies $f(0) = 1$ and

$$f(x - y) = f(x)f(y) - f(a - x)f(a + y)$$

for some fixed number a and all x and y. Without making any further assumptions about the nature of the function show that $f(a) = 0$.

Show that, for all t,

(i) $f(t) = f(-t)$,

(ii) $f(2a) = -1$,

(iii) $f(2a - t) = -f(t)$,

(iv) $f(4a + t) = f(t)$.

Give an example of a non-constant function satisfying the conditions of the first paragraph with $a = \frac{\pi}{2}$. Give an example of a non-constant function satisfying the conditions of the first paragraph with $a = -2$.

[STEP 2, 1994Q4]

By considering the area of the region defined in terms of Cartesian co-ordinates (x, y) by

$$\{(x, y): x^2 + y^2 \leq 1, 0 \leq y, 0 \leq x \leq c\},$$

show that

$$\int_0^c (1 - x^2)^{\frac{1}{2}}\,\mathrm{d}x = \frac{1}{2}\left(c(1 - c^2)^{\frac{1}{2}} + \sin^{-1} c\right),$$

if $0 < c \leq 1$.

Show that the area of the region defined by

$$\left\{(x, y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, 0 \leq y, 0 \leq x \leq c\right\},$$

is

$$\frac{ab}{2}\left(\frac{c}{a}\left(1 - \frac{c^2}{a^2}\right)^{\frac{1}{2}} + \sin^{-1}\frac{c}{a}\right),$$

if $0 < c \leq a$ and $0 < b$.

Suppose that $0 < b \leq a$. Show that the area of the intersection $E \cap F$ of the two regions defined by

$$E = \left\{(x, y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\} \quad \text{and} \quad F = \left\{(x, y): \frac{x^2}{b^2} + \frac{y^2}{a^2} \leq 1\right\}$$

is

$$4ab \sin^{-1}\left(\frac{b}{\sqrt{a^2 + b^2}}\right).$$

[STEP 2, 1994Q5]

(i) Show that the equation

$$(x-1)^4+(x+1)^4=c$$

has exactly two real roots if $c>2$, one root if $c=2$ and no roots if $c<2$.

(ii) How many real roots does the equation $(x-3)^4+(x-1)^4=c$ have?

(iii) How many real roots does the equation $|x-3|+|x-1|=c$ have?

(iv) How many real roots does the equation $(x-3)^3+(x-1)^3=c$ have?

[The answers to parts (ii), (iii) and (iv) may depend on the value of c. You should give reasons for your answers.]

[STEP 2, 1994Q6]

Prove by induction, or otherwise, that, if $0<\theta<\pi$,

$$\frac{1}{2}\tan\frac{\theta}{2}+\frac{1}{2^2}\tan\frac{\theta}{2^2}+\cdots+\frac{1}{2^n}\tan\frac{\theta}{2^n}=\frac{1}{2^n}\cot\frac{\theta}{2^n}-\cot\theta.$$

Deduce that

$$\sum_{r=1}^{\infty}\frac{1}{2^r}\tan\frac{\theta}{2^r}=\frac{1}{\theta}-\cot\theta.$$

[STEP 2, 1994Q7]

Show that the equation

$$ax^2+ay^2+2gx+2fy+c=0$$

where $a>0$ and $f^2+g^2>ac$ represents a circle in Cartesian coordinates and find its centre.

The smooth and level parade ground of the First Ruritanian Infantry Division is ornamented by two tall vertical flagpoles of heights h_1 and h_2 a distance d apart. As part of an initiative test a soldier has to march in such a way that he keeps the angles of elevation of the tops of the two flagpoles equal to one another. Show that if the two flagpoles are of different heights he will march in a circle. What happens if the two flagpoles have the same height?

To celebrate the King's birthday a third flagpole is added. Soldiers are then assigned to each of the three different pairs of flagpoles and are told to march in such a way that they always keep the tops of their two assigned flagpoles at equal angles of elevation to one another. Show that, if the three flagpoles have different heights h_1, h_2 and h_3 and the circles in which the soldiers march have centres at (x_{ij},y_{ij}) (for the flagpoles of height h_i and h_j) relative to Cartesian coordinates fixed in the parade ground, then the x_{ij} satisfy

$$h_3^2(h_1^2-h_2^2)x_{12}+h_1^2(h_2^2-h_3^2)x_{23}+h_2^2(h_3^2-h_1^2)x_{31}=0,$$

and the same equation connects the y_{ij}. Deduce that the three centres lie in a straight line.

[STEP 2, 1994Q8]

'24 Hour Spares' stocks a small, widely used and cheap component. Every T hours X units arrive by lorry from the wholesaler, for which the owner pays a total £$(a + qX)$. It costs the owner £b per hour to store one unit. If she has the units in stock she expects to sell r units per hour at £$(p + q)$ per unit. The other running costs of her business remain at £c pounds an hour irrespective of whether she has stock or not. (All of the quantities T, X, a, b, r, q, p and c are greater than 0.) Explain why she should take $X \leq rT$.

Given that the process may be assumed continuous (the items are very small and she sells many each hour), sketch $S(t)$ the amount of stock remaining as a function of t the time from the last delivery. Compute the total profit over each period of T hours. Show that, if T is fixed with $T \geq \frac{p}{b}$, the business can be made profitable if

$$p^2 > 2\frac{(a + cT)b}{r}.$$

Section B: Mechanics

[STEP 2, 1994Q9]

A light rod of length $2a$ is hung from a point O by two light inextensible strings OA and OB each of length b and each fixed at O. A particle of mass m is attached to the end A and a particle of mass $2m$ is attached to the end B. Show that, in equilibrium, the angle θ that the rod makes with the horizontal satisfes the equation

$$\tan\theta = \frac{a}{3\sqrt{b^2 - a^2}}.$$

Express the tension in the string AO in terms of m, g, a and b.

[STEP 2, 1994Q10]

A truck is towing a trailer of mass m across level ground by means of an elastic rope of natural length l whose modulus of elasticity is λ. At first the rope is slack and the trailer stationary. The truck then accelerates until the rope becomes taut and thereafter the truck travels in a straight line at a constant speed u. Assuming that the effect of friction on the trailer is negligible, show that the trailer will collide with the truck at a time

$$\pi\left(\frac{lm}{\lambda}\right)^{\frac{1}{2}} + \frac{l}{u}$$

after the rope first becomes taut.

[STEP 2, 1994Q11]

As part of a firework display a shell is fired vertically upwards with velocity v from a point on a level stretch of ground. When it reaches the top of its trajectory an explosion splits it into two equal fragments each travelling at speed u but (since momentum is conserved) in exactly opposite (not necessarily horizontal) directions. Show, neglecting air resistance, that the greatest possible distance between the points where the two fragments hit the ground is $\frac{2uv}{g}$ if $u \leq v$ and $\frac{(u^2+v^2)}{g}$ if $v \leq u$.

Section C: Probability and Statistics

[STEP 2, 1994Q12]

Calamity Jane sits down to play the game of craps with Buffalo Bill. In this game she rolls two fair dice. If, on the first throw, the sum of the dice is 2, 3 or 12 she loses, while if it is 7 or 11 she wins. Otherwise Calamity continues to roll the dice until either the first sum is repeated, in which case she wins, or the sum is 7, in which case she loses. Find the probability that she wins on the first throw.

Given that she throws more than once, show that the probability that she wins on the nth throw is

$$\frac{1}{48}\left(\frac{3}{4}\right)^{n-2} + \frac{1}{27}\left(\frac{13}{18}\right)^{n-2} + \frac{25}{432}\left(\frac{25}{36}\right)^{n-2}.$$

Given that she throws more than m times, where $m > 1$, what is the probability that she wins on the nth throw?

[STEP 2, 1994Q13]

The makers of Cruncho ('The Cereal Which Cares') are giving away a series of cards depicting n great mathematicians. Each packet of Cruncho contains one picture chosen at random. Show that when I have collected r different cards the expected number of packets I must open to find a new card is $\frac{n}{n-r}$ $[0 \leq r \leq n-1]$.

Show by means of a diagram, or otherwise, that

$$\frac{1}{r+1} \leq \int_r^{r+1} \frac{1}{x}\,\mathrm{d}x \leq \frac{1}{r}$$

and deduce that

$$\sum_{r=2}^{n} \frac{1}{r} \leq \ln n \leq \sum_{r=1}^{n-1} \frac{1}{r}$$

for all $n \geq 2$.

My children will give me no peace until we have the complete set of cards, but I am the only person in our household prepared to eat Cruncho and my spouse will only buy the stuff if I eat it. If n is large, roughly how many packets must I expect to consume before we have the set?

[STEP 2, 1994Q14]

When Septimus Moneybags throws darts at a dart board they are certain to end on the board (a disc of radius a) but, it must be admitted, otherwise are uniformly randomly distributed over the board.

(i) Show that the distance R that his shot lands from the centre of the board is a random variable with variance $\frac{a^2}{18}$.

(ii) At a charity fête he can buy m throws for £$(12 + m)$, but he must choose m before he starts to throw. If at least one of his throws lands within $\frac{a}{\sqrt{10}}$ of the centre he wins back £12. In order to show what a good sport he is, he determines to play but, being a careful man, he wishes to choose m so as to minimise his expected loss. What value of m should he choose?

STEP 2 1995

STEP 2 1995
On-line Exam

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TIME ALLOWED: 180 MINUTES

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Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 1995Q1]

(i) By considering $(1 + x + x^2 + \cdots + x^n)(1 - x)$ show that, if $x \neq 1$,

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

(ii) By differentiating both sides and setting $x = -1$ show that

$$1 - 2 + 3 - 4 + \cdots + (-1)^{n-1}n$$

takes the value $-\frac{n}{2}$ if n is even and the value $\frac{n+1}{2}$ if n is odd.

(iii) Show that

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}(An^2 + Bn)$$

where the constants A and B are to be determined.

[STEP 2, 1995Q2]

I have n fence posts placed in a line and, as part of my spouse's birthday celebrations, I wish to paint them using three different colours red, white and blue in such a way that no adjacent fence posts have the same colours. (This allows the possibility of using fewer than three colours as well as exactly three.) Let r_n be the number of ways (possibly zero) that I can paint them if I paint the first and last post red and let s_n be the number of ways that I can paint them if I paint the first post red but the last post either of the other two colours. Explain why $r_{n+1} = s_n$ and find $r_n + s_n$. Hence find the value of $r_{n+1} + r_n$ for all $n \geq 1$.

Prove, by induction, that

$$r_n = \frac{2^{n-1} + 2(-1)^{n-1}}{3}.$$

Find the number of ways of painting n fence posts (where $n \geq 3$) placed in a circle using three different colours in such a way that no adjacent fence posts have the same colours.

[STEP 2, 1995Q3]

The Tour de Clochemerle is not yet as big as the rival Tour de France. This year there were five riders, Arouet, Barthes, Camus, Diderot and Eluard, who took part in five stages. The winner of each stage got 5 points, the runner up 4 points and so on down to the last rider who got 1 point. The total number of points acquired over the five stages was the rider's score. Each rider obtained a different score overall and the riders finished the whole tour in alphabetical order with Arouet gaining a magnificent 24 points. Camus showed consistency by gaining the same position in four of the five stages and Eluard's rather dismal performance was relieved by a third place in the fourth stage and first place in the final stage. Explain why Eluard must have received 11 points in all and find the scores obtained by Barthes, Camus and Diderot.

Where did Barthes come in the final stage?

[STEP 2, 1995Q4]

Let

$$u_n = \int_0^{\frac{\pi}{2}} \sin^n t \, \mathrm{d}t$$

for each integer $n \geq 0$. By integrating

$$\int_0^{\frac{\pi}{2}} \sin t \sin^{n-1} t \, \mathrm{d}t$$

by parts, or otherwise, obtain a formula connecting u_n and u_{n-2} when $n \geq 2$ and deduce that

$$nu_n u_{n-1} = (n-1)u_{n-1}u_{n-2}$$

for all $n \geq 2$. Deduce that

$$nu_n u_{n-1} = \frac{\pi}{2}.$$

Sketch graphs of $\sin^n t$ and $\sin^{n-1} t$, for $0 \leq t \leq \frac{\pi}{2}$, on the same diagram and explain why $0 < u_n < u_{n-1}$. By using the result of the previous paragraph show that

$$nu_n^2 < \frac{\pi}{2} < nu_{n-1}^2$$

for all $n \geq 1$. Hence show that

$$\left(\frac{n}{n+1}\right)\frac{\pi}{2} < nu_n^2 < \frac{\pi}{2}$$

and deduce that $nu_n^2 \to \frac{\pi}{2}$ as $n \to \infty$.

[STEP 2, 1995Q5]

The famous film star Birkhoff Maclane is sunning herself by the side of her enormous circular swimming pool (with centre O) at a point A on its circumference. She wants a drink from a small jug of iced tea placed at the diametrically opposite point B. She has three choices:

(a) to swim directly to B.

(b) to choose θ with $0 < \theta < \pi$, to run round the pool to a point X with $\angle AOX = \theta$ and then to swim directly from X to B.

(c) to run round the pool from A to B.

She can run k times as fast as she can swim and she wishes to reach her tea as fast as possible. Explain, with reasons, which of (a), (b)and (c) she should choose for each value of k. Is there one choice from (a), (b) and (c) she will never take whatever the value of k?

[STEP 2, 1995Q6]

If u and v are the two roots of $z^2 + az + b = 0$, show that $a = -u - v$ and $b = uv$.

Let $\alpha = \cos\frac{2\pi}{7} + \mathbf{i}\sin\frac{2\pi}{7}$. Show that α is a root of $z^7 - 1 = 0$ and express the remaining roots in terms of α. The number $\alpha + \alpha^2 + \alpha^4$ is a root of a quadratic equation

$$z^2 + Az + B = 0$$

where A and B are real. By guessing the other root, or otherwise, find the numerical values of A and B.

Show that

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2},$$

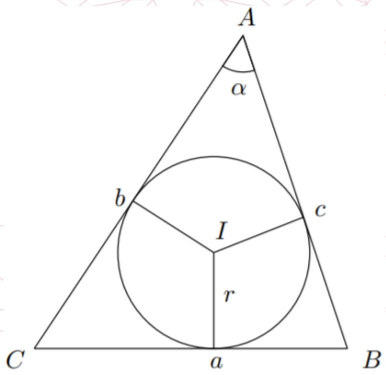
and evaluate

$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7},$$

making it clear how you determine the sign of your answer.

[STEP 2, 1995Q7]

The diagram shows a circle of radius r and centre I, touching the three sides of a triangle ABC. We write a for the length of BC and α for the angle $\angle BAC$ and so on. Let $s = \frac{(a+b+c)}{2}$ and let Δ be the area of the triangle.



(i) By considering the area of the triangles AIB, BIC and CIA, or otherwise, show that $\Delta = rs$.

(ii) By using the formula $\Delta = \frac{1}{2}bc\sin\alpha$, show that

$$\Delta^2 = \frac{1}{16}(4b^2c^2 - (2bc\cos\alpha)^2).$$

Now use the formula $a^2 = b^2 + c^2 - 2bc\cos\alpha$ to show that

$$\Delta^2 = (a^2 - (b-c)^2)((b+c)^2 - a^2)$$

and deduce that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

(iii) A hole in the shape of the triangle ABC is cut in the top of a level table. A sphere of radius R rests in the hole. Find the height of the centre of the sphere above the level of the table-top, expressing your answer in terms of a, b, c, s and R.

[STEP 2, 1995Q8]

If there are x micrograms of bacteria in a nutrient medium, the population of bacteria will grow at the rate $(2K - x)x$ micrograms per hour. Show that, if $x = K$ when $t = 0$, the population at time t is given by

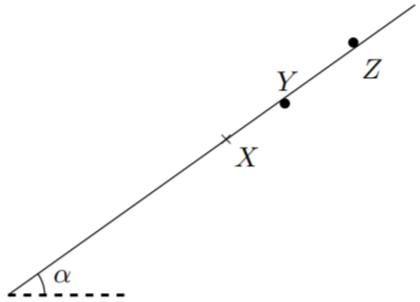
$$x(t) = K + K\frac{1 - \mathrm{e}^{-2Kt}}{1 + \mathrm{e}^{-2Kt}}.$$

Sketch, for $t \geq 0$, the graph of x against t. What happens to $x(t)$ as $t \to \infty$?

Now suppose that the situation is as described in the first paragraph, except that we remove bacteria from the nutrient medium at a rate L micrograms per hour where $K^2 > L$. We set $\alpha = \sqrt{K^2 - L}$. Write down the new differential equation for x. By considering a new variable $y = x - K + \alpha$, or otherwise, show that, if $x(0) = K$ then $x(t) \to K + \alpha$ as $t \to \infty$.

Section B: Mechanics

[STEP 2, 1995Q9]



Two thin horizontal bars are parallel and fixed at a distance d apart, and the plane containing them is at an angle α to the horizontal. A thin uniform rod rests in equilibrium in contact with the bars under one and above the other and perpendicular to both. The diagram shows the bars (in cross section and exaggerated in size) with the rod over one bar at Y and under the other at Z. (Thus YZ has length d.) The centre of the rod is at X and XZ has length l. The coefficient of friction between the rod and each bar is μ. Explain why we must have $l \geq d$.

Find, in terms of d, l and α, the least possible value of μ.Verify that, when l = 2d, your result shows that

$$\mu \geq \frac{1}{3}\tan\alpha.$$

[STEP 2, 1995Q10]

Three small spheres of masses m_1, m_2 and m_3, move in a straight line on a smooth horizontal table. (Their order on the straight line is the order given.) The coefficient of restitution between any two spheres is e. The first moves with velocity u towards the second whilst the second and third are at rest. After the first collision the second sphere hits the third after which the velocity of the second sphere is u. Find m_1 in terms of m_2, m_3 and e. Deduce that

$$m_2 e > m_3(1 + e + e^2).$$

Suppose that the relation between m_1, m_2 and m_3 is that in the formula you found above, but that now the first sphere initially moves with velocity u and the other two spheres with velocity v, all in the same direction along the line. If $u > v > 0$ use the first part to find the velocity of the second sphere after two collisions have taken place. (You should not need to make any substantial computations but you should state your argument clearly.)

[STEP 2, 1995Q11]

Two identical particles of unit mass move under gravity in a medium for which the magnitude of the retarding force on a particle is k times its speed. The first particle is allowed to fall from rest at a point A whilst, at the same time, the second is projected upwards with speed u from a point B a positive distance d vertically above A. Find their distance apart after a time t and show that this distance tends to the value

$$d + \frac{u}{k}$$

as $t \to \infty$.

Section C: Probability and Statistics

[STEP 2, 1995Q12]

Bread roll throwing duels at the Drones' Club are governed by a strict etiquette. The two duellists throw alternately until one is hit, when the other is declared the winner. If Percy has probability $p > 0$ of hitting his target and Rodney has probability $r > 0$ of hitting his, show that, if Perey throws first, the probability that he beats Rodney is

$$\frac{p}{p + r - pr}.$$

Algernon, Bertie and Cuthbert decide to have a three sided duel in which they throw in order $A, B, C, A, B, C, \ldots$ except that anyone who is hit must leave the game. Cuthbert always hits his target, Bertie hits his target with probability $\frac{3}{5}$ and Algernon hits his target with probability $\frac{2}{5}$. Bertie and Cuthbert will always aim at each other if they are both still in the duel. Otherwise they aim at Algernon. With his first shot Algernon may aim at either Bertie or Cuthbert or deliberately miss both. Faced with only one opponent Algernon will aim at him.

(i) What are Algernon's chances of winning if he hits Cuthbert with his first shot?

(ii) What are Algernon's chances of winning if he hits Bertie with his first shot?

(iii) What are Algernon's chances of winning if he misses with his first shot?

Advise Algernon as to his best plan and show that, if he uses this plan, his probability of winning is $\frac{226}{475}$.

[STEP 2, 1995Q13]

Fly By Night Airlines run jumbo jets which seat N passengers. From long experience they know that a very small proportion ϵ of their passengers fail to turn up. They decide to sell $N + k$ tickets for each flight. If k is very small compared with N explain why they might expect

$$\mathrm{P}(r \text{ passengers fail to turn up}) = \frac{\lambda^r}{r!}\mathrm{e}^{-\lambda}$$

approximately, with $\lambda = N\epsilon$. For the rest of the question you may assume that the formula holds exactly.

Each ticket sold represents £A profit, but the airline must pay each passenger that it cannot fly £B where $B > A > 0$. Explain why, if r passengers fail to turn up, its profit, in pounds, is

$$A(N + k) - B \max(0, k - r),$$

where $\max(0, k - r)$ is the larger of 0 and $k - r$. Write down the expected profit u_k when $k = 0, k = 1, k = 2, k = 3$. Find $v_k = u_{k+1} - u_k$ for general k and show that $v_k > v_{k+1}$. Show also that

$$v_k \to A - B$$

as $k \to \infty$.

Advise Fly By Night on how to choose k to maximise its expected profit u_k.

[STEP 2, 1995Q14]

Suppose X is a random variable with probability density

$$f(x) = Ax^2 \exp\left(-\frac{x^2}{2}\right)$$

for $-\infty < x < \infty$. Find A.

You belong to a group of scientists who believe that the outcome of a certain experiment is a random variable with the probability density just given, while other scientists believe that the probability density is the same except with different mean (i.e. the probability density is $f(x-\mu)$ with $\mu \neq 0$). In each of the following two cases decide whether the result given would shake your faith in your hypothesis, and justify your answer.

(i) A single trial produces the result 87.3.

(ii) 1000 independent trials produce results having a mean value 0.23.

[Great weight will be placed on clear statements of your reasons and none on the mere repetition of standard tests, however sophisticated, if unsupported by argument. There are several possible approaches to this question. For some of them it is useful to know that if Z is normal with mean 0 and variance 1 then $\mathrm{E}(Z^4) = 3$.]

STEP 2 1996

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On-line Exam

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Section A: Pure Mathematics

[STEP 2, 1996Q1]

(i) Find the coefficient of x^6 in

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4)^3.$$

You should set out your working clearly.

(ii) By considering the binomial expansions of $(1+x)^{-2}$ and $(1+x)^{-6}$, or otherwise, find the coefficient of x^6 in

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6)^3.$$

[STEP 2, 1996Q2]

Consider the system of equations

$$\begin{aligned} 2yz + zx - 5xy &= 2 \\ yz - zx + 2xy &= 1 \\ yz - 2zx + 6xy &= 3. \end{aligned}$$

Show that

$$xyz = \pm 6$$

and find the possible values of x, y and z.

[STEP 2, 1996Q3]

The Fibonacci numbers F_n are defined by the conditions $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}$$

for all $n \geq 1$. Show that $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ and compute F_5, F_6 and F_7.

Compute $F_{n+1}F_{n-1} - F_n^2$ for a few values of n; guess a general formula and prove it by induction, or otherwise.

By induction on k, or otherwise, show that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

for all positive integers n and k.

[STEP 2, 1996Q4]

Show that $\cos 4u = 8\cos^4 u - 8\cos^2 u + 1$.

If

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2}\,\mathrm{d}x,$$

Show, by using the change of variable $x = \cos t$, that

$$I = \int_{0}^{\pi} \frac{\sin t}{4\cos^2\left(\frac{t}{4} - \frac{\pi}{8}\right)}\,\mathrm{d}t.$$

By using the further change of variable $u = \frac{t}{4} - \frac{\pi}{8}$, or otherwise, show that

$$I = 4\sqrt{2} - \pi - 2.$$

[You may assume that $\tan\frac{\pi}{8} = \sqrt{2} - 1$.]

[STEP 2, 1996Q5]

If

$$z^4 + z^3 + z^2 + z + 1 = 0 \qquad (*)$$

and $u = z + z^{-1}$, find the possible values of u. Hence find the possible values of z. [Do not try to simplify your answers.]

show that, if z satisfies $(*)$, then

$$z^5 - 1 = 0.$$

Hence write the solutions of $(*)$ in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ for suitable real r and θ. Deduce that

$$\sin\frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4} \quad \text{and} \quad \cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}.$$

[STEP 2, 1996Q6]

A *proper factor* of a positive integer N is an integer M, with $M \neq 1$ and $M \neq N$, which divides N without remainder. Show that 12 has 4 proper factors and 16 has 3.

Suppose that N has the prime factorisation

$$N = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r},$$

where $p_1, p_2, \dots, p_r$ are distinct primes and $m_1, m_2, \dots, m_r$ are positive integers. How many proper factors does N have and why?

Find:

(i) the smallest positive integer which has precisely 12 proper factors.

(ii) the smallest positive integer which has at least 12 proper factors.

[STEP 2, 1996Q7]

Consider a fixed square $ABCD$ and a variable point P in the plane of the square. We write the perpendicular distance from P to AB as p, from P to BC as q, from P to CD as r and from P to DA as s. (Remember that distance is never negative, so $p, q, r, s \geq 0$.) If $pr = qs$, show that the only possible positions of P lie on two straight lines and a circle and that every point on these two lines and a circle is indeed a possible position of P.

[STEP 2, 1996Q8]

Suppose that

$$f''(x) + f(-x) = x + 3\cos 2x$$

and $f(0) = 1$, $f'(0) = -1$. If $g(x) = f(x) + f(-x)$, find $g(0)$ and show that $g'(0) = 0$. Show that

$$g''(x) + g(x) = 6\cos 2x,$$

and hence find $g(x)$.

Similarly, if $h(x) = f(x) - f(-x)$, find $h(x)$ and show that

$$f(x) = 2\cos x - \cos 2x - x.$$

Section B: Mechanics

[STEP 2, 1996Q9]

A child's toy consists of a solid cone of height λa and a solid hemisphere of radius a, made out of the same uniform material and fastened together so that their plane faces coincide. (Thus the diameter of the hemisphere is equal to that of the base of the cone.) Show that if $\lambda < \sqrt{3}$ the toy will always move to an upright position if placed with the surface of the hemisphere on a horizontal table, but that if $\lambda > \sqrt{3}$ the toy may overbalance.

Show, however, that if the toy is placed with the surface of the cone touching the table it will remain there whatever the value of λ.

[The centre of gravity of a uniform solid cone of height h is a height $\frac{h}{4}$ above its base. The centre of gravity of a uniform solid hemisphere of radius a is at distance $\frac{3a}{8}$ from the centre of its base.]

[STEP 2, 1996Q10]

The plot of 'Rhode Island Red and the Henhouse of Doom' calls for the heroine to cling on to the circumference of a fairground wheel of radius a rotating with constant angular velocity ω about its horizontal axis and then let go. Let ω_0 be the largest value of ω for which it is not possible for her subsequent path to carry her higher than the top of the wheel. Find ω_0 in terms of a and g.

If $\omega > \omega_0$ show that the greatest height above the top of the wheel to which she can rise is

$$\frac{a}{2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2.$$

[STEP 2, 1996Q11]

A particle hangs in equilibrium from the ceiling of a stationary lift, to which it is attached by an elastic string of natural length l extended to a length $l + a$. The lift now descends with constant acceleration f such that $0 < f < \frac{g}{2}$. Show that the extension y of the string from its equilibrium length satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{g}{a}y = g - f.$$

Hence show that the string never becomes slack and the amplitude of the oscillation of the particle is $\frac{af}{g}$.

After a time T the lift stops accelerating and moves with constant velocity. Show that the string never becomes slack and the amplitude of the oscillation is now

$$\frac{2af}{g}\left|\sin\frac{1}{2}\omega T\right|,$$

where $\omega^2 = \frac{g}{a}$.

Section C: Probability and Statistics

[STEP 2, 1996Q12]

(i) Let $X_1, X_2, \dots, X_n$ be independent random variables each of which is uniformly distributed on $[0, 1]$. Let Y be the largest of $X_1, X_2, \dots, X_n$. By using the fact that $Y < \lambda$ if and only if $X_j < \lambda$ for $1 \le j \le n$, find the probability density function of Y. Show that the variance of Y is

$$\frac{n}{(n+2)(n+1)^2}.$$

(ii) The probability that a neon light switched on at time 0 will have failed by a time $t > 0$ is $1 - \mathrm{e}^{-\frac{t}{\lambda}}$ where $\lambda > 0$. I switch on n independent neon lights at time zero. Show that the expected time until the first failure is $\frac{\lambda}{n}$.

[STEP 2, 1996Q13]

By considering the coefficients of t^n in the equation

$$(1+t)^n(1+t)^n = (1+t)^{2n},$$

or otherwise, show that

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{r}\binom{n}{n-r} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}.$$

The large American city of Triposville is laid out in a square grid with equally spaced streets running east-west and avenues running north-south. My friend is staying at a hotel n avenues west and n streets north of my hotel. Both hotels are at intersections. We set out from our own hotels at the same time. We walk at the same speed, taking 1 minute to go from one intersection to the next. Every time I reach an intersection I go north with probability $\frac{1}{2}$ or west with probability $\frac{1}{2}$. Every time my friend reaches an intersection she goes south with probability $\frac{1}{2}$ or east with probability $\frac{1}{2}$. Our choices are independent of each other and of our previous decisions. Indicate by a sketch or by a brief description the set of points where we could meet. Find the probability that we meet.

Suppose that I oversleep and leave my hotel $2k$ minutes later than my friend leaves hers. where k is an integer and $0 \le 2k \le n$. Find the probability that we meet. Have you any comment? If $n = 1$ and I leave my hotel 1 minute later than my friend leaves hers, what is the probability that we meet and why?

[STEP 2, 1996Q14]

The random variable X is uniformly distributed on $[0,1]$. A new random variable Y is defined by the rule

$$Y = \begin{cases} \frac{1}{4}, & \text{if } X \leq \frac{1}{4}, \\ X, & \text{if} \frac{1}{4} \leq X \leq \frac{3}{4}, \\ \frac{3}{4}, & \text{if } X \geq \frac{3}{4}. \end{cases}$$

Find $\mathrm{E}(Y^n)$ for all integers $n \geq 1$.

Show that $\mathrm{E}(Y) = \mathrm{E}(X)$ and that

$$\mathrm{E}(X^2) - \mathrm{E}(Y^2) = \frac{1}{24}.$$

By using the fact that $4^n = (3+1)^n$,or otherwise, show that $\mathrm{E}(X^n) > \mathrm{E}(Y^n)$ for $n \geq 2$.

Suppose that $Y_1, Y_2, \ldots$ are independent random variables each having the same distribution as Y. Find, to a good approximation, K such that

$$\mathrm{P}(Y_1 + Y_2 + \cdots + Y_{240000} < K) = \frac{3}{4}.$$

STEP 2 1997

STEP 2 1997
On-line Exam

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Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 1997Q1]

Find the sum of those numbers between 1000 and 6000 every one of whose digits is one of the numbers 0, 2, 5 or 7, giving your answer as a product of primes.

[STEP 2, 1997Q2]

Suppose that

$$3 = \frac{2}{x_1} = x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = x_3 + \frac{2}{x_4} = \cdots.$$

Guess an expression, in terms of n, for x_n. Then, by induction or otherwise, prove the correctness of your guess.

[STEP 2, 1997Q3]

Find constants a, b, c and d such that

$$\frac{ax+b}{x^2+2x+2} + \frac{cx+d}{x^2-2x+2} = \frac{1}{x^4+4}.$$

Show that

$$\int_0^1 \frac{1}{x^4+4}\,\mathrm{d}x = \frac{1}{16}\ln 5 + \frac{1}{8}\tan^{-1} 2.$$

[STEP 2, 1997Q4]

Show that, when the polynomial $p(x)$ is divided by $(x-a)$, where a is a real number, the remainder is $p(a)$.

(i) When the polynomial $p(x)$ is divided by $(x-1)$, $(x-2)$ and $(x-3)$ the remainders are 3, 1, and 5 respectively. Given that

$$p(x) = (x-1)(x-2)(x-3)q(x) + r(x),$$

where $q(x)$ and $r(x)$ are polynomials with $r(x)$ having degree less than three, find $r(x)$.

(ii) Find a polynomial $P(x)$ of degree $n+1$, where n is a given positive integer, such that for each integer a satisfying $0 \le a \le n$ the remainder when $P(x)$ is divided by $(x-a)$ is a.

[STEP 2, 1997Q5]

The complex numbers $w = u + \mathbf{i}v$ and $z = x + \mathbf{i}y$ are related by the equation

$$z = (\cos v + \mathbf{i}\sin v)\exp u.$$

Find all w which correspond to $z = \mathbf{i}\mathrm{e}$.

Find the loci in the x-y plane corresponding to the lines $u =$ constant in the u-v plane. Find also the loci corresponding to the lines $v =$ constant. Illustrate your answers with clearly labelled sketches.

Identify two subsets W_1 and W_2 of the u-v plane each of which is in one-to-one correspondence with the first quadrant $\{(x, y): x > 0, y > 0\}$ of the x-y plane. Identify also two subsets W_3 and W_4 each of which is in one-to-one correspondence with the set $\{z: 0 < |z| < 1\}$.

[**NB** 'one-to-one' means here that to each value of w there is only one corresponding value of z, and vice-versa.]

[STEP 2, 1997Q6]

Show that, if $\tan^2\phi = 2\tan\phi + 1$, then $\tan 2\phi = -1$.

Find all solutions of the equation

$$\tan\theta = 2 + \tan 3\theta$$

which satisfy $0 < \theta < 2\pi$, expressing your answers as rational multiples of π.

Find all solutions of the equation

$$\cot\theta = 2 + \cot 3\theta$$

which satisfy

$$-\frac{3\pi}{2} < \theta < \frac{\pi}{2}.$$

[Ignore values of θ for which tan or cot is undefined.]

[STEP 2, 1997Q7]

Let

$$y^2 = x^2(a^2 - x^2),$$

where a is a real constant. Find, in terms of a, the maximum and minimum values of y.

Sketch carefully on the same axes the graphs of y in the cases $a = 1$ and $a = 2$.

[STEP 2, 1997Q8]

If $f(t) \geq g(t)$ for $a \leq t \leq b$, explain very briefly why $\int_a^b f(t)\,\mathrm{d}t \geq \int_a^b g(t)\,\mathrm{d}t$.

Prove that if $p > q > 0$ and $x \geq 1$ then

$$\frac{x^p - 1}{p} \geq \frac{x^q - 1}{q}.$$

Show that this inequality also holds when $p > q > 0$ and $0 \leq x \leq 1$.

Prove that, if $p > q > 0$ and $x \geq 0$, then

$$\frac{1}{p}\left(\frac{x^p}{p+1} - 1\right) \geq \frac{1}{q}\left(\frac{x^q}{q+1} - 1\right).$$

Section B: Mechanics

[STEP 2, 1997Q9]

A uniform solid sphere of diameter d and mass m is drawn slowly and without slipping from horizontal ground onto a step of height $\frac{d}{4}$ by a horizontal force which is always applied to the highest point of the sphere and is always perpendicular to the vertical plane which forms the face of the step. Find the maximum horizontal force throughout the movement, and prove that the coefficient of friction between the sphere and the edge of the step must exceed $\frac{1}{\sqrt{3}}$.

[STEP 2, 1997Q10]

In this question the effect of gravity is to be neglected.

A small body of mass M is moving with velocity v along the axis of a long, smooth, fixed, circular cylinder of radius L. An internal explosion splits the body into two spherical fragments, with masses qM and $(1-q)M$, where $q \leq \frac{1}{2}$. After bouncing perfectly elastically off the cylinder (one bounce each) the fragments collide and coalesce at a distance $\frac{1}{2}L$ from the axis. Show that $q = \frac{3}{8}$.

The collision occurs at a time $\frac{5L}{v}$ after the explosion. Find the energy imparted to the fragments by the explosion, and find the velocity after coalescence.

[STEP 2, 1997Q11]

A tennis player serves from height H above horizontal ground, hitting the ball downwards with speed v at an angle α below the horizontal. The ball just clears the net of height h at a horizontal distance a from the server and hits the ground a further horizontal distance b beyond the net. Show that

$$v^2 = \frac{g(a+b)^2(1+\tan^2\alpha)}{2[H-(a+b)\tan\alpha)}$$

and

$$\tan\alpha = \frac{2a+b}{a(a+b)}H - \frac{a+b}{ab}h.$$

By considering the signs of v^2 and $\tan\alpha$, find, in terms of a, b, and h, upper and lower bounds on H for such a serve to be possible.

Section C: Probability and Statistics

[STEP 2, 1997Q12]

The game of Cambridge Whispers starts with the first participant Albert flipping an unbiased coin and whispering to his neighbour Bertha whether it fell ‘heads’ or ‘tails’. Bertha then whispers this information to her neighbour and so on. The game ends when the final player Zebedee whispers to Alfred and the game is won, by all players, if what Alfred hears is correct. The acoustics are such that the listeners have, independently at each stage, only a probability of $\frac{2}{3}$ of hearing correctly what is said. Find the probability that the game is won when there are just three players.

By considering the binomial expansion of $(a+b)^n + (a-b)^n$, or otherwise, find a concise expression for the probability P that the game is won when it is played by n players each having a probability p of hearing correctly.

To avoid the trauma of a lost game, the rules are now modified to require Alfred to whisper to Bertha what he hears from Zebedee, and so keep the game going, if what he hears from Zebedee is not correct. Find the expected total number of times that Alfred whispers to Bertha before the modified game ends.

[You may use without proof the fact that $\sum_{k=1}^{\infty} kx^{k-1} = (1-x)^{-2}$ for $|x| < 1$.]

[STEP 2, 1997Q13]

A needle of length 2 cm is dropped at random onto a large piece of paper ruled with parallel lines 2 cm apart.

(i) By considering the angle which the needle makes with the lines, find the probability that the needle crosses the nearest line given that its centre is x cm from it, where $0 < x < 1$.

(ii) Given that the centre of the needle is x cm from the nearest line and that the needle crosses that line, find the cumulative distribution function for the length of the shorter segment of the needle cut off by the line.

(iii) Find the probability that the needle misses all the lines.

[STEP 2, 1997Q14]

Traffic enters a tunnel which is 9600 metres long, and in which overtaking is impossible. The number of vehicles which enter in any given time is governed by the Poisson distribution with mean 6 cars per minute. All vehicles travel at a constant speed until forced to slow down on catching up with a slower vehicle ahead. I enter the tunnel travelling at 30 m s^{-1} and all the other traffic is travelling at 32 m s^{-1}. What is the expected number of vehicles in the queue behind me when I leave the tunnel?

Assuming again that I travel at 30 m s^{-1}, but that all the other vehicles are independently equally likely to be travelling at 30 m s^{-1} or 32 m s^{-1}, find the probability that exactly two vehicles enter the tunnel while I am in it and catch me up before I leave it. Find also the probability that there are exactly two vehicles queuing behind me when I leave the tunnel.

[Ignore the lengths of the vehicles.]

STEP 2 1998

STEP 2 1998
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Section A: Pure Mathematics

[STEP 2, 1998Q1]

Show that, if n is an integer such that

$$(n-3)^3 + n^3 = (n+3)^3, \qquad (*)$$

then n is even and n^2 is a factor of 54. Deduce that there is no integer n which satisfies the equation $(*)$.

Show that, if n is an integer such that

$$(n-6)^3 + n^3 = (n+6)^3, \qquad (**)$$

then n is even. Deduce that there is no integer n which satisfies the equation $(**)$.

[STEP 2, 1998Q2]

Use the first four terms of the binomial expansion of $\left(1-\frac{1}{50}\right)^{\frac{1}{2}}$, writing $\frac{1}{50} = \frac{2}{100}$ to simplify the calculation, to derive the approximation $\sqrt{2} \approx 1.414214$.

Calculate similarly an approximation to the cube root of 2 to six decimal places by considering $\left(1+\frac{N}{125}\right)^{a}$, where a and N are suitable numbers.

[You need not justify the accuracy of your approximations.]

[STEP 2, 1998Q3]

Show that the sum S_N of the first N terms of the series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots + \frac{2n-1}{n(n+1)(n+2)} + \cdots$$

is

$$\frac{1}{2}\left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2}\right).$$

What is the limit of S_N as $N \to \infty$?

The numbers a_n are such that

$$\frac{a_n}{a_{n-1}} = \frac{(n-1)(2n-1)}{(n+2)(2n-3)}.$$

Find an expression for $\frac{a_n}{a_1}$ and hence, or otherwise, evaluate $\sum_{n=1}^{\infty} a_n$ when $a_1 = \frac{2}{9}$.

[STEP 2, 1998Q4]

The integral I_n is defined by

$$I_n = \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \sin\left(nx + \frac{x}{2}\right) \operatorname{cosec}\left(\frac{x}{2}\right) \mathrm{d}x,$$

where n is a positive integer. Evaluate $I_n - I_{n-1}$, and hence evaluate I_n leaving your answer in the form of a sum.

[STEP 2, 1998Q5]

Define the modulus of a complex number z and give the geometric interpretation of $|z_1 - z_2|$ for two complex numbers z_1 and z_2. On the basis of this interpretation establish the inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Use this result to prove, by induction, the corresponding inequality for $|z_1 + \cdots + z_n|$.

The complex numbers $a_1, a_2, \ldots, a_n$ satisfy $|a_i| \leq 3$ $(i = 1, 2, \ldots, n)$. Prove that the equation

$$a_1 z + a_2 z^2 + \cdots + a_n z^n = 1$$

has no solution z with $|z| \leq \frac{1}{4}$.

[STEP 2, 1998Q6]

Two curves are given parametrically by

$$x_1 = (\theta + \sin\theta), \qquad y_1 = (1 + \cos\theta), \qquad (1)$$

and

$$x_2 = (\theta - \sin\theta), \qquad y_2 = -(1 + \cos\theta). \qquad (2)$$

Find the gradients of the tangents to the curves at the points where $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

Sketch, using the same axes, the curves for $0 \leq \theta \leq 2\pi$.

Find the equation of the normal to the curve (1) at the point with parameter θ. Show that this normal is a tangent to the curve (2).

[STEP 2, 1998Q7]

Let

$$f(x) = \tan x - x\,,$$
$$g(x) = 2 - 2\cos x - x\sin x\,,$$
$$h(x) = 2x + x\cos 2x - \frac{3}{2}\sin 2x\,,$$
$$F(x) = \frac{x(\cos x)^{\frac{1}{3}}}{\sin x}\,.$$

(i) By considering $f(0)$ and $f'(x)$, show that $f(x) > 0$ for $0 < x < \frac{\pi}{2}$.

(ii) Show similarly that $g(x) > 0$ for $0 < x < \frac{\pi}{2}$.

(iii) Show that $h(x) > 0$ for $0 < x < \frac{\pi}{4}$, and hence that

$$x(\sin^2 x + 3\cos^2 x) - 3\sin x\cos x > 0$$

for $0 < x < \frac{\pi}{4}$.

(iv) By considering $\frac{F'(x)}{F(x)}$, show that $F'(x) < 0$ for $0 < x < \frac{\pi}{4}$.

[STEP 2, 1998Q8]

Points A, B, C in three dimensions have coordinate vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, respectively. Show that the lines joining the vertices of the triangle ABC to the mid-points of the opposite sides meet at a point R.

P is a point which is **not** in the plane ABC. Lines are drawn through the mid-points of BC, CA and AB parallel to PA, PB and PC respectively. Write down the vector equations of the lines and show by inspection that these lines meet at a common point Q.

Prove further that the line PQ meets the plane ABC at R.

Section B: Mechanics

[STEP 2, 1998Q9]

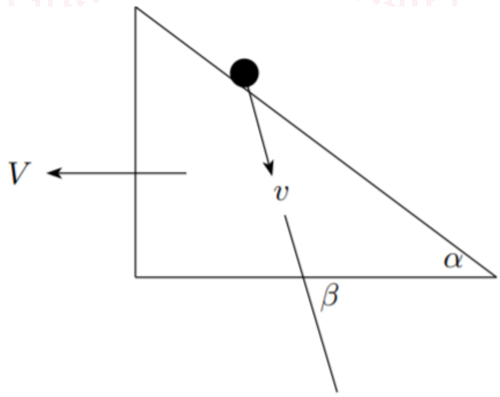
A light smoothly jointed planar framework in the form of a regular hexagon $ABCDEF$ is suspended smoothly from A and a weight 1 kg is suspended from C. The framework is kept rigid by three light rods BD, BE and BF. What is the direction and magnitude of the supporting force which must be exerted on the framework at A?

Indicate on a labelled diagram which rods are in thrust (compression) and which are in tension.

Find the magnitude of the force in BE.

[STEP 2, 1998Q10]

A wedge of mass M rests on a smooth horizontal surface. The face of the wedge is a smooth plane inclined at an angle α to the horizontal. A particle of mass m slides down the face of the wedge, starting from rest. At a later time t, the speed V of the wedge, the speed v of the particle and the angle β of the velocity of the particle below the horizontal are as shown in the diagram.



Let y be the vertical distance descended by the particle. Derive the following results, stating in (ii) and (iii) the mechanical principles you use:

(i) $V\sin\alpha = v\sin(\beta-\alpha)$.

(ii) $\tan\beta = \left(1+\frac{m}{M}\right)\tan\alpha$.

(iii) $2gy = \frac{v^2(M+m\cos^2\beta)}{M}$.

Write down a differential equation for y and hence show that

$$y = \frac{gMt^2\sin^2\beta}{2(M+m\cos^2\beta)}.$$

[STEP 2, 1998Q11]

A fielder, who is perfectly placed to catch a ball struck by the batsman in a game of cricket, watches the ball in flight. Assuming that the ball is struck at the fielder's eye level and is caught just in front of her eye, show that $\frac{\mathrm{d}}{\mathrm{d}t}(\tan\theta)$ is constant, where θ is the angle between the horizontal and the fielder's line of sight.

In order to catch the next ball, which is also struck towards her but at a different velocity, the fielder runs at constant speed v towards the batsman. Assuming that the ground is horizontal, show that the fielder should choose v so that $\frac{\mathrm{d}}{\mathrm{d}t}(\tan\theta)$ remains constant.

Section C: Probability and Statistics

[STEP 2, 1998Q12]

The diagnostic test AL has a probability 0.9 of giving a positive result when applied to a person suffering from the rare disease mathematitis. It also has a probability $\frac{1}{11}$ of giving a false positive result when applied to a non-sufferer. It is known that only 1% of the population suffer from the disease. Given that the test AL is positive when applied to Frankie, who is chosen at random from the population, what is the probability that Frankie is a sufferer?

In an attempt to identify sufferers more accurately, a second diagnostic test STEP is given to those for whom the test AL gave a positive result. The probability of STEP giving a positive result on a sufferer is 0.9, and the probability that it gives a false positive result on a non-sufferer is p. Half of those for whom AL was positive and on whom STEP then also gives a positive result are sufferers. Find p.

[STEP 2, 1998Q13]

A random variable X has the probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that

$$P(X > s + t \mid X > t) = P(X > s).$$

The time it takes an assistant to serve a customer in a certain shop is a random variable with the above distribution and the times for different customers are independent. If, when I enter the shop, the only two assistants are serving one customer each, what is the probability that these customers are both still being served at time t after I arrive?

One of the assistants finishes serving his customer and immediately starts serving me. What is the probability that I am still being served when the other customer has finished being served?

[STEP 2, 1998Q14]

The staff of Catastrophe College are paid a salary of A pounds per year. With a Teaching Assessment Exercise impending it is decided to try to lower the student failure rate by offering each lecturer an alternative salary of $\frac{B}{1+X}$ pounds, where X is the number of his or her students who fail the end of year examination. Dr Doom has N students, each with independent probability p of failure. Show that she should accept the new salary scheme if

$$A(N+1)p < B(1 - (1-p)^{N+1}).$$

Under what circumstances could X, for Dr Doom, be modelled by a Poisson random variable? What would Dr Doom's expected salary be under this model?

STEP 2 1999

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Section A: Pure Mathematics

[STEP 2, 1999Q1]

Let $x = 10^{100}$, $y = 10^x$, $z = 10^y$, and let

$$a_1 = x!, \qquad a_2 = x^y, \qquad a_3 = y^x, \qquad a_4 = z^x, \qquad a_5 = e^{xyz}, \qquad a_6 = z^{\frac{1}{y}}, \qquad a_7 = y^{\frac{z}{x}}.$$

(i) Use Stirling's approximation $n! \approx \sqrt{2\pi}n^{n+\frac{1}{2}}\,e^{-n}$, which is valid for large n, to show that $\log_{10}(\log_{10} a_1) \approx 102$.

(ii) Arrange the seven numbers $a_1, \ldots, a_7$ in ascending order of magnitude, justifying your result.

[STEP 2, 1999Q2]

Consider the quadratic equation

$$nx^2 + 2x\sqrt{pn^2 + q} + rn + s = 0, \qquad (*)$$

where $p > 0$, $p \neq r$ and $n = 1, 2, 3, \ldots$.

(i) For the case where $p = 3$, $q = 50$, $r = 2$, $s = 15$, find the set of values of n for which equation $(*)$ has no real roots.

(ii) Prove that if $p < r$ and $4q(p - r) > s^2$, then $(*)$ has no real roots for any value of n.

(iii) If $n = 1$, $p - r = 1$ and $q = \frac{s^2}{8}$, show that $(*)$ has real roots if, and only if, $s \leq 4 - 2\sqrt{2}$ or $s \geq 4 + 2\sqrt{2}$.

[STEP 2, 1999Q3]

Let

$$S_n(x) = e^{x^3}\frac{d^n}{dx^n}(e^{-x^3}).$$

Show that $S_2(x) = 9x^4 - 6x$ and find $S_3(x)$.

Prove by induction on n that $S_n(x)$ is a polynomial. By means of your induction argument, determine the order of this polynomial and the coefficient of the highest power of x.

Show also that if $\frac{dS_n}{dx} = 0$ for some value a of x, then $S_n(a)S_{n+1}(a) \leq 0$.

[STEP 2, 1999Q4]

By considering the expansions in powers of x of both sides of the identity

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

show that

$$\sum_{s=0}^{n}\binom{n}{s}^2 = \binom{2n}{n},$$

where $\binom{n}{s} = \frac{n!}{s!(n-s)!}$.

By considering similar identities, or otherwise, show also that:

(i) if n is an even integer, then

$$\sum_{s=0}^{n}(-1)^s\binom{n}{s}^2 = (-1)^{\frac{n}{2}}\binom{n}{\frac{n}{2}}.$$

(ii)

$$\sum_{t=1}^{n}2t\binom{n}{t}^2 = n\binom{2n}{n}.$$

[STEP 2, 1999Q5]

Show that if α is a solution of the equation

$$5\cos x + 12\sin x = 7,$$

then either

$$\cos\alpha = \frac{35 - 12\sqrt{120}}{169}$$

or $\cos\alpha$ has one other value which you should find.

Prove carefully that if $\frac{\pi}{2} < \alpha < \pi$, then $\alpha < \frac{3\pi}{4}$.

[STEP 2, 1999Q6]

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if

$$y = \frac{ax+b}{cx+d}. \qquad (*)$$

By using changes of variable of the form $(*)$, or otherwise, show that

$$\int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x+1}{x+3}\right) \mathrm{d}x = \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12},$$

and evaluate the integrals

$$\int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x^2+3x+2}{(x+3)^2}\right) \mathrm{d}x$$

and

$$\int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x+1}{x+2}\right) \mathrm{d}x.$$

[STEP 2, 1999Q7]

The curve C has equation

$$y = \frac{x}{\sqrt{x^2 - 2x + a}},$$

where the square root is positive. Show that, if $a > 1$, then C has exactly one stationary point.

Sketch C when (i) $a = 2$ and (ii) $a = 1$.

[STEP 2, 1999Q8]

Prove that

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos\frac{1}{2}\theta - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}. \qquad (*)$$

(i) Deduce that, when n is large,

$$\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right) \approx \frac{2n}{\pi}.$$

(ii) By differentiating $(*)$ with respect to θ, or otherwise, show that, when n is large,

$$\sum_{k=0}^{n} k\sin^2\left(\frac{k\pi}{2n}\right) \approx \left(\frac{1}{4} + \frac{1}{\pi^2}\right)n^2.$$

[The approximations, valid for small θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ may be assumed.]

Section B: Mechanics

[STEP 2, 1999Q9]

In the Z–universe, a star of mass M suddenly blows up, and the fragments, with various initial speeds, start to move away from the centre of mass G which may be regarded as a fixed point. In the subsequent motion the acceleration of each fragment is directed towards G. Moreover, in accordance with the laws of physics of the Z–universe, there are positive constants k_1, k_2 and R such that when a fragment is at a distance x from G, the magnitude of its acceleration is k_1x^3 if $x < R$ and is k_2x^{-4} if $x \geq R$. The initial speed of a fragment is denoted by u.

(i) For $x < R$, write down a differential equation for the speed v, and hence determine v in terms of u, k_1 and x for $x < R$.

(ii) Show that if $u < a$, where $2a^2 = k_1R^4$, then the fragment does not reach a distance R from G.

(iii) Show that if $u \geq b$, where $6b^2 = 3k_1R^4 + \frac{4k_2}{R^3}$, then from the moment of the explosion the fragment is always moving away from G.

(iv) If $a < u < b$, determine in terms of k_2, b and u the maximum distance from G attained by the fragment.

[STEP 2, 1999Q10]

N particles $P_1, P_2, P_3, \ldots, P_N$ with masses $m, qm, q^2m, \ldots, q^{N-1}m$, respectively, are at rest at distinct points along a straight line in gravity-free space. The particle P_1 is set in motion towards P_2 with velocity V and in every subsequent impact the coefficient of restitution is e, where $0 < e < 1$. Show that after the first impact the velocities of P_1 and P_2 are

$$\left(\frac{1-eq}{1+q}\right)V \quad \text{and} \quad \left(\frac{1+e}{1+q}\right)V,$$

respectively.

Show that if $q \leq e$, then there are exactly $N-1$ impacts and that if $q = e$, then the total loss of kinetic energy after all impacts have occurred is equal to

$$\frac{1}{2}me(1-e^{N-1})V^2.$$

[STEP 2, 1999Q11]

An automated mobile dummy target for gunnery practice is moving anti-clockwise around the circumference of a large circle of radius R in a horizontal plane at a constant angular speed ω. A shell is fired from O, the centre of this circle, with initial speed V and angle of elevation α. Show that if $V^2 < gR$, then no matter what the value of α, or what vertical plane the shell is fired in, the shell cannot hit the target.

Assume now that $V^2 > gR$ and that the shell hits the target, and let β be the angle through which the target rotates between the time at which the shell is fired and the time of impact. Show that β satisfies the equation

$$g^2\beta^4 - 4\omega^2V^2\beta^2 + 4R^2\omega^4 = 0.$$

Deduce that there are exactly two possible values of β.

Let β_1 and β_2 be the possible values of β and let P_1 and P_2 be the corresponding points of impact. By considering the quantities $(\beta_1^2 + \beta_2^2)$ and $\beta_1^2\beta_2^2$, or otherwise, show that the linear distance between P_1 and P_2 is

$$2R\sin\left(\frac{\omega}{g}\sqrt{V^2 - Rg}\right).$$

Section C: Probability and Statistics

[STEP 2, 1999Q12]

It is known that there are three manufacturers A, B, C, who can produce micro chip MB666. The probability that a randomly selected MB666 is produced by A is $2p$, and the corresponding probabilities for B and C are p and $1-3p$, respectively, where $0 \leq p \leq \frac{1}{3}$. It is also known that 70% of MB666 micro chips from A are sound and that the corresponding percentages for B and C are 80% and 90%, respectively.

Find in terms of p, the conditional probability, $\mathrm{P}(A|S)$ that if a randomly selected MB666 chip is found to be sound then it came from A, and also the conditional probability, $\mathrm{P}(C|S)$, that if it is sound then it came from C.

A quality inspector took a random sample of one MB666 micro chip and found it to be sound. She then traced its place of manufacture to be A, and so estimated p by calculating the value of p that corresponds to the greatest value of $\mathrm{P}(A|S)$. A second quality inspector also a took random sample of one MB666 chip and found it to be sound. Later he traced its place of manufacture to be C and so estimated p by applying the procedure of his colleague to $\mathrm{P}(C|S)$.

Determine the values of the two estimates and comment briefly on the results obtained.

[STEP 2, 1999Q13]

A stick is broken at a point, chosen at random, along its length. Find the probability that the ratio, R, of the length of the shorter piece to the length of the longer piece is less than r.

Find the probability density function for R, and calculate the mean and variance of R.

[STEP 2, 1999Q14]

You play the following game. You throw a six-sided fair die repeatedly. You may choose to stop after any throw, except that you must stop if you throw a 1. Your score is the number obtained on your last throw. Determine the strategy that you should adopt in order to maximize your expected score, explaining your reasoning carefully.

STEP 2 2000

STEP 2 2000
On-line Exam

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Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 2000Q1]

A number of the form $\frac{1}{N}$, where N is an integer greater than 1, is called a *unit fraction*.

Noting that

$$\frac{1}{2}=\frac{1}{3}+\frac{1}{6} \quad \text{and} \quad \frac{1}{3}=\frac{1}{4}+\frac{1}{12},$$

guess a general result of the form

$$\frac{1}{N}=\frac{1}{a}+\frac{1}{b} \qquad (*)$$

and hence prove that any unit fraction can be expressed as the sum of two distinct unit fractions.

By writing $(*)$ in the form

$$(a-N)(b-N)=N^2$$

and by considering the factors of N^2, show that if N is prime, then there is only one way of expressing $\frac{1}{N}$ as the sum of two distinct unit fractions.

Prove similarly that any fraction of the form $\frac{2}{N}$, where N is prime number greater than 2, can be expressed uniquely as the sum of two distinct unit fractions.

[STEP 2, 2000Q2]

Prove that if $(x-a)^2$ is a factor of the polynomial $p(x)$, then $p'(a)=0$. Prove a corresponding result if $(x-a)^4$ is a factor of $p(x)$.

Given that the polynomial

$$x^6+4x^5-5x^4-40x^3-40x^2+32x+k$$

has a factor of the form $(x-a)^4$, find k.

[STEP 2, 2000Q3]

The lengths of the sides BC, CA, AB of the triangle ABC are denoted by a, b, c, respectively.

Given that

$$b=8+\epsilon_1\,, \qquad c=3+\epsilon_2\,, \qquad a=\frac{\pi}{3}+\epsilon_3\,,$$

where ϵ_1, ϵ_2, and ϵ_3 are small, show that $a\approx 7+\eta$, where $\eta=\frac{13\epsilon_1-2\epsilon_2+24\sqrt{3}\epsilon_3}{14}$.

Given now that

$$|\epsilon_1|\le 2\times10^{-3}, \qquad |\epsilon_2|\le 4.9\times10^{-2}, \qquad |\epsilon_3|\le\sqrt{3}\times10^{-3},$$

find the range of possible values of η.

[STEP 2, 2000Q4]

Prove that

$$(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) = \cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)$$

and that, for every positive integer n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

By considering $(5-\mathrm{i})^2(1+\mathrm{i})$, or otherwise, prove that

$$\arctan\left(\frac{7}{17}\right) + 2\arctan\left(\frac{1}{5}\right) = \frac{\pi}{4}.$$

Prove also that

$$3\arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{20}\right) + \arctan\left(\frac{1}{1985}\right) = \frac{\pi}{4}.$$

[Note that $\arctan\theta$ is another notation for $\tan^{-1}\theta$.]

[STEP 2, 2000Q5]

It is required to approximate a given function $f(x)$, over the interval $0 \le x \le 1$, by the linear function λx, where λ is chosen to minimise

$$\int_0^1 (f(x) - \lambda x)^2\,\mathrm{d}x.$$

Show that

$$\lambda = 3\int_0^1 xf(x)\,\mathrm{d}x.$$

The residual error, R, of this approximation process is such that

$$R^2 = \int_0^1 (f(x) - \lambda x)^2\,\mathrm{d}x.$$

Show that

$$R^2 = \int_0^1 \big(f(x)\big)^2\,\mathrm{d}x - \frac{1}{3}\lambda^2.$$

Given now that $f(x) = \sin\left(\frac{\pi x}{n}\right)$, show that (i) for large n, $\lambda \approx \frac{\pi}{n}$ and (ii) $\lim_{n\to\infty} R = 0$.

Explain why, prior to any calculation, these results are to be expected.

[You may assume that, when θ is small, $\sin\theta \approx \theta - \frac{\theta^3}{6}$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$.]

[STEP 2, 2000Q6]

Show that

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad \frac{1+\cos\theta}{\sin\theta} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right),$$

where $t = \tan\left(\frac{\theta}{2}\right)$.

Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to show that, for $0 < \alpha < \frac{\pi}{2}$,

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\sin\theta}\,\mathrm{d}\theta = \frac{\alpha}{\sin\alpha},$$

and deduce a similar result for

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\alpha\cos\theta}\,\mathrm{d}\theta.$$

[STEP 2, 2000Q7]

The line l has vector equation $\mathbf{r} = \lambda\mathbf{s}$, where

$$\mathrm{s} = \left(\cos\theta + \sqrt{3}\right)\hat{\mathbf{i}} + \left(\sqrt{2}\sin\theta\right)\hat{\mathbf{j}} + \left(\cos\theta - \sqrt{3}\right)\hat{\mathbf{k}}$$

and λ is a scalar parameter. Find an expression for the angle between l and the line $\mathbf{r} = \mu\left(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}\right)$. Show that there is a line m through the origin such that, whatever the value of θ, the acute angle between l and m is $\frac{\pi}{6}$.

A plane has equation $x - z = 4\sqrt{3}$. The line l meets this plane at P. Show that, as θ varies, P describes a circle, with its centre on m. Find the radius of this circle.

[STEP 2, 2000Q8]

(i) Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\mathrm{e}^{-x^2}(y+3)^{\frac{1}{2}} = 0 \qquad (x \geq 0),$$

that satisfies the condition $y = 6$ when $x = 0$. Find y in terms of x and show that $y \to 1$ as $x \to \infty$.

(ii) Let y be any solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - x\mathrm{e}^{6x^2}(y+3)^{1-k} = 0 \qquad (x \geq 0).$$

Find a value of k such that, as $x \to \infty$, $\mathrm{e}^{-3x^2}y$ tends to a finite non-zero limit, which you should determine.

Section B: Mechanics

[STEP 2, 2000Q9]

In an aerobatics display, Jane and Karen jump from a great height and go through a period of free fall before opening their parachutes. While in free fall at speed v, Jane experiences air resistance kv per unit mass but Karen, who spread-eagles, experiences air resistance $kv + \left(\frac{2k^2}{g}\right)v^2$ per unit mass. Show that Jane's speed can never reach $\frac{g}{k}$. Obtain the corresponding result for Karen.

Jane opens her parachute when her speed is $\frac{g}{3k}$. Show that she has then been in free fall for time $k^{-1}\ln\left(\frac{3}{2}\right)$.

Karen also opens her parachute when her speed is $\frac{g}{3k}$. Find the time she has then been in free fall.

[STEP 2, 2000Q10]

A long light inextensible string passes over a fixed smooth light pulley. A particle of mass 4 kg is attached to one end A of this string and the other end is attached to a second smooth light pulley. A long light inextensible string BC passes over the second pulley and has a particle of mass 2 kg attached at B and a particle of mass of 1 kg attached at C. The system is held in equilibrium in a vertical plane. The string BC is then released from rest. Find the accelerations of the two moving particles.

After T seconds, the end A is released so that all three particles are now moving in a vertical plane. Find the accelerations of A, B and C in this second phase of the motion. Find also, in terms of g and T, the speed of A when B has moved through a total distance of $0.6gT^2$ metres.

[STEP 2, 2000Q11]

The string AP has a natural length of 1.5 metres and modulus of elasticity equal to $5g$ newtons. The end A is attached to the ceiling of a room of height 2.5 metres and a particle of mass 0.5 kg is attached to the end P. The end P is released from rest at a point 0.5 metres above the floor and vertically below A. Show that the string becomes slack, but that P does not reach the ceiling.

Show also that while the string is in tension, P executes simple harmonic motion, and that the time in seconds that elapses from the instant when P is released to the instant when P first returns to its original position is

$$\left(\frac{8}{3g}\right)^{\frac{1}{2}} + \left(\frac{3}{5g}\right)^{\frac{1}{2}}\left(\pi - \arccos\left(\frac{3}{7}\right)\right).$$

[Note that arccos x is another notation for $\cos^{-1} x$.]

Section C: Probability and Statistics

[STEP 2, 2000Q12]

Tabulated values of $\Phi(\cdot)$, *the cumulative distribution function of a standard normal variable, should not be used in this question.*

Henry the commuter lives in Cambridge and his working day starts at his office in London at 0900. He catches the 0715 train to King's Cross with probability p, or the 0720 to Liverpool Street with probability $1-p$. Measured in minutes, journey times for the first train are $\mathrm{N}(55, 25)$ and for the second are $\mathrm{N}(65, 16)$. Journey times from King's Cross and Liverpool Street to his office are $\mathrm{N}(30, 144)$ and $\mathrm{N}(25, 9)$, respectively. Show that Henry is more likely to be late for work if he catches the first train.

Henry makes M journeys, where M is large. Writing A for $1-\Phi\left(\frac{20}{13}\right)$ and B for $1-\Phi(2)$, find, in terms of A, B, M and p, the expected number, L, of times that Henry will be late and show that for all possible values of p,

$$BM \leq L \leq AM.$$

Henry noted that in $\frac{3}{5}$ of the occasions when he was late, he had caught the King's Cross train. Obtain an estimate of p in terms of A and B.

[A random variable is said to be $\mathrm{N}(\mu, \sigma^2)$ if it has a normal distribution with mean μ and variance σ^2.]

[STEP 2, 2000Q13]

A group of biologists attempts to estimate the magnitude, N, of an island population of voles (*Microtus agrestis*). Accordingly, the biologists capture a random sample of 200 voles, mark them and release them. A second random sample of 200 voles is then taken of which 11 are found to be marked. Show that the probability, p_N, of this occurrence is given by

$$p_N = k\frac{\left((N-200)!\right)^2}{N!\,(N-389)!},$$

where k is independent of N.

The biologists then estimate N by calculating the value of N for which p_N is a maximum. Find this estimate.

All unmarked voles in the second sample are marked and then the entire sample is released. Subsequently a third random sample of 200 voles is taken. Write down the probability that this sample contains exactly j marked voles, leaving your answer in terms of binomial coefficients.

Deduce that

$$\sum_{j=0}^{200}\binom{389}{j}\binom{3247}{200-j} = \binom{3636}{200}.$$

[STEP 2, 2000Q14]

The random variables $X_1, X_2, \dots, X_{2n+1}$ are independently and uniformly distributed on the interval $0 \le x \le 1$. The random variable Y is defined to be the median of $X_1, X_2, \dots, X_{2n+1}$. Given that the probability density function of Y is $g(y)$, where

$$g(y) = \begin{cases} ky^n(1-y)^n, & \text{if } 0 \le y \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

use the result

$$\int_0^1 y^r(1-y)^s \, \mathrm{d}y = \frac{r!\,s!}{(r+s+1)!}$$

to show that $k = \frac{(2n+1)!}{(n!)^2}$, and evaluate $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. Hence show that, for any given positive number d, the inequality

$$\mathrm{P}\left(\left|Y - \frac{1}{2}\right| < \frac{d}{\sqrt{n}}\right) < \mathrm{P}\left(\left|\bar{X} - \frac{1}{2}\right| < \frac{d}{\sqrt{n}}\right)$$

holds provided n is large enough, where $\bar{X}$ is the mean of $X_1, X_2, \dots, X_{2n+1}$.

[You may assume that Y and $\bar{X}$ are normally distributed for large n.]

STEP 2 2001

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On-line Exam

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Section A: Pure Mathematics

[STEP 2, 2001Q1]

Use the binomial expansion to obtain a polynomial of degree 2 which is a good approximation to $\sqrt{1-x}$ when x is small.

(i) By taking $x = \frac{1}{100}$, show that $\sqrt{11} \approx \frac{79599}{24000}$, and estimate, correct to 1 significant figure, the error in this approximation. (You may assume that the error is given approximately by the first neglected term in the binomial expansion.)

(ii) Find a rational number which approximates $\sqrt{1111}$ with an error of about 2×10^{-12}.

[STEP 2, 2001Q2]

Sketch the graph of the function $\left[\frac{x}{N}\right]$, for $0 < x < 2N$, where the notation $[y]$ means the integer part of y. (Thus $[2.9] = 2$, $[4] = 4$.)

(i) Prove that

$$\sum_{k=1}^{2N} (-1)^{\left[\frac{k}{N}\right]} k = 2N - N^2.$$

(ii) Let

$$S_N = \sum_{k=1}^{2N} (-1)^{\left[\frac{k}{N}\right]} 2^{-k}.$$

Find S_N in terms of N and determine the limit of S_N as $N \to \infty$.

[STEP 2, 2001Q3]

The cuboid $ABCDEFGH$ is such AE, BF, CG, DH are perpendicular to the opposite faces $ABCD$ and $EFGH$, and $AB = 2$, $BC = 1$, $AE = \lambda$. Show that if α is the acute angle between the diagonals AG and BH then

$$\cos\alpha = \left|\frac{3-\lambda^2}{5+\lambda^2}\right|.$$

Let R be the ratio of the volume of the cuboid to its surface area. Show that $R < \frac{1}{3}$ for all possible values of λ.

Prove that, if $R \geq \frac{1}{4}$, then $\alpha \leq \arccos\left(\frac{1}{9}\right)$.

[STEP 2, 2001Q4]

Let

$$f(x) = P\sin x + Q\sin 2x + R\sin 3x\,.$$

Show that if $Q^2 < 4R(P-R)$, then the only values of x for which $f(x) = 0$ are given by $x = m\pi$, where m is an integer.

[You may assume that $\sin 3x = \sin x\,(4\cos^2 x - 1)$.]

Now let

$$g(x) = \sin 2nx + \sin 4nx - \sin 6nx,$$

where n is a positive integer and $0 < x < \frac{\pi}{2}$. Find an expression for the largest root of the equation $g(x) = 0$, distinguishing between the cases where n is even and n is odd.

[STEP 2, 2001Q5]

The curve C_1 passes through the origin in the x-y plane and its gradient is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1-x^2)\mathrm{e}^{-x^2}.$$

Show that C_1 has a minimum point at the origin and a maximum point at $\left(1, \frac{1}{2}\mathrm{e}^{-1}\right)$. Find the coordinates of the other stationary point. Give a rough sketch of C_1.

The curve C_2 passes through the origin and its gradient is given by

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1-x^2)\mathrm{e}^{-x^3}.$$

Show that C_2 has a minimum point at the origin and a maximum point at $(1, k)$, where $k > \frac{1}{2}\mathrm{e}^{-1}$. (You need not find k.)

[STEP 2, 2001Q6]

Show that

$$\int_0^1 \frac{x^4}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4} - \frac{2}{3}.$$

Determine the values of

(i) $\int_0^1 x^3 \tan^{-1}\left(\frac{1-x}{1+x}\right)\mathrm{d}x$,

(ii) $\int_0^1 \frac{(1-y)^3}{(1+y)^5}\tan^{-1} y\,\mathrm{d}y$.

[STEP 2, 2001Q7]

In an Argand diagram, O is the origin and P is the point $2 + 0\mathbf{i}$. The points Q, R and S are such that the lengths OP, PQ, QR and RS are all equal, and the angles OPQ, PQR and QRS are all equal to $\frac{5\pi}{6}$, so that the points O, P, Q, R and S are five vertices of a regular 12-sided polygon lying in the upper half of the Argand diagram. Show that Q is the point $2 + \sqrt{3} + \mathbf{i}$ and find S.

The point C is the centre of the circle that passes through the points O, P and Q. Show that, if the polygon is rotated anticlockwise about O until C first lies on the real axis, the new position of S is

$$-\frac{1}{2}(3\sqrt{2} + \sqrt{6})(\sqrt{3} - \mathbf{i}).$$

[STEP 2, 2001Q8]

The function f satisfies $f(x + 1) = f(x)$ and $f(x) > 0$ for all x.

(i) Give an example of such a function.

(ii) The function F satisfies

$$\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$$

and $F(0) = 0$. Show that $F(n) = nF(1)$, for any positive integer n.

(iii) Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = 0$$

that satisfies $y = 1$ when $x = 0$. Show that $y(n) \to 0$ as $n \to \infty$, where $n = 1, 2, 3, \ldots$.

Section B: Mechanics

[STEP 2, 2001Q9]

A particle of unit mass is projected vertically upwards with speed u. At height x, while the particle is moving upwards, it is found to experience a total force F, due to gravity and air resistance, given by $F = \alpha e^{-\beta x}$, where α and β are positive constants. Calculate the energy expended in reaching this height. Show that

$$F = \frac{1}{2}\beta v^2 + \alpha - \frac{1}{2}\beta u^2,$$

where v is the speed of the particle, and explain why $\alpha = \frac{1}{2}\beta u^2 + g$, where g is the acceleration due to gravity.

Determine an expression, in terms of y, g and β, for the air resistance experienced by the particle on its downward journey when it is at a distance y below its highest point.

[STEP 2, 2001Q10]

Two particles A and B of masses m and km, respectively, are at rest on a smooth horizontal surface. The direction of the line passing through A and B is perpendicular to a vertical wall which is on the other side of B from A. The particle A is now set in motion towards B with speed u. The coefficient of restitution between A and B is e_1 and between B and the wall is e_2. Show that there will be a second collision between A and B provided

$$k < \frac{1 + e_2(1 + e_1)}{e_1}.$$

Show that, if $e_1 = \frac{1}{3}$, $e_2 = \frac{1}{2}$ and $k < 5$, then the kinetic energy of A and B immediately after B rebounds from the wall is greater than $\frac{mu^2}{27}$.

[STEP 2, 2001Q11]

A two-stage missile is projected from a point A on the ground with horizontal and vertical velocity components u and v, respectively. When it reaches the highest point of its trajectory an internal explosion causes it to break up into two fragments. Immediately after this explosion one of these fragments, P, begins to move vertically upwards with speed v_e, but retains the previous horizontal velocity. Show that P will hit the ground at a distance R from A given

by

$$\frac{gR}{u} = v + v_e + \sqrt{v_e^2 + v^2}.$$

It is required that the range R should be greater than a certain distance D (where $D > \frac{2uv}{g}$).

Show that this requirement is satisfied if

$$v_e > \frac{gD}{2u}\left(\frac{gD - 2uv}{gD - uv}\right).$$

[*The effect of air resistance is to be neglected.*]

Section C: Probability and Statistics

[STEP 2, 2001Q12]

The national lottery of Ruritania is based on the positive integers from 1 to N, where N is very large and fixed. Tickets cost £1 each. For each ticket purchased, the punter (i.e. the purchaser) chooses a number from 1 to N. The winning number is chosen at random, and the jackpot is shared equally amongst those punters who chose the winning number.

A syndicate decides to buy N tickets, choosing every number once to be sure of winning a share of the jackpot. The total number of tickets purchased in this draw is $3.8N$ and the jackpot is £W. Assuming that the non-syndicate punters choose their numbers independently and at random, find the most probable number of winning tickets and show that the expected net loss of the syndicate is approximately

$$N - \frac{5(1 - \mathrm{e}^{-2.8})}{14} W.$$

[STEP 2, 2001Q13]

The life times of a large batch of electric light bulbs are independently and identically distributed. The probability that the life time, T hours, of a given light bulb is greater than t hours is given by

$$\mathrm{P}(T > t) = \frac{1}{(1 + kt)^{\alpha}},$$

where α and k are constants, and $\alpha > 1$. Find the median M and the mean m of T in terms of α and k.

Nine randomly selected bulbs are switched on simultaneously and are left until all have failed. The fifth failure occurs at 1000 hours and the mean life time of all the bulbs is found to be 2400 hours. Show that $\alpha \approx 2$ and find the approximate value of k. Hence estimate the probability that, if a randomly selected bulb is found to last M hours, it will last a further $m - M$ hours.

[STEP 2, 2001Q14]

Two coins A and B are tossed together. A has probability p of showing a head, and B has probability $2p$, independent of A, of showing a head, where $0 < p < \frac{1}{2}$. The random variable X takes the value 1 if A shows a head and it takes the value 0 if A shows a tail. The random variable Y takes the value 1 if B shows a head and it takes the value 0 if B shows a tail. The random variable T is defined by

$$T = \lambda X + \frac{1}{2}(1 - \lambda)Y.$$

Show that $\mathrm{E}(T) = p$ and find an expression for $\mathrm{Var}(T)$ in terms of p and λ. Show that as λ varies, the minimum of $\mathrm{Var}(T)$ occurs when

$$\lambda = \frac{1 - 2p}{3 - 4p}.$$

The two coins are tossed n times, where $n > 30$, and $\overline{T}$ is the mean value of T. Let b be a fixed positive number. Show that the maximum value of $\mathrm{P}(|\overline{T} - p| < b)$ as λ varies is approximately $2\phi\left(\frac{b}{s}\right) - 1$, where ϕ is the cumulative distribution function of a standard normal variate and

$$s^2 = \frac{p(1-p)(1-2p)}{(3-4p)n}.$$

STEP 2 2002

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Section A: Pure Mathematics

[STEP 2, 2002Q1]

Show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{1-\cos 2\theta}\,\mathrm{d}\theta = \frac{\sqrt{3}}{2} - \frac{1}{2}.$$

By using the substitution $x = \sin 2\theta$, or otherwise, show that

$$\int_{\frac{\sqrt{3}}{2}}^{1} \frac{1}{1-\sqrt{1-x^2}}\,\mathrm{d}x = \sqrt{3} - 1 - \frac{\pi}{6}.$$

Hence evaluate the integral

$$\int_{1}^{\frac{2}{\sqrt{3}}} \frac{1}{y\left(y-\sqrt{y^2-1^2}\right)}\,\mathrm{d}y\,.$$

[STEP 2, 2002Q2]

Show that setting $z - z^{-1} = w$ in the quartic equation

$$z^4 + 5z^3 + 4z^2 - 5z + 1 = 0$$

results in the quadratic equation $w^2 + 5w + 6 = 0$. Hence solve the above quartic equation.

Solve similarly the equation

$$2z^8 - 3z^7 - 12z^6 + 12z^5 + 22z^4 - 12z^3 - 12z^2 + 3z + 2 = 0.$$

[STEP 2, 2002Q3]

The nth Fermat number, F_n, is defined by

$$F_n = 2^{2^n} + 1, \qquad n = 0, 1, 2, \ldots,$$

where 2^{2^n} means 2 raised to the power 2^n. Calculate F_0, F_1, F_2 and F_3. Show that, for $k = 1$, $k = 2$ and $k = 3$,

$$F_0 F_1 \ldots F_{k-1} = F_k - 2. \qquad (*)$$

Prove, by induction, or otherwise, that $(*)$ holds for all $k \geq 1$. Deduct that no two Fermat numbers have a common factor greater than 1.

Hence show that there are infinitely many prime numbers.

[STEP 2, 2002Q4]

Give a sketch to show that, if $f(x) > 0$ for $p < x < q$, then $\int_p^q f(x)\,\mathrm{d}x > 0$.

(i) By considering $f(x) = ax^2 - bx + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $3b < 2a + 6c$.

(ii) By considering $f(x) = a\sin^2 x - b\sin x + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $4b < (a + 2c)\pi$.

(iii) Show that, if $a > 0$, $b^2 < 4ac$ and $q > p > 0$, then

$$b\ln\left(\frac{q}{p}\right) < a\left(\frac{1}{p} - \frac{1}{q}\right) + c(q - p)\,.$$

[STEP 2, 2002Q5]

The numbers x_n, where $n = 0, 1, 2, \ldots$, satisfy

$$x_{n+1} = kx_n(1 - x_n).$$

(i) Prove that, if $0 < k < 4$ and $0 < x_0 < 1$, then $0 < x_n < 1$ for all n.

(ii) Given that $x_0 = x_1 = x_2 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, find k in terms of a.

(iii) Given instead that $x_0 = x_2 = x_4 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, show that $ab^3 - b^2 + (1 - a) = 0$, where $b = k(1 - a)$. Given, in addition, that $x_1 \neq a$, find the possible values of k in terms of a.

[STEP 2, 2002Q6]

The lines l_1, l_2 and l_3 lie in an inclined plane P and pass through a common point A. The line l_2 is a line of greatest slope in P. The line l_1 is perpendicular to l_3 and makes an acute angle α with l_2. The angles between the horizontal and l_1, l_2 and l_3 are $\frac{\pi}{6}$, β and $\frac{\pi}{4}$, respectively. Show that $\cos\alpha\sin\beta = \frac{1}{2}$ and find the value of $\sin\alpha\sin\beta$. Deduce that $\beta = \frac{\pi}{3}$.

The lines l_1 and l_3 are rotated in P about A so that l_1 and l_3 remain perpendicular to each other. The new acute angle between l_1 and l_2 is θ. The new angles which l_1 and l_3 make with the horizontal are ϕ and 2ϕ, respectively. Show that

$$\tan^2\theta = \frac{3 + \sqrt{13}}{2}.$$

[STEP 2, 2002Q7]

In 3-dimensional space, the lines m_1 and m_2 pass through the origin and have directions $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{k}}$, respectively. Find the directions of the two lines m_3 and m_4 that pass through the origin and make angles of $\frac{\pi}{4}$ with both m_1 and m_2. Find also the cosine of the acute angle between m_3 and m_4.

The points A and B lie on m_1 and m_2 respectively, and are each at distance $\lambda\sqrt{2}$ units from O. The points P and Q lie on m_3 and m_4 respectively, and are each at distance 1 unit from O. If all the coordinates (with respect to axes $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$) of A, B, P and Q are non-negative, prove that:

(i) there are only two values of λ for which AQ is perpendicular to BP.

(ii) there are no non-zero values of λ for which AQ and BP intersect.

[STEP 2, 2002Q8]

Find y in terms of x, given that:

$$\text{for } x < 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -y \quad \text{and} \quad y = a \text{ when } x = -1;$$
$$\text{for } x > 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = y \quad \text{and} \quad y = b \text{ when } x = 1.$$

Sketch a solution curve. Determine the condition on a and b for the solution curve to be continuous (that is, for there to be no 'jump' in the value of y) at $x = 0$.

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = |\mathrm{e}^x - 1|y$$

given that $y = \mathrm{e}^{\mathrm{e}}$ when $x = 1$ and that y is continuous at $x = 0$. Write down the following limits:

(i) $\lim_{x \to +\infty} y \exp(-\mathrm{e}^x)$.

(ii) $\lim_{x \to -\infty} y\mathrm{e}^{-x}$.

Section B: Mechanics

[STEP 2, 2002Q9]

A particle is projected from a point O on a horizontal plane with speed V and at an angle of elevation α. The vertical plane in which the motion takes place is perpendicular to two vertical walls, both of height h, at distances a and b from O. Given that the particle just passes over the walls, find $\tan\alpha$ in terms of a, b and h and show that

$$\frac{2V^2}{g} = \frac{ab}{h} + \frac{(a+b)^2 h}{ab}.$$

The heights of the walls are now increased by the same small positive amount δh. A second particle is projected so that it just passes over both walls, and the new angle and speed of projection are $\alpha + \delta\alpha$ and $V + \delta V$, respectively. Show that

$$\sec^2\alpha\,\delta\alpha \approx \frac{a+b}{ab}\delta h\,,$$

and deduce that $\delta\alpha > 0$. Show also that δV is positive if $h > \frac{ab}{a+b}$ and negative if $h < \frac{ab}{a+b}$.

[STEP 2, 2002Q10]

A competitor in a Marathon of $42\frac{3}{8}$ km runs the first t hours of the race at a constant speed of 13 km h^{-1} and the remainder at a constant speed of $14 + \frac{2t}{T}$ km h^{-1}, where T hours is her time for the race. Show that the minimum possible value of T over all possible values of t is 3.

The speed of another competitor decreases linearly with respect to time from 16 km h^{-1} at the start of the race. If both of these competitors have a run time of 3 hours, find the maximum distance between them at any stage of the race.

[STEP 2, 2002Q11]

A rigid straight beam AB has length l and weight W. Its weight per unit length at a distance x from B is $\alpha W l^{-1}\left(\frac{x}{l}\right)^{\alpha-1}$, where α is a positive constant. Show that the centre of mass of the beam is at a distance $\frac{\alpha l}{\alpha+1}$ from B.

The beam is placed with the end A on a rough horizontal floor and the end B resting against a rough vertical wall. The beam is in a vertical plane at right angles to the plane of the wall and makes an angle of θ with the floor. The coefficient of friction between the floor and the beam is μ and the coefficient of friction between the wall and the beam is also μ. Show that, if the equilibrium is limiting at both A and B, then

$$\tan\theta = \frac{1-\alpha\mu^2}{(1+\alpha)\mu}.$$

Given that $\alpha = \frac{3}{2}$ and given also that the beam slides for any $\theta < \frac{\pi}{4}$ find the greatest possible value of μ.

Section C: Probability and Statistics

[STEP 2, 2002Q12]

On K consecutive days each of L identical coins is thrown M times. For each coin, the probability of throwing a head in any one throw is p (where $0 < p < 1$). Show that the probability that on exactly k of these days more than l of the coins will each produce fewer than m heads can be approximated by

$$\binom{K}{k} q^k (1-q)^{K-k},$$

where

$$q = \Phi\left(\frac{2h - 2l - 1}{2\sqrt{h}}\right), \qquad h = L\Phi\left(\frac{2m - 1 - 2Mp}{2\sqrt{Mp(1-p)}}\right)$$

and $\Phi(\,.\,)$ is the cumulative distribution function of a standard normal variate.

Would you expect this approximation to be accurate in the case $K = 7$, $k = 2$, $L = 500$, $l = 4$, $M = 100$, $m = 48$ and $p = 0.6$?

[STEP 2, 2002Q13]

Let $F(x)$ be the cumulative distribution function of a random variable X, which satisfies $F(a) = 0$ and $F(b) = 1$, where $a > 0$. Let

$$G(y) = \frac{F(y)}{2 - F(y)}.$$

Show that $G(a) = 0$, $G(b) = 1$ and that $G'(y) \geq 0$. Show also that

$$\frac{1}{2} \leq \frac{2}{\left(2 - F(y)\right)^2} \leq 2.$$

The random variable Y has cumulative distribution function $G(y)$. Show that

$$\frac{1}{2}\mathrm{E}(X) \leq \mathrm{E}(Y) \leq 2\mathrm{E}(X),$$

and that

$$\mathrm{Var}(Y) \leq 2\mathrm{Var}(X) + \frac{7}{4}\left(\mathrm{E}(X)\right)^2.$$

[STEP 2, 2002Q14]

A densely populated circular island is divided into N concentric regions $R_1, R_2, \ldots, R_N$, such that the inner and outer radii of R_n are $n-1$ km and n km, respectively. The average number of road accidents that occur in any one day in R_n is $2-\frac{n}{N}$, independently of the number of accidents in any other region.

Each day an observer selects a region at random, with a probability that is proportional to the area of the region, and records the number of road accidents, X, that occur in it. Show that, in the long term, the average number of recorded accidents per day will be

$$2-\frac{1}{6}\left(1+\frac{1}{N}\right)\left(4-\frac{1}{N}\right).$$

[Note: $\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$.]

Show also that

$$P(X=k) = \frac{\mathrm{e}^{-2}N^{-k-2}}{k!}\sum_{n=1}^{N}(2n-1)(2N-n)^k\,\mathrm{e}^{\frac{n}{N}}.$$

Suppose now that $N=3$ and that, on a particular day, two accidents were recorded. Show that the probability that R_2 had been selected is

$$\frac{48}{48+45\mathrm{e}^{\frac{1}{3}}+25\mathrm{e}^{-\frac{1}{3}}}.$$

STEP 2 2003

STEP 2 2003
On-line Exam

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Section A: Pure Mathematics

[STEP 2, 2003Q1]

Consider the equations

$$\begin{aligned} ax - y - z &= 3, \\ 2ax - y - 3z &= 7, \\ 3ax - y - 5z &= b, \end{aligned}$$

where a and b are given constants.

(i) In the case $a = 0$, show that the equations have a solution if and only if $b = 11$.

(ii) In the case $a \neq 0$ and $b = 11$ show that the equations have a solution with $z = \lambda$ for any given number λ.

(iii) In the case $a = 2$ and $b = 11$ find the solution for which $x^2 + y^2 + z^2$ is least.

(iv) Find a value for a for which there is a solution such that $x > 10^6$ and $y^2 + z^2 < 1$.

[STEP 2, 2003Q2]

Write down a value of θ in the interval $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ that satisfies the equation

$$4\cos\theta + 2\sqrt{3}\sin\theta = 5.$$

Hence, or otherwise, show that

$$\pi = 3\arccos\left(\frac{5}{\sqrt{28}}\right) + 3\arctan\left(\frac{\sqrt{3}}{2}\right).$$

Show that

$$\pi = 4\arcsin\left(\frac{7\sqrt{2}}{10}\right) - 4\arctan\left(\frac{3}{4}\right).$$

[STEP 2, 2003Q3]

Prove that the cube root of any irrational number is an irrational number.

Let $u_n = 5^{\frac{1}{3^n}}$. Given that $\sqrt[3]{5}$ is an irrational number, prove by induction that u_n is an irrational number for every positive integer n.

Hence, or otherwise, give an example of an infinite sequence of irrational numbers which converges to a given integer m.

[An irrational number is a number that cannot be expressed as the ratio of two integers.]

[STEP 2, 2003Q4]

The line $y = d$, where $d > 0$, intersects the circle $x^2 + y^2 = R^2$ at G and H. Show that the area of the minor segment GH is equal to

$$R^2 \arccos\left(\frac{d}{R}\right) - d\sqrt{R^2 - d^2}. \qquad (*)$$

In the following cases, the given line intersects the given circle. Determine how, in each case, the expression $(*)$ should be modified to give the area of the minor segment.

(i) Line: $y = c$; circle: $(x-a)^2 + (y-b)^2 = R^2$.

(ii) Line: $y = mx + c$; circle: $x^2 + y^2 = R^2$.

(iii) Line: $y = mx + c$; circle: $(x-a)^2 + (y-b)^2 = R^2$.

[STEP 2, 2003Q5]

The position vectors of the points A, B and P with respect to an origin O are $a\hat{\mathbf{i}}$, $b\hat{\mathbf{j}}$ and $l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$, respectively, where a, b, and n are all non-zero. The points E, F, G and H are the midpoints of OA, BP, OB and AP, respectively. Show that the line EF and GH intersect.

Let D be the point with position vector $d\hat{\mathbf{k}}$, where d is non-zero, and let S be the point of intersection of EF and GH. The point T is such that the mid-point of DT is S. Find the position vector of T and hence find d in terms of n if T lies in the plane OAB.

[STEP 2, 2003Q6]

The function f is defined by

$$f(x) = |x - 1|,$$

where the domain is $\mathbf{R}$, the set of all real numbers. The function $g_n = f^n$, with domain $\mathbf{R}$, so for example $g_3(x) = f\left(f\left(f(x)\right)\right)$. In separate diagrams, sketch graphs of g_1, g_2, g_3 and g_4.

The function h is defined by

$$h(x) = \left|\sin\frac{\pi x}{2}\right|,$$

where the domain is $\mathbf{R}$. Show that if n is even,

$$\int_0^n \left(h(x) - g_n(x)\right)\mathrm{d}x = \frac{2n}{\pi} - \frac{n}{2}.$$

[STEP 2, 2003Q7]

Show that, if $n > 0$, then

$$\int_{e^{\frac{1}{n}}}^{\infty} \frac{\ln x}{x^{n+1}}\,dx = \frac{2}{n^2 e}.$$

You may assume that $\frac{\ln x}{x} \to 0$ as $x \to \infty$.

Explain why, if $1 < a < b$, then

$$\int_b^{\infty} \frac{\ln x}{x^{n+1}}\,dx < \int_a^{\infty} \frac{\ln x}{x^{n+1}}\,dx.$$

Deduce that

$$\sum_{n=1}^{N} \frac{1}{n^2} < \frac{e}{2} \int_{e^{\frac{1}{N}}}^{\infty} \left(\frac{1 - x^{-N}}{x^2 - x} \right) \ln x \, dx,$$

where N is any integer greater than 1.

[STEP 2, 2003Q8]

It is given that y satisfies

$$\frac{dy}{dt} + k\left(\frac{t^2 - 3t + 2}{t + 1}\right) y = 0,$$

where k is a constant, and $y = A$ when $t = 0$, where A is a positive constant. Find y in terms of t, k and A.

Show that y has two stationary values whose ratio is $\left(\frac{3}{2}\right)^{6k} e^{-\frac{5k}{2}}$.

Describe the behaviour of y as $t \to +\infty$ for the case where $k > 0$ and for the case where $k < 0$.

In separate diagrams, sketch the graph of y for $t > 0$ for each of these cases.

Section B: Mechanics

[STEP 2, 2003Q9]

AB is a uniform rod of weight W. The point C on AB is such that $AC > CB$. The rod is in contact with a rough horizontal floor at A and with a cylinder at C. The cylinder is fixed to the floor with its axis horizontal. The rod makes an angle α with the horizontal and lies in a vertical plane perpendicular to the axis of the cylinder. The coefficient of friction between the rod and the floor is $\tan\lambda_1$ and the coefficient of friction between the rod and the cylinder is $\tan\lambda_2$.

Show that if friction is limiting both at A and at C, and $\alpha \neq \lambda_2 - \lambda_1$, then the frictional force acting on the rod at A has magnitude

$$\frac{W\sin\lambda_1\sin(\alpha-\lambda_2)}{\sin(\alpha+\lambda_1-\lambda_2)}.$$

[STEP 2, 2003Q10]

A bead B of mass m can slide along a rough horizontal wire. A light inextensible string of length $2l$ has one end attached to a fixed point A of the wire and the other to B. A particle P of mass $3m$ is attached to the mid-point of the string and B is held at a distance l from A. The bead is released from rest.

Let a_1 and a_2 be the magnitudes of the horizontal and vertical components of the initial acceleration of P. Show by considering the motion of P relative to A, or otherwise, that $a_1 = \sqrt{3}a_2$. Show also that the magnitude of the initial acceleration of B is $2a_1$.

Given that the frictional force opposing the motion of B is equal to $\frac{\sqrt{3}}{6}R$, where R is the normal reaction between B and the wire, show that the magnitude of the initial acceleration of P is $\frac{g}{18}$.

[STEP 2, 2003Q11]

A particle P_1 is projected with speed V at an angle of elevation α ($> 45°$), from a point in a horizontal plane. Find T_1, the flight time of P_1, in terms of α, V and g. Show that the time after projection at which the direction of motion of P_1 first makes an angle of $45°$ with the horizontal is $\frac{1}{2}(1-\cot\alpha)T_1$.

A particle P_2 is projected under the same conditions. When the direction of the motion of P_2 first makes an angle of $45°$ with the horizontal, the speed of P_2 is instantaneously doubled. If T_2 is the total flight time of P_2, show that

$$\frac{2T_2}{T_1} = 1 + \cot\alpha + \sqrt{1+3\cot^2\alpha}.$$

Section C: Probability and Statistics

[STEP 2, 2003Q12]

The life of a certain species of elementary particles can be described as follows. Each particle has a life time of T seconds, after which it disintegrates into X particles of the same species, where X is a random variable with binomial distribution $\mathrm{B}(2, p)$. A population of these particles starts with the creation of a single such particle at $t = 0$. Let X_n be the number of particles in existence in the time interval $n_T < t < (n+1)T$, where $n = 1, 2, \ldots$.

Show that $\mathrm{P}(X_1 = 2 \text{ and } X_2 = 2) = 6p^4q^2$, where $q = 1 - p$. Find the possible values of p if it is known that $\mathrm{P}(X_1 = 2 | X_2 = 2) = \frac{9}{25}$.

Explain briefly why $\mathrm{E}(X_n) = 2p\mathrm{E}(X_{n-1})$ and hence determine $\mathrm{E}(X_n)$ in terms of p. Show that for one of the values of p found above $\lim_{n\to\infty} \mathrm{E}(X_n) = 0$ and that for the other $\lim_{n\to\infty} \mathrm{E}(X_n) = +\infty$.

[STEP 2, 2003Q13]

The random variable X takes the values $k = 1, 2, 3, \ldots$, and has probability distribution

$$\mathrm{P}(X = k) = A\frac{\lambda^k \mathrm{e}^{-\lambda}}{k!},$$

where λ is a positive constant. Show that $A = \left(1 - \mathrm{e}^{-\lambda}\right)^{-1}$. Find the mean μ in terms of λ and show that

$$\mathrm{Var}\,(X) = \mu(1 - \mu + \lambda).$$

Deduce that $\lambda < \mu < 1 + \lambda$.

Use a normal approximation to find the value of $\mathrm{P}(X = \lambda)$ in the case where $\lambda = 100$, giving your answer to 2 decimal places.

[STEP 2, 2003Q14]

The probability of throwing a 6 with a biased die is p. It is known that p is equal to one or other of the numbers A and B where $0 < A < B < 1$. Accordingly the following statistical test of the hypothesis $H_0 : p = B$ against the alternative hypothesis $H_1 : p = A$ is performed.

The die is thrown repeatedly until a 6 is obtained. Then if X is the total number of throws, H_0 is accepted if $X \le M$, where M is a given positive integer; otherwise H_1 is accepted. Let α be the probability that H_1 is accepted if H_0 is true, and let β be the probability that H_0 is accepted if H_1 is true.

Show that $\beta = 1 - \alpha^K$, where K is independent of M and is to be determined in terms of A and B. Sketch the graph of β against α.

STEP 2 2004

STEP 2 2004
On-line Exam

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Section A: Pure Mathematics

[STEP 2, 2004Q1]

Find all real values of x that satisfy:

(i) $\sqrt{3x^2+1}+\sqrt{x}-2x-1=0$.

(ii) $\sqrt{3x^2+1}-2\sqrt{x}+x-1=0$.

(iii) $\sqrt{3x^2+1}-2\sqrt{x}-x+1=0$.

[STEP 2, 2004Q2]

Prove that, if $|\alpha|<2\sqrt{2}$, then there is no value of x for which

$$x^2-\alpha|x|+2<0. \qquad (*)$$

Find the solution set of $(*)$ for $\alpha=3$.

For $\alpha>2\sqrt{2}$, the sum of the lengths of the intervals in which x satisfies $(*)$ is denoted by S. Find S in terms of α and deduce that $S<2\alpha$.

Sketch the graph of S against α.

[STEP 2, 2004Q3]

The curve C has equation

$$y=x(x+1)(x-2)^4.$$

Show that the gradient of C is $(x-2)^3(6x^2+x-2)$ and find the coordinates of all the stationary points. Determine the nature of each stationary point and sketch C.

In separate diagrams draw sketches of the curves whose equations are:

(i) $y^2=x(x+1)(x-2)^4$.

(ii) $y=x^2(x^2+1)(x^2-2)^4$.

In each case, you should pay particular attention to the points where the curve meets the x axis.

[STEP 2, 2004Q4]

Figure 1

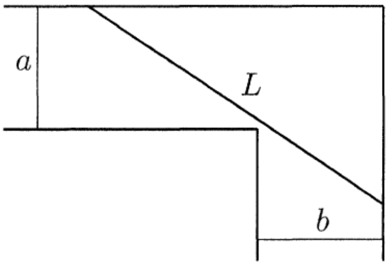
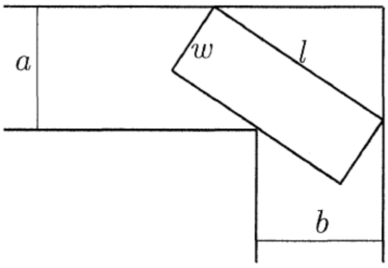


Figure 2



(i) An attempt is made to move a rod of length L from a corridor of width a into a corridor of width b, where $a \neq b$. The corridors meet at right angles, as shown in Figure 1 and the rod remains horizontal. Show that if the attempt is to be successful then

$$L \leq a \operatorname{cosec} \alpha + b \sec \alpha \,,$$

where α satisfies

$$\tan^3 \alpha = \frac{a}{b}.$$

(ii) An attempt is made to move a rectangular table-top, of width w and length l, from one corridor to the other, as shown in Figure 2. The table-top remains horizontal. Show that if the attempt is to be successful then

$$l \leq a \operatorname{cosec} \beta + b \sec \beta - 2w \operatorname{cosec} 2\beta,$$

where β satisfies

$$w = \left(\frac{a - b \tan^3 \beta}{1 - \tan^2 \beta} \right) \cos \beta.$$

[STEP 2, 2004Q5]

Evaluate $\int_0^\pi x \sin x \, dx$ and $\int_0^\pi x \cos x \, dx$.

The function f satisfies the equation

$$f(t) = t + \int_0^\pi f(x) \sin(x + t) \, dx. \qquad (*)$$

Show that

$$f(t) = t + A \sin t + B \cos t, \qquad (**)$$

where $A = \int_0^\pi f(x) \cos x \, dx$ and $B = \int_0^\pi f(x) \sin x \, dx$.

Use the expression $(**)$ to find A and B by substituting for $f(t)$ and $f(x)$ in $(*)$ and equating coefficients of $\sin t$ and $\cos t$.

[STEP 2, 2004Q6]

The vectors $\mathbf{a}$ and $\mathbf{b}$ lie in the plane Π. Given that $|\mathbf{a}| = 1$ and $\mathbf{a}.\mathbf{b} = 3$, find, in terms of $\mathbf{a}$ and $\mathbf{b}$, a vector $\mathbf{p}$ parallel to $\mathbf{a}$ and a vector $\mathbf{q}$ perpendicular to $\mathbf{a}$, both lying in the plane Π, such that

$$\mathbf{p} + \mathbf{q} = \mathbf{a} + \mathbf{b}.$$

The vector $\mathbf{c}$ is not parallel to the plane Π and is such that $\mathbf{a}.\mathbf{c} = -2$ and $\mathbf{b}.\mathbf{c} = 2$. Given that $|\mathbf{b}| = 5$, find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, vectors $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ such that $\mathbf{P}$ and $\mathbf{Q}$ are parallel to $\mathbf{p}$ and $\mathbf{q}$, respectively, $\mathbf{R}$ is perpendicular to the plane Π and

$$\mathbf{P} + \mathbf{Q} + \mathbf{R} = \mathbf{a} + \mathbf{b} + \mathbf{c}.$$

[STEP 2, 2004Q7]

The function f is defined by

$$f(x) = 2\sin x - x.$$

Show graphically that the equation $f(x) = 0$ has exactly one root in the interval $\left[\frac{1}{2}\pi,\ \pi\right]$. This interval is denoted I_0.

In order to determine the root, a sequence of intervals $I_1, I_2, \ldots$ is generated in the following way. If the interval $I_n = [a_n,\ b_n]$, and $c_n = \frac{a_n+b_n}{2}$, then

$$I_{n+1} = \begin{cases} [a_n, c_n], & \text{if } f(a_n)f(c_n) < 0; \\ [c_n, b_n], & \text{if } f(c_n)f(b_n) < 0. \end{cases}$$

By using the approximations $\frac{1}{\sqrt{2}} \approx 0.7$ and $\pi \approx \sqrt{10}$, show that $I_2 = \left[\frac{1}{2}\pi,\ \frac{5}{8}\pi\right]$ and find I_3.

[STEP 2, 2004Q8]

Let x satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1 - x^n)^{\frac{1}{n}}$$

and the condition $x = 0$ when $t = 0$.

(i) Solve the equation in the case $n = 1$ and sketch the graph of the solution for $t > 0$.

(ii) Prove that $1 - x < (1 - x^2)^{\frac{1}{2}}$ for $0 < x < 1$.

Use this result to sketch the graph of the solution in the case $n = 2$ for $0 < t < \frac{1}{2}\pi$, using the same axes as your previous sketch.

By setting $x = \sin y$, solve the equation in this case.

(iii) Use the result (which you need not prove)

$$(1 - x^2)^{\frac{1}{2}} < (1 - x^3)^{\frac{1}{3}} \quad \text{for} \quad 0 < x < 1,$$

to sketch, without solving the equation, the graph of the solution of the equation in the case $n = 3$ using the same axes as your previous sketches. Use your sketch to show that $x = 1$ at a value of t less than $\frac{1}{2}\pi$.

Section B: Mechanics

[STEP 2, 2004Q9]

The base of a non-uniform solid hemisphere, of mass M, has radius r. The distance of the centre of gravity, G, of the hemisphere from the base is p and from the centre of the base is $\sqrt{p^2+q^2}$. The hemisphere rests in equilibrium with its curved surface on a horizontal plane.

A particle of mass m, where m is small, is attached to A, the lowest point of the circumference of the base. In the new position of equilibrium, find the angle, α, that the base makes with the horizontal.

The particle is removed and attached to the point B of the base which is at the other end of the diameter through A. In the new position of equilibrium the base makes an angle β with the horizontal. Show that

$$\tan(\alpha-\beta)=\frac{2mMrp}{M^2(p^2+q^2)-m^2r^2}.$$

[STEP 2, 2004Q10]

In this question take g=10 m s^{-2}.

The point A lies on a fixed rough plane inclined at 30° to the horizontal and l is the line of greatest slope through A. A partical P is projected up l from A with initial speed 6 m s^{-1}. At time T seconds later, a partical Q is projected from A up l, also with speed 6 m s^{-1}. The coefficient of friction between each particle and the plane is $\frac{1}{5\sqrt{3}}$ and the mass of each particle is 4 kg.

(i) Given that $T<1+\sqrt{\frac{3}{2}}$, show that the particle collide at a time $(3-\sqrt{6})T+1$ seconds after A is projected.

(ii) In the case $T=1+\sqrt{\frac{2}{3}}$, determine the energy lost due to friction from the instant at which P is projected to the time of the collision.

[STEP 2, 2004Q11]

The maximum power that can be developed by the engine of train A, of mass m, when travelling at speed v is $Pv^{\frac{3}{2}}$, where P is a constant. The maximum power that can be developed by the engine of train B, of mass $2m$, when travelling at speed v is $2Pv^{\frac{3}{2}}$. For both A and B resistance to motion is equal to kv, where k is a constant.

For $t \leq 0$, the engines are crawling along at very low equal speeds. At $t = 0$, both drivers switch on full power and at time t the speeds of A and B are v_A and v_B, respectively.

(i) Show that

$$v_A = \frac{P^2\left(1 - e^{-\frac{kt}{2m}}\right)^2}{k^2}$$

and write down the corresponding result for v_B.

(ii) Find v_A and v_B when $9v_A = 4v_B$. [You may find the substitution $v_A = u^2$ useful.]

(iii) Both engines are switched off when $9v_A = 4v_B$. Show that thereafter $k^2v_B^2 = 4P^2v_A$.

Section C: Probability and Statistics

[STEP 2, 2004Q12]

Sketch the graph, for $x \geq 0$, of

$$y = kxe^{-ax^2},$$

where a and k are positive constants.

The random variable X has probability density function $f(x)$ given by

$$f(x) = \begin{cases} kxe^{-ax^2} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $k = \frac{2a}{1-e^{-a}}$ and find the mode m in terms of a, distinguishing between the cases $a < \frac{1}{2}$ and $a > \frac{1}{2}$.

Find the median h in terms of a and show that $h > m$ if $a > -\ln\left(2e^{-\frac{1}{2}} - 1\right)$.

Show that $-\ln\left(2e^{-\frac{1}{2}} - 1\right) > \frac{1}{2}$. Show that, if $a > -\ln\left(2e^{-\frac{1}{2}} - 1\right)$, then

$$\mathrm{P}(X > m | X < h) = \frac{2e^{-\frac{1}{2}} - e^{-a} - 1}{1 - e^{-a}}.$$

[STEP 2, 2004Q13]

A bag contains b balls, r of them red and the rest white. In a game the player must remove balls one at a time from the bag (without replacement). She may remove as many balls as she wishes, but if she removes any red ball, she loses and get no reward at all. If she does not remove a red ball, she is rewarded with £1 for each white ball she has removed.

If she removes n white balls on her first n draws, calculate her expected gain on the next draw and show that it is zero if $n = \frac{b-r}{r+1}$.

Hence, or otherwise, show that she will maximise her expected total reward if she aims to remove n balls, where

$$n = \text{the integer part of } \frac{b-r}{r+1} + 1.$$

With this value of n, show that in the case $r = 1$ and b even, her expected total reward is £$\frac{1}{4}b$, and find her expected total reward in the case $r = 1$ and b odd.

[STEP 2, 2004Q14]

Explain why, if A, B and C are three events,

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A) + \mathrm{P}(A \cap B \cap C),$$

where $\mathrm{P}(\mathrm{X})$ denotes the probability of event X.

A cook makes three plum puddings for Christmas. He stirs r silver sixpences thoroughly into the pudding mixture before dividing it into three equal portions. Find an expression for the probability that at least one pudding contains no sixpence. Show that the cook must stir 6 or more sixpences into the mixture if there is to be less than $\frac{1}{3}$ chance that at least one of the puddings contains no sixpence.

Given that the cook stirs 6 sixpences into the mixture and that each pudding contains at least one sixpence, find the probability that there are two sixpences in each pudding.

STEP 2 2005

STEP 2 2005

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Section A: Pure Mathematics

[STEP 2, 2005Q1]

Find the three values of x for which the derivative of $x^2\mathrm{e}^{-x^2}$ is zero.

Given that a and b are distinct positive numbers, find a polynomial $P(x)$ such that the derivative of $P(x)\mathrm{e}^{-x^2}$ is zero for $x = 0$, $x = \pm a$ and $x = \pm b$, but for no other values of x.

[STEP 2, 2005Q2]

For any positive integer N, the function $f(N)$ is defined by

$$f(N) = N\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

where $p_1, p_2, \dots, p_k$ are the only prime numbers that are factors of N.

Thus $f(80) = 80\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)$.

(a) (i) Evaluate $f(12)$ and $f(180)$.

(ii) Show that $f(N)$ is an integer for all N.

(b) Prove, or disprove by means of a counterexample, each of the following:

(i) $f(m)f(n) = f(mn)$.

(ii) $f(p)f(q) = f(pq)$ if p and q are distinct prime numbers.

(iii) $f(p)f(q) = f(pq)$ only if p and q are distinct prime numbers.

(c) Find a positive integer m and a prime number p such that $f(p^m) = 146410$.

[STEP 2, 2005Q3]

Give a sketch, for $0 \le x \le \frac{\pi}{2}$, of the curve

$$y = \sin x - x\cos x\,,$$

and show that $0 \le y \le 1$.

Show that:

(i) $\int_0^{\frac{\pi}{2}} y\,\mathrm{d}x = 2 - \frac{\pi}{2}$.

(ii) $\int_0^{\frac{\pi}{2}} y^2\,\mathrm{d}x = \frac{\pi^3}{48} - \frac{\pi}{8}$.

Deduce that $\pi^3 + 18\pi < 96$.

[STEP 2, 2005Q4]

The positive numbers a, b and c satisfy $bc = a^2 + 1$. Prove that

$$\arctan\left(\frac{1}{a+b}\right) + \arctan\left(\frac{1}{a+c}\right) = \arctan\left(\frac{1}{a}\right).$$

The positive numbers p, q, r, s, t, u and v satisfy

$$st = (p+q)^2 + 1, \qquad uv = (p+r)^2 + 1, \qquad qr = p^2 + 1.$$

Prove that

$$\arctan\left(\frac{1}{p+q+s}\right) + \arctan\left(\frac{1}{p+q+t}\right) + \arctan\left(\frac{1}{p+r+u}\right) + \arctan\left(\frac{1}{p+r+v}\right) = \arctan\left(\frac{1}{p}\right).$$

Hence show that

$$\arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{1}{21}\right) + \arctan\left(\frac{1}{82}\right) + \arctan\left(\frac{1}{187}\right) = \arctan\left(\frac{1}{7}\right).$$

[Note that $\arctan x$ is another notation for $\tan^{-1} x$.]

[STEP 2, 2005Q5]

The angle A of triangle ABC is a right angle and the sides BC, CA and AB are of lengths a, b and c, respectively. Each side of the triangle is tangent to the circle S_1 which is of radius r. Show that $2r = b + c - a$.

Each vertex of the triangle lies on the circle S_2. The ratio of the area of the region between S_1 and the triangle to the area of S_2 is denoted by R. Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1),$$

where $q = \frac{b+c}{a}$. Deduce that

$$R \leq \frac{1}{\pi(\pi - 1)}.$$

[STEP 2, 2005Q6]

(i) Write down the general term in the expansion in powers of x of $(1-x)^{-1}$, $(1-x)^{-2}$ and $(1-x)^{-3}$, where $|x|<1$.

Evaluate

$$\sum_{n=1}^{\infty} n2^{-n} \quad \text{and} \quad \sum_{n=1}^{\infty} n^2 2^{-n}.$$

(ii) Show that

$$(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!\ x^n}{(n!)^2\ 2^{2n}}, \qquad \text{for } |x|<1.$$

Evaluate

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n} 3^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2 2^{2n} 3^n}.$$

[STEP 2, 2005Q7]

The position vectors, relative to an origin O, at time t of the particles P and Q are

$$\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$$

and

$$\cos\left(t+\frac{1}{4}\pi\right)\left[\frac{3}{2}\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{k}}\right] + 3\sin\left(t+\frac{1}{4}\pi\right)\hat{\mathbf{j}},$$

respectively, where $0 \le t \le 2\pi$.

(i) Give a geometrical description of the motion of P and Q.

(ii) Let θ be the angle POQ at time t that satisfies $0 \le \theta \le 2\pi$. Show that

$$\cos\theta = \frac{3\sqrt{2}}{8} - \frac{1}{4}\cos\left(2t+\frac{1}{4}\pi\right).$$

(iii) Show that the total time for which $\theta \ge \frac{1}{4}\pi$ is $\frac{3}{2}\pi$.

[STEP 2, 2005Q8]

For $x \geq 0$ the curve C is defined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3y^2}{(1+x^2)^{\frac{5}{2}}}$$

with $y = 1$ when $x = 0$. Show that

$$\frac{1}{y} = \frac{2+3x^2}{3(1+x^2)^{\frac{3}{2}}} + \frac{1}{3}$$

and hence that for large positive x

$$y \approx 3 - \frac{9}{x}.$$

Draw a sketch of C.

On a separate diagram draw a sketch of the two curves defined for $x \geq 0$ by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x^3z^3}{2(1+x^2)^{\frac{5}{2}}}$$

with $z = 1$ at $x = 0$ on one curve, and $z = -1$ at $x = 0$ on the other.

Section B: Mechanics

[STEP 2, 2005Q9]

Two particles, A and B, of masses m and $2m$, respectively, are placed on a line of greatest slope, l, of a rough inclined plane which makes an angle of 30° with the horizontal. The coefficient of friction between A and the plane is $\frac{1}{6}\sqrt{3}$ and the coefficient of friction between B and the plane is $\frac{1}{3}\sqrt{3}$. The particles are at rest with B higher up l than A and are connected by a light inextensible string which is taut. A force P is applied to B.

(i) Show that the least magnitude of P for which the two particles move upwards along l is $\frac{11}{8}\sqrt{3}mg$ and give, in this case, the direction in which P acts.

(ii) Find the least magnitude of P for which the particles do not slip downwards along l.

[STEP 2, 2005Q10]

The points A and B are 180 metres apart and lie on horizontal ground. A missile is launched from A at speed of 100 m s^{-1} and at an acute angle of elevation to the line AB of $\arcsin\frac{3}{5}$. A time T seconds later, an anti-missile missile is launched from B, at speed of 200 m s^{-1} and at an acute angle of elevation to the line BA of $\arcsin\frac{4}{5}$. The motion of both missiles takes place in the vertical plane containing A and B, and the missiles collide.

Taking g=10 m s^{-2} and ignoring air resistance, find T.

[Note that $\arcsin\frac{3}{5}$ is another notation for $\sin^{-1}\frac{3}{5}$.]

[STEP 2, 2005Q11]

A plane is inclined at an angle $\arctan\frac{3}{4}$ to the horizontal and a small, smooth, light pulley P is fixed to the top of the plane. A string, APB, passes over the pulley. A particle of mass m_1 is attached to the string at A and rests on the inclined plane with AP parallel to a line of greatest slope in the plane. A particle of mass m_2, where $m_2 > m_1$, is attached to the string at B and hangs freely with BP vertical. The coefficient of friction between the particle at A and the plane is $\frac{1}{2}$.

The system is released from rest with the string taut. Show that the acceleration of the particles is $\frac{m_2-m_1}{m_2+m_1}g$.

At a time T after release, the string breaks. Given that the particle at A does not reach the pulley at any point in its motion, find an expression in terms of T for the time after release at which the particle at A reaches its maximum height. It is found that, regardless of when the string broke, this time is equal to the time taken by the particle at A to descend from its point of maximum height to the point at which it was released. Find the ratio $m_1 : m_2$.

[Note that $\arctan\frac{3}{4}$ is another notation for $\tan^{-1}\frac{3}{4}$.]

Section C: Probability and Statistics

[STEP 2, 2005Q12]

The twins Anna and Bella share a computer and never sign their e-mails. When I email them, only the twin currently online responds. The probability that it is Anna who is online is p and she answers each question I ask her truthfully with probability a, independently of all her other answers, even if a question is repeated. The probability that it is Bella who is online is q, where $q = 1 - p$, and she answers each question truthfully with probability b, independently of all her other answers, even if a question is repeated.

(i) I send the twins the email: 'Toss a fair coin and answer the following question. Did the coin come down heads?'. I receive the answer 'yes'. Show that the probability that the coin did come down heads is $\frac{1}{2}$ if and only if $2(ap + bq) = 1$.

(ii) I send the twins the email: 'Toss a fair coin and answer the following question. Did the coin come down heads?' I receive the answer 'yes'. I then send the e-mail: 'Did the coin come down heads?' and I receive the answer 'no'. Show that the probability (taking into account these answers) that the coin did come down heads is $\frac{1}{2}$.

(iii) I send the twins the email: 'Toss a fair coin and answer the following question. Did the coin come down heads?' I receive the answer 'yes'. I then send the e-mail: 'Did the coin come down heads?' and I receive the answer 'yes'. Show that, if $2(ap + bq) = 1$, the probability (taking into account these answers) that the coin did come down heads is $\frac{1}{2}$.

[STEP 2, 2005Q13]

The number of printing errors on any page of a large book of N pages is modelled by a Poisson variate with parameter λ and is statistically independent of the number of printing errors on any other page. The number of pages in a random sample of n pages (where n is much smaller than N and $n \geq 2$) which contains fewer than two error is denoted by Y. Show that $\mathrm{P}(Y = k) = \binom{n}{k} p^k q^{n-k}$ where $p = (1 + \lambda)\mathrm{e}^{-\lambda}$ and $q = 1 - p$.

Show also that, if λ is sufficiently small,

(i) $q \approx \frac{1}{2}\lambda^2$.

(ii) the largest value of n for which $\mathrm{P}(Y = n) \geq 1 - \lambda$ is approximately $\frac{2}{\lambda}$.

(iii) $\mathrm{P}(Y > 1 | Y > 0) \approx 1 - n\left(\frac{\lambda^2}{2}\right)^{n-1}$.

[STEP 2, 2005Q14]

The probability density function $f(x)$ of the random variable X is given by

$$f(x) = k[\phi(x) + \lambda g(x)],$$

where $\phi(x)$ is the probability density function of a normal variate with mean 0 and variance 1, λ is a positive constant, and $g(x)$ is a probability density function defined by

$$g(x) = \begin{cases} \dfrac{1}{\lambda}, & \text{for } 0 \le x \le \lambda; \\ 0, & \text{otherwise.} \end{cases}$$

Find μ, the mean of X, in terms of λ, and prove that σ, the standard deviation of X, satisfies

$$\sigma^2 = \frac{\lambda^4 + 4\lambda^3 + 12\lambda + 12}{12(1+\lambda)^2}.$$

In the case $\lambda = 2$:

(i) draw a sketch of the curve $y = f(x)$.

(ii) express the cumulative distribution function of X in terms of $\Phi(x)$, the cumulative distribution function corresponding to $\phi(x)$.

(iii) evaluate $\mathrm{P}(0 < X < \mu + 2\sigma)$, given that $\Phi\left(\frac{2}{3} + \frac{2}{3}\sqrt{7}\right) = 0.9921$.

STEP 2 2006

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Section A: Pure Mathematics

[STEP 2, 2006Q1]

The sequence of real numbers $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2, \qquad \text{and} \qquad u_{n+1} = k - \frac{36}{u_n} \quad \text{for} \quad n \geq 1, \qquad (*)$$

where k is a constant.

(i) Determine the values of k for which the sequence $(*)$ is:

(a) constant.

(b) periodic with period 2.

(c) periodic with period 4.

(ii) In the case $k = 37$, show that $u_n \geq 2$ for all n. Given that in this case the sequence $(*)$ converges to a limit l, find the value of l.

[STEP 2, 2006Q2]

Using the series

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

show that $\mathrm{e} > \frac{8}{3}$.

Show that $n! > 2^n$ for $n \geq 4$ and hence show that $\mathrm{e} < \frac{67}{24}$.

Show that the curve with equation

$$y = 3\mathrm{e}^{2x} + 14\ln\left(\frac{4}{3} - x\right), \qquad x < \frac{4}{3},$$

has a minimum turning point between $x = \frac{1}{2}$ and $x = 1$ and a maximum turning point between $x = 1$ and $x = \frac{5}{4}$.

[STEP 2, 2006Q3]

(i) Show that

$$\left(5+\sqrt{24}\right)^4 + \frac{1}{\left(5+\sqrt{24}\right)^4}$$

is an integer.

Show also that

$$0.1 < \frac{1}{5+\sqrt{24}} < \frac{2}{19} < 0.11.$$

Hence determine, with clear reasoning, the value of $\left(5+\sqrt{24}\right)^4$ correct to four decimal places.

(ii) If N is an integer greater than 1, show that $\left(N+\sqrt{N^2-1}\right)^k$, where k is a positive integer, differs from the integer nearest to it by less than $\left(2N-\frac{1}{2}\right)^{-k}$.

[STEP 2, 2006Q4]

By making the substitution $x = \pi - t$, show that

$$\int_0^\pi x f(\sin x)\,\mathrm{d}x = \frac{1}{2}\pi\int_0^\pi f(\sin x)\,\mathrm{d}x,$$

where $f(\sin x)$ is a given function of $\sin x$.

Evaluate the following integrals:

(i) $\int_0^\pi \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x$.

(ii) $\int_0^{2\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x$.

(iii) $\int_0^\pi \frac{x|\sin 2x|}{3+\sin^2 x}\,\mathrm{d}x$.

[STEP 2, 2006Q5]

The notation $\lfloor x\rfloor$ denotes the greatest integer less than or equal to the real number x. Thus, for example, $\lfloor\pi\rfloor = 3$, $\lfloor 18\rfloor = 18$ and $\lfloor -4.2\rfloor = -5$.

(i) Two curves are given by $y = x^2+3x-1$ and $y = x^2+3\lfloor x\rfloor - 1$. Sketch the curves, for $1 \le x \le 3$, on the same axes.

Find the area between the two curves for $1 \le x \le n$, where n is a positive integer.

(ii) Two curves are given by $y = x^2+3x-1$ and $y = \lfloor x\rfloor^2 + 3\lfloor x\rfloor - 1$. Sketch the curves, for $1 \le x \le 3$, on the same axes.

Show that the area between the two curves for $1 \le x \le n$, where n is a positive integer, is

$$\frac{1}{6}(n-1)(3n+11).$$

[STEP 2, 2006Q6]

By considering a suitable scalar product, prove that

$$(ax + by + cz)^2 \le (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$$

for any real numbers a, b, c, x, y and z. Deduce a necessary and sufficient condition on a, b, c, x, y and z for the following equation to hold:

$$(ax + by + cz)^2 = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2).$$

(i) Show that $(x + 2y + 2z)^2 \le 9(x^2 + y^2 + z^2)$ for any real numbers x, y and z, and use this result to solve the equation $(x + 56)^2 = 9(x^2 + 392)$.

(ii) Find real numbers p, q and r that satisfy both

$$p^2 + 4q^2 + 9r^2 = 729$$

and

$$8p + 8q + 3r = 243.$$

[STEP 2, 2006Q7]

An ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are positive. Show that the equation of the tangent at, the point $(a\cos\alpha\,, b\sin\alpha)$ is

$$y = -\frac{b\cot\alpha}{a}x + b\operatorname{cosec}\alpha.$$

The point A has coordinates $(-a, -b)$. The point E has coordinates $(-a, 0)$ and the point P has coordinates (a, kb), where $0 < k < 1$. The line through E parallel to AP meets the line $y = b$ at the point Q. Show that the line PQ is tangent to the above ellipse at the point given by $\tan\left(\frac{\alpha}{2}\right) = k$.

Determine by means of sketches, or otherwise, whether this result holds also for $k = 0$ and $k = 1$.

[STEP 2, 2006Q8]

Show that the line through the points with position vectors $\mathbf{x}$ and $\mathbf{y}$ has equation

$$\mathbf{r} = (1 - \alpha)\mathbf{x} + \alpha\mathbf{y},$$

where α is a scalar parameter.

The sides OA and CB of a trapezium $OABC$ are parallel, and $OA > CB$. The point E on OA is such that $OE : EA = 1 : 2$, and F is the midpoint of CB. The point D is the intersection of OC produced and AB produced; the point G is the intersection of OB and EF; and the point H is the intersection of DG produced and OA. Let $\mathbf{a}$ and $\mathbf{c}$ be the position vectors of the points A and C, respectively, with respect to the origin O.

(i) Show that B has position vector $\lambda\mathbf{a} + \mathbf{c}$ for some scalar parameter λ.

(ii) Find, in terms of $\mathbf{a}$, $\mathbf{c}$ and λ only, the position vectors of D, E, F, G and H. Determine the ratio $OH : HA$.

Section B: Mechanics

[STEP 2, 2006Q9]

A painter of weight kW uses a ladder to reach the guttering on the outside wall of a house. The wall is vertical and the ground is horizontal. The ladder is modelled as a uniform rod of weight W and length $6a$.

The ladder is not long enough, so the painter stands the ladder on a uniform table. The table consists of a square top of side $\frac{1}{2}a$ with a leg of length a at each corner. The weight of the table is $2W$. The foot of the ladder is at the centre of the table top and the ladder is inclined at an angle $\arctan 2$ to the horizontal. The edge of the table nearest the wall is parallel to the wall.

The coefficient of friction between the foot of the ladder and the table top is $\frac{1}{2}$. The contact between the ladder and the wall is sufficiently smooth for the effects of friction to be ignored.

(i) Show that, if the legs of the table are fixed to the ground, the ladder does not slip on the table however high the painter stands on the ladder.

(ii) It is given that $k = 9$ and that the coefficient of friction between each table leg and the ground is $\frac{1}{3}$. If the legs of the table are not fixed to the ground, so that the table can tilt or slip, determine which occurs first when the painter slowly climbs the ladder.

[**Note:** arctan 2 is another notation for $\tan^{-1} 2$.]

[STEP 2, 2006Q10]

Three particles, A, B, and C, of masses m, km and $3m$ respectively, are initially at rest lying in a straight line on a smooth horizontal surface. Then A is projected towards B at speed u. After the collision, B collides with C. The coefficient of restitution between A and B is $\frac{1}{2}$ and the coefficient of restitution between B and C is $\frac{1}{4}$.

(i) Find the range of values of k for which A and B collide for a second time.

(ii) Given that $k = 1$ and that B and C are initially a distance d apart, show that the time that elapses between the two collisions of A and B is $\frac{60d}{13u}$.

[STEP 2, 2006Q11]

A projectile of unit mass is fired in a northerly direction from a point O on a horizontal plain at speed u and an angle θ above the horizontal. It lands at a point A on the plain. In flight, the projectile experiences two forces: gravity, of magnitude g; and a horizontal force of constant magnitude f due to a wind blowing from North to South. Derive an expression, in terms of u, g, f and θ for the distance OA.

(i) Determine the acute angle α such that, for any acute angle θ with $\theta > \alpha$, the wind starts to blow the projectile back towards O before it lands at A.

(ii) An identical projectile, which experiences the same forces, is fired from O in a northerly direction at speed u and angle $45°$ above the horizontal and lands at a point B on the plain. Given that θ is chosen to maximise OA, show that

$$\frac{OB}{OA} = \frac{g - f}{\sqrt{g^2 + f^2} - f}.$$

Describe carefully the motion of the second projectile when $f = g$.

Section C: Probability and Statistics

[STEP 2, 2006Q12]

A cricket team has only three bowlers, Arthur, Betty and Cuba, each of whom bowls 30 balls in any match. Past performance reveals that, on average, Arthur takes one wicket for every 36 balls bowled, Betty takes one wicket for every 25 balls bowled, and Cuba takes one wicket for every 41 balls bowled.

(i) In one match, the team took exactly one wicket, but the name of the bowler was not recorded. Using a binomial model, find the probability that Arthur was the bowler.

(ii) Using a binomial model, show that the average number of wickets taken by the team in any given match is approximately 3.

(iii) By means of an approximation for the binomial distribution, the use of which you should justify briefly, show that the probability of the team taking at least five wickets in any given match is approximately $\frac{1}{5}$.

[You may use the approximation $e^3 = 20$.]

[STEP 2, 2006Q13]

I know that ice-creams come in n different sizes, but I don't know what the sizes are. I am offered one ice-cream of each size in succession, in random order. I am certainly going to choose one – the bigger the better – but I am not allowed more than one. My strategy is to reject the first ice-cream I am offered and choose the first one thereafter that is bigger than the first one I was offered; but if the first ice-cream offered is in fact the biggest one, then I will choose the last one offered, however small it is.

Let $P_n(k)$ be the probability that the ice-cream I choose is the kth biggest, where $k = 1$ is the biggest and $k = n$ is the smallest.

(i) Show that $P_4(1) = \frac{11}{24}$ and find $P_4(2)$, $P_4(3)$ and $P_4(4)$.

(ii) Find and expression for $P_n(1)$.

[STEP 2, 2006Q14]

Sketch the graph of $y = \frac{1}{x \ln x}$ for $x > 0$, $x \neq 1$. You may assume that $x \ln x \to 0$ as $x \to 0$.

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \dfrac{\lambda}{x \ln x} & \text{for } a \leq x \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

where a, b and λ are suitably chosen constants.

(i) In the case $a = \frac{1}{4}$ and $b = \frac{1}{2}$, find λ.

(ii) In the case $\lambda = 1$ and $a > 1$, show that $b = a^{\mathrm{e}}$.

(iii) In the case $\lambda = 1$ and $a = \mathrm{e}$, show that $\mathrm{P}\left(\mathrm{e}^{\frac{3}{2}} \leq X \leq \mathrm{e}^2\right) \approx \frac{31}{108}$.

(iv) In the case $\lambda = 1$ and $a = \mathrm{e}^{\frac{1}{2}}$, find $\mathrm{P}\left(\mathrm{e}^{\frac{3}{2}} \leq X \leq \mathrm{e}^2\right)$.

STEP 2 2007

STEP 2 2007
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 180 MINUTES

INSTRUCTIONS TO CANDIDATES

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Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

Section A	Pure Mathematics
Section B	Mechanics
Section C	Probability and Statistics

There are 14 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 2007Q1]

In this question, you are not required to justify the accuracy of the approximations.

(i) Write down the binomial expansion of $\left(1+\frac{k}{100}\right)^{\frac{1}{2}}$ in ascending powers of k, up to and including the k^3 term.

(a) Use the value $k = 8$ to find an approximation to five decimal places for $\sqrt{3}$.

(b) By choosing a suitable integer value of k, find an approximation to five decimal places for $\sqrt{6}$.

(ii) By considering the first two terms of the binomial expansion of $\left(1+\frac{k}{1000}\right)^{\frac{1}{3}}$, show that $\frac{3029}{2100}$ is an approximation to $\sqrt[3]{3}$.

[STEP 2, 2007Q2]

A curve has equation $y = 2x^3 - bx^2 + cx$. It has a maximum point at (p, m) and a minimum point at (q, n) where $p > 0$ and $n > 0$. Let R be the region enclosed by the curve, the line $x = p$ and the line $y = n$.

(i) Express b and c in terms of p and q.

(ii) Sketch the curve. Mark on your sketch the point of inflection and shade the region R. Describe the symmetry of the curve.

(iii) Show that $m - n = (q - p)^3$.

(iv) Show that the area of R is $\frac{1}{2}(q-p)^4$.

[STEP 2, 2007Q3]

By writing $x = a\tan\theta$, show that, for $a \neq 0$, $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + \text{constant}$.

(i) Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x$.

(a) Evaluate I.

(b) Use the substitution $t = \tan\frac{1}{2}x$ to show that $\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\,\mathrm{d}t = \frac{1}{2}I$.

(ii) Evaluate $\int_0^1 \frac{1-t^2}{1+14t^2+t^4}\,\mathrm{d}t$.

[STEP 2, 2007Q4]

Given that $\cos A$, $\cos B$ and β are non-zero, show that the equation

$$\alpha \sin(A-B) + \beta \cos(A+B) = \gamma \sin(A+B)$$

reduces to the form

$$(\tan A - m)(\tan B - n) = 0,$$

where m and n are independent of A and B, if and only if $\alpha^2 = \beta^2 + \gamma^2$.

Determine all values of x, in the range $0 \le x < 2\pi$, for which:

(i) $2\sin\left(x - \frac{1}{4}\pi\right) + \sqrt{3}\cos\left(x + \frac{1}{4}\pi\right) = \sin\left(x + \frac{1}{4}\pi\right)$.

(ii) $2\sin\left(x - \frac{1}{6}\pi\right) + \sqrt{3}\cos\left(x + \frac{1}{6}\pi\right) = \sin\left(x + \frac{1}{6}\pi\right)$.

(iii) $2\sin\left(x + \frac{1}{3}\pi\right) + \sqrt{3}\cos(3x) = \sin(3x)$.

[STEP 2, 2007Q5]

In this question, $f^2(x)$ denotes $f(f(x))$, $f^3(x)$ denotes $f\big(f(f(x))\big)$, and so on.

(i) The function f is defined, for $x \neq \pm\frac{1}{\sqrt{3}}$, by

$$f(x) = \frac{x+\sqrt{3}}{1-\sqrt{3}x}.$$

Find by direct calculation $f^2(x)$ and $f^3(x)$, and determine $f^{2007}(x)$.

(ii) Show that $f^n(x) = \tan\left(\theta + \frac{1}{3}n\pi\right)$, where $x = \tan\theta$ and n is any positive integer.

(iii) The function $g(t)$ is defined, for $|t| \le 1$ by $g(t) = \frac{\sqrt{3}}{2}t + \frac{1}{2}\sqrt{1-t^2}$. Find an expression for $g^n(t)$ for any positive integer n.

[STEP 2, 2007Q6]

(i) Differentiate $\ln\left(x + \sqrt{3+x^2}\right)$ and $x\sqrt{3+x^2}$ and simplify your answers.

Hence find $\int \sqrt{3+x^2}\,\mathrm{d}x$.

(ii) Find the two solutions of the differential equation

$$3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

that satisfy $y = 0$ when $x = 1$.

[STEP 2, 2007Q7]

A function $f(x)$ is said to be concave on some interval if $f''(x) < 0$ in that interval. Show that $\sin x$ is concave for $0 < x < \pi$ and that $\ln x$ is concave for $x > 0$.

Let $f(x)$ be concave on a given interval and let $x_1, x_2, \ldots, x_n$ lie in the interval. *Jensen's inequality* states that

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \leq f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_n$. You may use this result without proving it.

(i) Given that A, B and C are angles of a triangle, show that

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

(ii) By choosing a suitable function f, prove that

$$\sqrt[n]{t_1 t_2 \cdots t_n} \leq \frac{t_1 + t_2 + \cdots + t_n}{n}$$

for any positive integer n and for any positive numbers $t_1, t_2, \ldots, t_n$.

Hence:

(a) Show that $x^4 + y^4 + z^4 + 16 \geq 8xyz$, where x, y and z are any positive numbers.

(b) find the minimum value of $x^5 + y^5 + z^5 - 5xyz$, where x, y and z are any positive numbers.

[STEP 2, 2007Q8]

The points B and C have position vectors $\mathbf{b}$ and $\mathbf{c}$, respectively, relative to the origin A, and A, B and C are not collinear.

(i) The point X has position vector $s\mathbf{b} + t\mathbf{c}$. Describe the locus of X when $s + t = 1$.

(ii) The point P has position vector $\beta\mathbf{b} + \gamma\mathbf{c}$, where β and γ are non-zero, and $\beta + \gamma \neq 1$. The line AP cuts the line BC at D. Show that $BD : DC = \gamma : \beta$.

(iii) The line BP cuts the line CA at E, and the line CP cuts the line AB at F. Show that

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1.$$

Section B: Mechanics

[STEP 2, 2007Q9]

A solid right circular cone, of mass M, has semi-vertical angle α and smooth surfaces. It stands with its base on a smooth horizontal table. A particle of mass m is projected so that it strikes the curved surface of the cone at speed u. The coefficient of restitution between the particle and the cone is e. The impact has no rotational effect on the cone and the cone has no vertical velocity after the impact.

(i) The particle strikes the cone in the direction of the normal at the point of impact. Explain why the trajectory of the particle immediately after the impact is parallel to the normal to the surface of the cone. Find an expression, in terms of M, m, α, e and u, for the speed at which the cone slides along the table immediately after impact.

(ii) If instead the particle falls vertically onto the cone, show that the speed w at which the cone slides along the table immediately after impact is given by

$$w = \frac{mu(1+e)\sin\alpha\cos\alpha}{M + m\cos^2\alpha}.$$

Show also that the value of α for which w is greatest is given by

$$\cos\alpha = \sqrt{\frac{M}{2M+m}}.$$

[STEP 2, 2007Q10]

A solid figure is composed of a uniform solid cylinder of density ρ and a uniform solid hemisphere of density 3ρ. The cylinder has circular cross-section, with radius r, and height $3r$, and the hemisphere has radius r. The flat face of the hemisphere is joined to one end of the cylinder, so that their centres coincide.

The figure is held in equilibrium by a force P so that one point of its flat base is in contact with a rough horizontal plane and its base is inclined at an angle α to the horizontal. The force P is horizontal and acts through the highest point of the base. The coefficient of friction between the solid and the plane is μ. Show that

$$\mu \geq \left|\frac{9}{8} - \frac{1}{2}\cot\alpha\right|.$$

[STEP 2, 2007Q11]

In this question take the acceleration due to gravity to be 10 m s^{-2} *and neglect air resistance.*

The point O lies in a horizontal field. The point B lies 50 m east of O. A particle is projected from B at speed 25 m s^{-1} at an angle $\arctan\frac{1}{2}$ above the horizontal and in a direction that makes an angle 60° with OB; it passes to the north of O.

(i) Taking unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ in the directions east, north and vertically upwards, respectively, find the position vector of the particle relative to O at time t seconds after the particle was projected, and show that its distance from O is

$$5\left(t^2 - \sqrt{5}t + 10\right) \text{ m.}$$

When this distance is shortest, the particle is at point P. Find the position vector of P and its horizontal bearing from O.

(ii) Show that the particle reaches its maximum height at P.

(iii) When the particle is at P, a marksman fires a bullet from O directly at P. The initial speed of the bullet 350 m s^{-1}. Ignoring the effect of gravity on the bullet show that, when it passes through P, the distance between P and the particle is approximately 3 m.

Section C: Probability and Statistics

[STEP 2, 2007Q12]

I have two identical dice. When I throw either one of them, the probability of it showing a 6 is p and the probability of it not showing a 6 is q, where $p + q = 1$. As an experiment to determine p, I throw the dice simultaneously until at least one die shows a 6. If both dice show a six on this throw, I stop. If just one die shows a six, I throw the other die until it shows a 6 and then stop.

(i) Show that the probability that I stop after r throws is $pq^{r-1}(2 - q^{r-1} - q^r)$, and find an expression for the expected number of throws.

[**Note**: You may use the result $\sum_{r=0}^{\infty} rx^r = x(1-x)^{-2}$.]

(ii) In a large number of such experiments, the mean number of throws was m. Find an estimate for p in terms of m.

[STEP 2, 2007Q13]

Given that $0 < r < n$ and r is much smaller than n, show that $\frac{n-r}{n} \approx \mathrm{e}^{-\frac{r}{n}}$.

There are k guests at a party. Assuming that there are exactly 365 days in the year, and that the birthday of any guest is equally likely to fall on any of these days, show that the probability that there are at least two guests with the same birthday is approximately $1 - \mathrm{e}^{-\frac{k(k-1)}{730}}$.

Using the approximation $\frac{253}{365} \approx \ln 2$, find the smallest value of k such that the probability that at least two guests share the same birthday is at least $\frac{1}{2}$.

How many guests must there be at the party for the probability that at least one guest has the same birthday as the host to be at least $\frac{1}{2}$?

[STEP 2, 2007Q14]

The random variable X has a continuous probability density function $f(x)$ given by

$$f(x) = \begin{cases} 0 & \text{for } x \le 1 \\ \ln x & \text{for } 1 \le x \le k \\ \ln k & \text{for } k \le x \le 2k \\ a - bx & \text{for } 2k \le x \le 4k \\ 0 & \text{for } x \ge 4k \end{cases}$$

where k, a and b are constants.

(i) Sketch the graph of $y = f(x)$.

(ii) Determine a and b in terms of k and find the numerical values of k, a and b.

(iii) Find the median value of X.

STEP 2 2008

STEP 2 2008
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Section A: Pure Mathematics

[STEP 2, 2008Q1]

A sequence of points $(x_1, y_1), (x_2, y_2), \ldots$ in the cartesian plane is generated by first choosing (x_1, y_1) then applying the rule, for $n = 1, 2, \ldots$,

$$(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + a, 2x_n y_n + b + 2),$$

where a and b are given real constants.

(i) In the case $a = 1$ and $b = -1$, find the values of (x_1, y_1) for which the sequence is constant.

(ii) Given that $(x_1, y_1) = (-1, 1)$, find the values of a and b for which the sequence has period 2.

[STEP 2, 2008Q2]

Let a_n be the coefficient of x^n in the series expansion, in ascending powers of x, of

$$\frac{1+x}{(1-x)^2(1+x^2)},$$

where $|x| < 1$. Show, using partial fractions, that either $a_n = n + 1$ or $a_n = n + 2$ according to the value of n.

Hence find a decimal approximation, to nine significant figures, for the fraction $\frac{11\,000}{8181}$.

[You are not required to justify the accuracy of your approximation.]

[STEP 2, 2008Q3]

(i) Find the coordinates of the turning points of the curve $y = 27x^3 - 27x^2 + 4$. Sketch the curve and deduce that $x^2(1 - x) \le \frac{4}{27}$ for all $x \ge 0$.

Given that each of the numbers a, b and c lies between 0 and 1, prove by contradiction that at least one of the numbers $bc(1 - a)$, $ca(1 - b)$ and $ab(1 - c)$ is less than or equal to $\frac{4}{27}$.

(ii) Given that each of the numbers p and q lies between 0 and 1, prove that at least one of the numbers $p(1 - q)$ and $q(1 - p)$ is less than or equal to $\frac{1}{4}$.

[STEP 2, 2008Q4]

A curve is given by

$$x^2 + y^2 + 2axy = 1,$$

where a is a constant satisfying $0 < a < 1$. Show that the gradient of the curve at the point P with coordinates (x, y) is

$$-\frac{x + ay}{ax + y},$$

provided $ax + y \neq 0$. Show that θ, the acute angle between OP and the normal to the curve at P, satisfies

$$\tan\theta = a|y^2 - x^2|.$$

Show further that, if $\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0$ at P, then:

(i) $a(x^2 + y^2) + 2xy = 0.$

(ii) $(1 + a)(x^2 + y^2 + 2xy) = 1.$

(iii) $\tan\theta = \frac{a}{\sqrt{1-a^2}}.$

[STEP 2, 2008Q5]

Evaluate the integrals

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \sin^2 x}\,\mathrm{d}x$$

and

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin^2 x}\,\mathrm{d}x.$$

Show, using the binomial expansion, that $\left(1 + \sqrt{2}\right)^5 < 99$. Show also that $\sqrt{2} > 1.4$. Deduce that $2^{\sqrt{2}} > 1 + \sqrt{2}$. Use this result to determine which of the above integrals is greater.

[STEP 2, 2008Q6]

A curve has the equation $y = f(x)$, where

$$f(x) = \cos\left(2x + \frac{\pi}{3}\right) + \sin\left(\frac{3x}{2} - \frac{\pi}{4}\right).$$

(i) Find the period of $f(x)$.

(ii) Determine all values of x in the interval $-\pi \leq x \leq \pi$ for which $f(x) = 0$. Find a value of x in this interval at which the curve touches the x-axis without crossing it.

(iii) Find the value or values of x in the interval $0 \leq x \leq 2\pi$ for which $f(x) = 2$.

[STEP 2, 2008Q7]

(i) By writing $y = u(1+x^2)^{\frac{1}{2}}$, where u is a function of x, find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = xy + \frac{x}{1+x^2}$$

for which $y = 1$ when $x = 0$.

(ii) Find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y + \frac{x^2}{1+x^3}$$

for which $y = 1$ when $x = 0$.

(iii) Give, without proof, a conjecture for the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^{n-1}y + \frac{x^{n-1}}{1+x^n}$$

for which $y = 1$ when $x = 0$, where n is an integer greater than 1.

[STEP 2, 2008Q8]

The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, relative to the origin O. The points A, B and O are not collinear. The point P lies on AB between A and B such that

$$AP : PB = (1-\lambda) : \lambda.$$

Write down the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. Given that OP bisects $\angle AOB$, determine λ in terms of a and b, where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$.

The point Q also lies on AB between A and B, and is such that $AP = BQ$. Prove that

$$OQ^2 - OP^2 = (b-a)^2\,.$$

Section B: Mechanics

[STEP 2, 2008Q9]

In this question, use $g = 10 \text{ m s}^{-2}$.

In cricket, a fast bowler projects a ball at 40 m s^{-1} from a point h m above the ground, which is horizontal, and at an angle α above the horizontal. The trajectory is such that the ball will strike the stumps at ground level a horizontal distance of 20 m from the point of projection.

(i) Determine, in terms of h, the two possible values of $\tan\alpha$.

Explain which of these two values is the more appropriate one, and deduce that the ball hits the stumps after approximately half a second.

(ii) State the range of values of h for which the bowler projects the ball below the horizontal.

(iii) In the case $h = 2.5$, give an approximate value in degrees, correct to two significant figures, for α. You need not justify the accuracy of your approximation.

[You may use the small-angle approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$.]

[STEP 2, 2008Q10]

The lengths of the sides of a rectangular billiards table $ABCD$ are given by $AB = DC = a$ and $AD = BC = 2b$. There are small pockets at the midpoints M and N of the sides AD and BC, respectively. The sides of the table may be taken as smooth vertical walls.

A small ball is projected along the table from the corner A. It strikes the side BC at X, then the side DC at Y and then goes directly into the pocket at M. The angles BAX, CXY and DYM are α, β and γ respectively. On each stage of its path, the ball moves with constant speed in a straight line, the speeds being u, v and w respectively. The coefficient of restitution between the ball and the sides is e, where $e > 0$.

(i) Show that $\tan\alpha\tan\beta = e$ and find γ in terms of α.

(ii) Show that $\tan\alpha = \frac{(1+2e)b}{(1+e)a}$ and deduce that the shot is possible whatever the value of e.

(iii) Find an expression in terms of e for the fraction of the kinetic energy of the ball that is lost during the motion.

[STEP 2, 2008Q11]

A wedge of mass km has the shape (in cross-section) of a right-angled triangle. It stands on a smooth horizontal surface with one face vertical. The inclined face makes an angle θ with the horizontal surface. A particle P, of mass m, is placed on the inclined face and released from rest. The horizontal face of the wedge is smooth, but the inclined face is rough and the coefficient of friction between P and this face is μ.

(i) When P is released, it slides down the inclined plane at an acceleration a relative to the wedge. Show that the acceleration of the wedge is

$$\frac{a\cos\theta}{k+1}.$$

To a stationary observer, P appears to descend along a straight line inclined at an angle $45°$ to the horizontal. Show that

$$\tan\theta = \frac{k}{k+1}.$$

In the case $k = 3$, find an expression for a in terms of g and μ.

(ii) What happens when P is released if $\tan\theta \le \mu$?

Section C: Probability and Statistics

[STEP 2, 2008Q12]

In the High Court of Farnia, the outcome of each case is determined by three judges: the ass, the beaver and the centaur. Each judge decides its verdict independently. Being simple creatures, they make their decisions entirely at random. Past verdicts show that the ass gives a guilty verdict with probability p, the beaver gives a guilty verdict with probability $\frac{p}{3}$ and the centaur gives a guilty verdict with probability p^2.

Let X be the number of guilty verdicts given by the three judges in a case. Given that $\mathrm{E}(X) = \frac{4}{3}$, find the value of p.

The probability that a defendant brought to trial is guilty is t. The King pronounces that the defendant is guilty if at least two of the judges give a guilty verdict; otherwise, he pronounces the defendant not guilty. Find the value of t such that the probability that the King pronounces correctly is $\frac{1}{2}$.

[STEP 2, 2008Q13]

Bag P and bag Q each contain n counters, where $n \geq 2$. The counters are identical in shape and size, but coloured either black or white. First, k counters $(0 \leq k \leq n)$ are drawn at random from bag P and placed in bag Q. Then, k counters are drawn at random from bag Q and placed in bag P.

(i) If initially $n-1$ counters in bag P are white and one is black, and all n counters in bag Q are white, find the probability in terms of n and k that the black counter ends up in bag P.

Find the value or values of k for which this probability is maximised.

(ii) If initially $n-1$ counters in bag P are white and one is black, and $n-1$ counters in bag Q are white and one is black, find the probability in terms of n and k that the black counters end up in the same bag.

Find the value or values of k for which this probability is maximised.

STEP 2 2009

STEP 2 2009
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Section A: Pure Mathematics

[STEP 2, 2009Q1]

Two curves have equations $x^4 + y^4 = u$ and $xy = v$, where u and v are positive constants. State the equations of the lines of symmetry of each curve.

The curves intersect at the distinct points A, B, C and D (taken anticlockwise from A). The coordinates of A are (α, β), where $\alpha > \beta > 0$. Write down, in terms of α and β, the coordinates of B, C and D.

Show that the quadrilateral $ABCD$ is a rectangle and find its area in terms of u and v only. Verify that, for the case $u = 81$ and $v = 4$, the area is 14.

[STEP 2, 2009Q2]

The curve C has equation

$$y = a^{\sin(\pi e^x)},$$

where $a > 1$.

(i) Find the coordinates of the stationary points on C.

(ii) Use the approximations $e^t \approx 1 + t$ and $\sin t \approx t$ (both valid for small values of t) to show that

$$y \approx 1 - \pi x \ln a$$

for small values of x.

(iii) Sketch C.

(iv) By approximating C by means of straight lines joining consecutive stationary points, show that the area between C and the x-axis between the kth and $(k+1)$th maxima is approximately

$$\left(\frac{a^2+1}{2a}\right) \ln\left(1 + \left(k - \frac{3}{4}\right)^{-1}\right).$$

[STEP 2, 2009Q3]

Prove that

$$\tan\left(\frac{1}{4}\pi - \frac{1}{2}x\right) \equiv \sec x - \tan x. \qquad (*)$$

(i) Use $(*)$ to find the value of $\tan\frac{1}{8}\pi$. Hence show that

$$\tan\frac{11}{24}\pi = \frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{6}+1}.$$

(ii) Show that

$$\frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{6}+1} = 2+\sqrt{2}+\sqrt{3}+\sqrt{6}.$$

(iii) Use $(*)$ to show that

$$\tan\frac{1}{48}\pi = \sqrt{16+10\sqrt{2}+8\sqrt{3}+6\sqrt{6}} - 2 - \sqrt{2} - \sqrt{3} - \sqrt{6}.$$

[STEP 2, 2009Q4]

The polynomial $p(x)$ is of degree 9 and $p(x)-1$ is exactly divisible by $(x-1)^5$.

(i) Find the value of $p(1)$.

(ii) Show that $p'(x)$ is exactly divisible by $(x-1)^4$.

(iii) Given also that $p(x)+1$ is exactly divisible by $(x+1)^5$, find $p(x)$.

[STEP 2, 2009Q5]

Expand and simplify $\left(\sqrt{x-1}+1\right)^2$.

(i) Evaluate

$$\int_5^{10} \frac{\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}}\,dx.$$

(ii) Find the total area between the curve

$$y = \frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}}$$

and the x-axis between the points $x=\frac{5}{4}$ and $x=10$.

(iii) Evaluate

$$\int_{\frac{5}{4}}^{10} \frac{\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x+1}+2}}{\sqrt{x^2-1}}\,dx.$$

[STEP 2, 2009Q6]

The Fibonacci sequence $F_1, F_2, F_3, \ldots$ is defined by $F_1 = 1, F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \qquad (n \geq 2).$$

Write down the values of $F_3, F_4, \ldots, F_{10}$.

Let $S = \sum_{i=1}^{\infty} \frac{1}{F_i}$.

(i) Show that $\frac{1}{F_i} > \frac{1}{2F_{i-1}}$ for $i \geq 4$ and deduce that $S > 3$.

Show also that $S < 3\frac{2}{3}$.

(ii) Show further that $3.2 < S < 3.5$.

[STEP 2, 2009Q7]

Let $y = (x-a)^n \mathrm{e}^{bx}\sqrt{1+x^2}$, where n and a are constants and b is a non-zero constant. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-a)^{n-1}\mathrm{e}^{bx}q(x)}{\sqrt{1+x^2}},$$

where $q(x)$ is a cubic polynomial.

Using this result, determine:

(i) $\int \frac{(x-4)^{14}\mathrm{e}^{4x}(4x^3-1)}{\sqrt{1+x^2}}\,\mathrm{d}x$.

(ii) $\int \frac{(x-1)^{21}\mathrm{e}^{12x}(12x^4-x^2-11)}{\sqrt{1+x^2}}\,\mathrm{d}x$.

(iii) $\int \frac{(x-2)^{6}\mathrm{e}^{4x}(4x^4+x^3-2)}{\sqrt{1+x^2}}\,\mathrm{d}x$.

[STEP 2, 2009Q8]

The non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, given by

$$\mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b} \qquad \text{and} \qquad \mathbf{q} = \mu\mathbf{a} + (1-\mu)\mathbf{c}$$

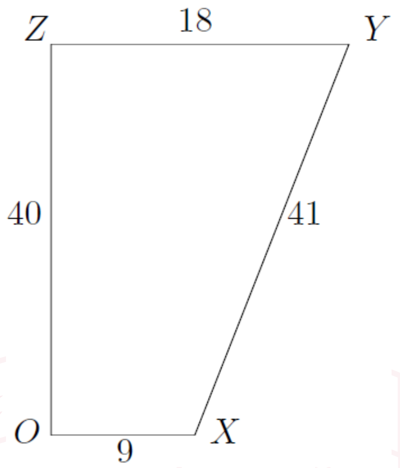
where $0 < \lambda < 1$ and $\mu > 1$. Draw a diagram showing A, B, C, P and Q.

Given that $CQ \times BP = AB \times AC$, find μ in terms of λ, and show that, for all values of λ, the line PQ passes through the fixed point D, with position vector $\mathbf{d}$ given by $\mathbf{d} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$. What can be said about the quadrilateral $ABDC$?

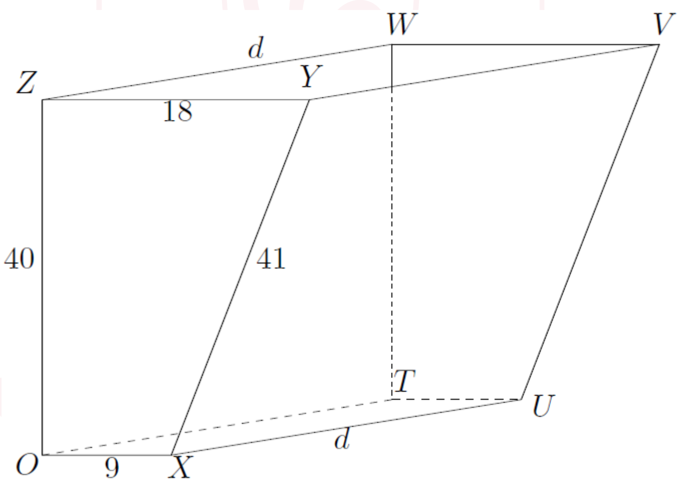
Section B: Mechanics

[STEP 2, 2009Q9]

(i) A uniform lamina $OXYZ$ is in the shape of the trapezium shown in the diagram. It is right-angled at O and Z, and OX is parallel to YZ. The lengths of the sides are given by $OX = 9$ cm, $XY = 41$ cm, $YZ = 18$ cm and $ZO = 40$ cm. Show that its centre of mass is a distance 7 cm from the edge OZ.



(ii) The diagram shows a tank with no lid made of thin sheet metal. The base $OXUT$, the back $OTWZ$ and the front $XUVY$ are rectangular, and each end is a trapezium as in part (i). The width of the tank is d cm.



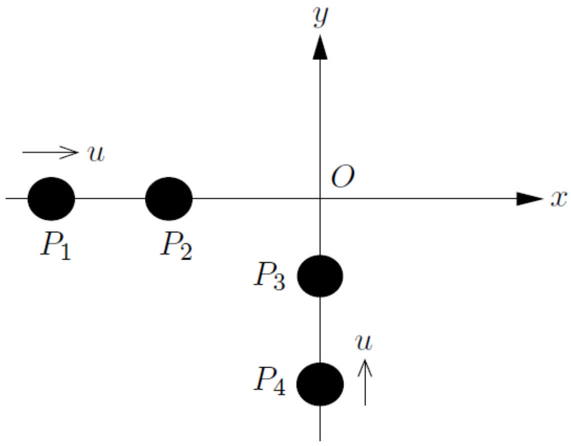
Show that the centre of mass of the tank, when empty, is a distance

$$\frac{3(140 + 11d)}{5(12 + d)} \text{ cm}$$

from the back of the tank.

The tank is then filled with a liquid. The mass per unit volume of this liquid is k times the mass per unit area of the sheet metal. In the case $d = 20$, find an expression for the distance of the centre of mass of the filled tank from the back of the tank.

[STEP 2, 2009Q10]



Four particles P_1, P_2, P_3 and P_4, of masses m_1, m_2, m_3 and m_4, respectively, are arranged on smooth horizontal axes as shown in the diagram.

Initially, P_2 and P_3 are stationary, and both P_1 and P_4 are moving towards O with speed u. Then P_1 and P_2 collide, at the same moment as P_4 and P_3 collide. Subsequently, P_2 and P_3 collide at O, as do P_1 and P_4 some time later. The coefficient of restitution between each pair of particles is e, and $e > 0$.

Show that initially P_2 and P_3 are equidistant from O.

[STEP 2, 2009Q11]

A train consists of an engine and n trucks. It is travelling along a straight horizontal section of track. The mass of the engine and of each truck is M. The resistance to motion of the engine and of each truck is R, which is constant. The maximum power at which the engine can work is P.

Obtain an expression for the acceleration of the train when its speed is v and the engine is working at maximum power.

The train starts from rest with the engine working at maximum power. Obtain an expression for the time T taken to reach a given speed V, and show that this speed is only achievable if

$$P > (n+1)RV.$$

(i) In the case when $\frac{(n+1)RV}{P}$ is small, use the approximation $\ln(1-x) \approx -x - \frac{1}{2}x^2$ (valid for small x) to obtain the approximation

$$PT \approx \frac{1}{2}(n+1)MV^2$$

and interpret this result.

(ii) In the general case, the distance moved from rest in time T is X. *Write down*, with explanation, an equation relating P, T, X, M, V, R and n and hence show that

$$X = \frac{2PT - (n+1)MV^2}{2(n+1)R}.$$

Section C: Probability and Statistics

[STEP 2, 2009Q12]

A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ k\mathrm{e}^{-2x^2} & \text{for } 0 \leq x < \infty, \end{cases}$$

where k is a constant.

(i) Sketch the graph of $f(x)$.

(ii) Find the value of k.

(iii) Determine $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(iv) Use statistical tables to find, to three significant figures, the median value of X.

[STEP 2, 2009Q13]

Satellites are launched using two different types of rocket: the Andover and the Basingstoke. The Andover has four engines and the Basingstoke has six. Each engine has a probability p of failing during any given launch. After the launch, the rockets are retrieved and repaired by replacing some or all of the engines. The cost of replacing each engine is K.

For the Andover, if more than one engine fails, all four engines are replaced. Otherwise, only the failed engine (if there is one) is replaced. Show that the expected repair cost for a single launch using the Andover is

$$4Kp(1 + q + q^2 - 2q^3) \qquad (q = 1 - p) \qquad (*)$$

For the Basingstoke, if more than two engines fail, all six engines are replaced. Otherwise only the failed engines (if there are any) are replaced. Find, in a form similar to $(*)$, the expected repair cost for a single launch using the Basingstoke.

Find the values of p for which the expected repair cost for the Andover is $\frac{2}{3}$ of the expected repair cost for the Basingstoke.

STEP 2 2010

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Section A: Pure Mathematics

[STEP 2, 2010Q1]

Let P be a given point on a given curve C. The *osculating circle* to C at P is defined to be the circle that satisfies the following two conditions at P: it touches C; and the rate of change of its gradient is equal to the rate of change of the gradient of C.

Find the centre and radius of the osculating circle to the curve $y = 1 - x + \tan x$ at the point on the curve with x-coordinate $\frac{1}{4}\pi$.

[STEP 2, 2010Q2]

Prove that

$$\cos 3x = 4\cos^3 x - 3\cos x.$$

Find and prove a similar result for $\sin 3x$ in terms of $\sin x$.

(i) Let

$$I(\alpha) = \int_0^\alpha (7\sin x - 8\sin^3 x)\,dx.$$

Show that

$$I(\alpha) = -\frac{8}{3}c^3 + c + \frac{5}{3},$$

where $c = \cos\alpha$. Write down one value of c for which $I(\alpha) = 0$.

(ii) Useless Eustace believes that

$$\int \sin^n x\,dx = \frac{\sin^{n+1} x}{n+1}$$

for $n = 1, 2, 3, \ldots$. Show that Eustace would obtain the correct value of $I(\beta)$, where $\cos\beta = -\frac{1}{6}$.

Find all values of α for which he would obtain the correct value of $I(\alpha)$.

[STEP 2, 2010Q3]

The first four terms of a sequence are given by $F_0 = 0$, $F_1 = 1$, $F_2 = 1$ and $F_3 = 2$. The general term is given by

$$F_n = a\lambda^n + b\mu^n, \qquad (*)$$

where a, b, λ and μ are independent of n, and a is positive.

(i) Show that $\lambda^2 + \lambda\mu + \mu^2 = 2$, and find the values of λ, μ, a and b.

(ii) Use $(*)$ to evaluate F_6.

(iii) Evaluate

$$\sum_{n=0}^{\infty} \frac{F_n}{2^{n+1}}.$$

[STEP 2, 2010Q4]

(i) Let

$$I = \int_0^a \frac{f(x)}{f(x) + f(a-x)}\,\mathrm{d}x.$$

Use a substitution to show that

$$I = \int_0^a \frac{f(a-x)}{f(x) + f(a-x)}\,\mathrm{d}x$$

and hence evaluate I in terms of a.

Use this result to evaluate the integrals

$$\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)}\,\mathrm{d}x$$

and

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin\left(x + \frac{\pi}{4}\right)}\,\mathrm{d}x.$$

(ii) Evaluate

$$\int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x + \sin\frac{1}{x}\right)}\,\mathrm{d}x.$$

[STEP 2, 2010Q5]

The points A and B have position vectors $\mathbf{\hat{i}} + \mathbf{\hat{j}} + \mathbf{\hat{k}}$ and $5\mathbf{\hat{i}} - \mathbf{\hat{j}} - \mathbf{\hat{k}}$, respectively, relative to the origin O. Find $\cos 2\alpha$, where 2α is the angle $\angle AOB$.

(i) The line L_1 has equation $\mathbf{r} = \lambda(m\mathbf{\hat{i}} + n\mathbf{\hat{j}} + p\mathbf{\hat{k}})$. Given that L_1 is inclined equally to OA and to OB, determine a relationship between m, n and p. Find also values of m, n and p for which L_1 is the angle bisector of $\angle AOB$.

(ii) The line L_2 has equation $\mathbf{r} = \mu(u\mathbf{\hat{i}} + v\mathbf{\hat{j}} + w\mathbf{\hat{k}})$. Given that L_2 is inclined at an angle α to OA, where $2\alpha = \angle AOB$, determine a relationship between u, v and w.

Hence describe the surface with Cartesian equation $x^2 + y^2 + z^2 = 2(yz + zx + xy)$.

[STEP 2, 2010Q6]

Each edge of the tetrahedron $ABCD$ has unit length. The face ABC is horizontal, and P is the point in ABC that is vertically below D.

(i) Find the length of PD.

(ii) Show that the cosine of the angle between adjacent faces of the tetrahedron is $\frac{1}{3}$.

(iii) Find the radius of the largest sphere that can fit inside the tetrahedron.

[STEP 2, 2010Q7]

(i) By considering the positions of its turning points, show that the curve with equation

$$y = x^3 - 3qx - q(1+q),$$

where $q > 0$ and $q \neq 1$, crosses the x-axis once only.

(ii) Given that x satisfies the cubic equation

$$x^3 - 3qx - q(1+q) = 0,$$

and that

$$x = u + \frac{q}{u},$$

obtain a quadratic equation satisfied by u^3. Hence find the real root of the cubic equation in the case $q > 0$, $q \neq 1$.

(iii) The quadratic equation

$$t^2 - pt + q = 0$$

has roots α and β. Show that

$$\alpha^3 + \beta^3 = p^3 - 3qp.$$

It is given that one of these roots is the square of the other. By considering the expression $(\alpha^2 - \beta)(\beta^2 - \alpha)$, find a relationship between p and q. Given further that $q > 0$, $q \neq 1$ and p is real, determine the value of p in terms of q.

[STEP 2, 2010Q8]

The curves C_1 and C_2 are defined by

$$y = \mathrm{e}^{-x} \quad (x > 0)$$

and

$$y = \mathrm{e}^{-x}\sin x \quad (x > 0),$$

respectively. Sketch roughly C_1 and C_2 on the same diagram.

Let x_n denote the x-coordinate of the nth point of contact between the two curves, where $0 < x_1 < x_2 < \cdots$, and let A_n denote the area of the region enclosed by the two curves between x_n and x_{n+1}. Show that

$$A_n = \frac{1}{2}(\mathrm{e}^{2\pi} - 1)e^{-\frac{(4n+1)\pi}{2}}$$

and hence find $\sum_{n=1}^{\infty} A_n$.

Section B: Mechanics

[STEP 2, 2010Q9]

Two points A and B lie on horizontal ground. A particle P_1 is projected from A towards B at an acute angle of elevation α and simultaneously a particle P_2 is projected from B towards A at an acute angle of elevation β. Given that the two particles collide in the air a horizontal distance b from B, and that the collision occurs after P_1 has attained its maximum height h, show that

$$2h\cot\beta < b < 4h\cot\beta$$

and

$$2h\cot\alpha < a < 4h\cot\alpha,$$

where a is the horizontal distance from A to the point of collision.

[STEP 2, 2010Q10]

(i) In an experiment, a particle A of mass m is at rest on a smooth horizontal table. A particle B of mass bm, where $b > 1$, is projected along the table directly towards A with speed u. The collision is perfectly elastic.

Find an expression for the speed of A after the collision in terms of b and u, and show that, irrespective of the relative masses of the particles, A cannot be made to move at twice the initial speed of B.

(ii) In a second experiment, a particle B_1 is projected along the table directly towards A with speed u. This time, particles $B_2, B_3, \dots, B_n$ are at rest in order on the line between B_1 and A. The mass of B_i $(i = 1, 2, \dots, n)$ is $\lambda^{n+1-i}m$, where $\lambda > 1$. All collisions are perfectly elastic. Show that, by choosing n sufficiently large, there is no upper limit on the speed at which A can be made to move.

In the case $\lambda = 4$, determine the least value of n for which A moves at more than $20u$. You may use the approximation $\log_{10} 2 \approx 0.30103$.

[STEP 2, 2010Q11]

A uniform rod AB of length $4L$ and weight W is inclined at an angle θ to the horizontal. Its lower end A rests on a fixed support and the rod is held in equilibrium by a string attached to the rod at a point C which is $3L$ from A. The reaction of the support on the rod acts in a direction α to AC and the string is inclined at an angle β to CA. Show that

$$\cot\alpha = 3\tan\theta + 2\cot\beta.$$

Given that $\theta = 30°$ and $\beta = 45°$, show that $\alpha = 15°$.

Section C: Probability and Statistics

[STEP 2, 2010Q12]

The continuous random variable X has probability density function $f(x)$, where

$$f(x) = \begin{cases} a & \text{for } 0 \le x < k \\ b & \text{for } k \le x \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$\mathrm{E}(X) = \frac{1 - 2b + ab}{2(a-b)}.$$

(ii) Show that the median, M, of X is given by $M = \frac{1}{2a}$ if $a + b \ge 2ab$ and obtain an expression for the median if $a + b \le 2ab$.

(iii) Show that $M < \mathrm{E}(X)$.

[STEP 2, 2010Q13]

Rosalind wants to join the Stepney Chess Club. In order to be accepted, she must play a challenge match consisting of several games against Pardeep (the Club champion) and Quentin (the Club secretary), in which she must win at least one game against each of Pardeep and Quentin. From past experience, she knows that the probability of her winning a single game against Pardeep is p and the probability of her winning a single game against Quentin is q, where $0 < p < q < 1$.

(i) The challenge match consists of three games. Before the match begins, Rosalind must choose either to play Pardeep twice and Quentin once or to play Quentin twice and Pardeep once. Show that she should choose to play Pardeep twice.

(ii) In order to ease the entry requirements, it is decided instead that the challenge match will consist of four games. Now, before the match begins, Rosalind must choose whether to play Pardeep three times and Quentin once (strategy 1), or to play Pardeep twice and Quentin twice (strategy 2) or to play Pardeep once and Quentin three times (strategy 3). Show that, if $q - p > \frac{1}{2}$, Rosalind should choose strategy 1.

If $q - p < \frac{1}{2}$ give examples of values of p and q to show that strategy 2 can be better or worse than strategy 1.

STEP 2 2011

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Section A: Pure Mathematics

[STEP 2, 2011Q1]

(i) Sketch the curve $y = \sqrt{1-x} + \sqrt{3+x}$.

Use your sketch to show that only one real value of x satisfies

$$\sqrt{1-x} + \sqrt{3+x} = x + 1,$$

and give this value.

(ii) Determine graphically the number of real values of x that satisfy

$$2\sqrt{1-x} = \sqrt{3+x} + \sqrt{3-x}.$$

Solve this equation.

[STEP 2, 2011Q2]

Write down the cubes of the integers $1, 2, \ldots, 10$.

The positive integers x, y and z, where $x < y$, satisfy

$$x^3 + y^3 = kz^3, \qquad (*)$$

where k is a given positive integer.

(i) In the case $x + y = k$, show that

$$z^3 = k^2 - 3kx + 3x^2.$$

Deduce that $\frac{(4z^3 - k^2)}{3}$ is a perfect square and that $\frac{1}{4}k^2 \le z^3 < k^2$.

Use these results to find a solution of $(*)$ when $k = 20$.

(ii) By considering the case $x + y = z^2$, find two solutions of $(*)$ when $k = 19$.

[STEP 2, 2011Q3]

In this question, you may assume without proof that any function f for which $f'(x) \geq 0$ is *increasing*; that is, $f(x_2) \geq f(x_1)$ if $x_2 \geq x_1$.

(i) (a) Let $f(x) = \sin x - x\cos x$. Show that $f(x)$ is increasing for $0 \leq x \leq \frac{1}{2}\pi$ and deduce that $f(x) \geq 0$ for $0 \leq x \leq \frac{1}{2}\pi$.

(b) Given that $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) \geq 1$ for $0 \leq x < 1$, show that

$$\arcsin x \geq x \qquad (0 \leq x < 1).$$

(c) Let $g(x) = x\ \operatorname{cosec} x$ for $0 < x < \frac{1}{2}\pi$. Show that g is increasing and deduce that

$$(\arcsin x)x^{-1} \geq x\ \operatorname{cosec} x \qquad (0 < x < 1).$$

(ii) Given that $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) \leq 1$ for $x \geq 0$, show by considering the function $x^{-1}\tan x$ that

$$(\tan x)(\arctan x) \geq x^2 \qquad \left(0 < x < \frac{1}{2}\pi\right).$$

[STEP 2, 2011Q4]

(i) Find all the values of θ, in the range $0° < \theta < 180°$, for which $\cos\theta = \sin 4\theta$. Hence show that

$$\sin 18° = \frac{1}{4}(\sqrt{5} - 1).$$

(ii) Given that

$$4\sin^2 x + 1 = 4\sin^2 2x,$$

find all possible values of $\sin x$, giving your answers in the form $p + q\sqrt{5}$ where p and q are rational numbers.

(iii) Hence find two values of α with $0° < \alpha < 90°$ for which

$$\sin^2 3\alpha + \sin^2 5\alpha = \sin^2 6\alpha.$$

[STEP 2, 2011Q5]

The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O, and O, A and B are non-collinear. The point C, with position vector $\mathbf{c}$, is the reflection of B in the line through O and A. Show that $\mathbf{c}$ can be written in the form

$$\mathbf{c} = \lambda\mathbf{a} - \mathbf{b}$$

where $\lambda = \frac{2\mathbf{a}.\mathbf{b}}{\mathbf{a}.\mathbf{a}}$.

The point D, with position vector $\mathbf{d}$, is the reflection of C in the line through O and B. Show that $\mathbf{d}$ can be written in the form

$$\mathbf{d} = \mu\mathbf{b} - \lambda\mathbf{a}$$

for some scalar μ to be determined.

Given that A, B and D are collinear, find the relationship between λ and μ. In the case $\lambda = -\frac{1}{2}$, determine the cosine of $\angle AOB$ and describe the relative positions of A, B and D.

[STEP 2, 2011Q6]

For any given function f, let

$$I = \int [f'(x)]^2 [f(x)]^n \, dx, \qquad (*)$$

where n is a positive integer. Show that, if $f(x)$ satisfies $f''(x) = kf(x)f'(x)$ for some constant k, then $(*)$ can be integrated to obtain an expression for I in terms of $f(x)$, $f'(x)$, k and n.

(i) Verify your result in the case $f(x) = \tan x$. Hence find

$$\int \frac{\sin^4 x}{\cos^8 x} \, dx.$$

(ii) Find

$$\int \sec^2 x \, (\sec x + \tan x)^6 \, dx.$$

[STEP 2, 2011Q7]

The two sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ have general terms

$$a_n = \lambda^n + \mu^n \quad \text{and} \quad b_n = \lambda^n - \mu^n,$$

respectively, where $\lambda = 1 + \sqrt{2}$ and $\mu = 1 - \sqrt{2}$.

(i) Show that $\sum_{r=0}^{n} b_r = -\sqrt{2} + \frac{1}{\sqrt{2}} a_{n+1}$, and give a corresponding result for $\sum_{r=0}^{n} a_r$.

(ii) Show that, if n is odd,

$$\sum_{m=0}^{2n} \left(\sum_{r=0}^{m} a_r \right) = \frac{1}{2} b_{n+1}^2 ,$$

and give a corresponding result when n is even.

(iii) Show that, if n is even,

$$\left(\sum_{r=0}^{n} a_r \right)^2 - \sum_{r=0}^{n} a_{2r+1} = 2,$$

and give a corresponding result when n is odd.

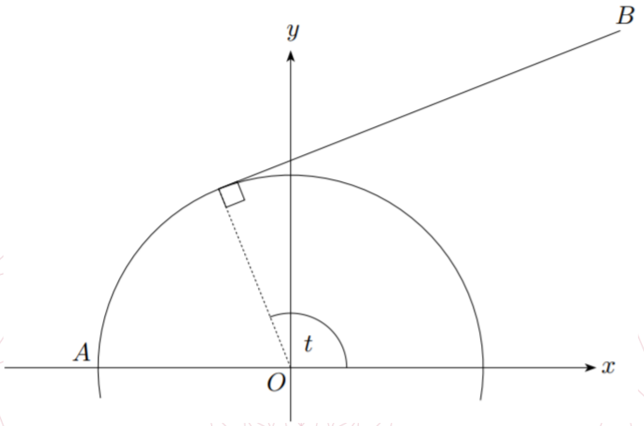
[STEP 2, 2011Q8]

The end A of an inextensible string AB of length π is attached to a point on the circumference of a fixed circle of unit radius and centre O. Initially the string is straight and tangent to the circle. The string is then wrapped round the circle until the end B comes into contact with the circle. The string remains taut during the motion, so that a section of the string is in contact with the circumference and the remaining section is straight.

Taking O to be the origin of cartesian coordinates with A at $(-1, 0)$ and B initially at $(-1, \pi)$, show that the curve described by B is given parametrically by

$$x = \cos t + t \sin t\,, \qquad y = \sin t - t \cos t,$$

where t is the angle shown in the diagram.



Find the value, t_0, of t for which x takes its maximum value on the curve, and sketch the curve.

Use the area integral $\int y \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$ to find the area between the curve and the x axis for $\pi \geq t \geq t_0$.

Find the area swept out by the string (that is, the area between the curve described by B and the semicircle shown in the diagram).

Section B: Mechanics

[STEP 2, 2011Q9]

Two particles, A of mass $2m$ and B of mass m, are moving towards each other in a straight line on a smooth horizontal plane, with speeds $2u$ and u respectively. They collide directly. Given that the coefficient of restitution between the particles is e, where $0 < e \leq 1$, determine the speeds of the particles after the collision.

After the collision, B collides directly with a smooth vertical wall, rebounding and then colliding directly with A for a second time. The coefficient of restitution between B and the wall is f, where $0 < f \leq 1$. Show that the velocity of B after its second collision with A is

$$\frac{2}{3}(1-e^2)u - \frac{1}{3}(1-4e^2)fu$$

towards the wall and that B moves towards (not away from) the wall for all values of e and f.

[STEP 2, 2011Q10]

A particle is projected from a point on a horizontal plane, at speed u and at an angle θ above the horizontal. Let H be the maximum height of the particle above the plane. Derive an expression for H in terms of u, g and θ.

A particle P is projected from a point O on a smooth horizontal plane, at speed u and at an angle θ above the horizontal. At the same instant, a second particle R is projected horizontally from O in such a way that R is vertically below P in the ensuing motion. A light inextensible string of length $\frac{1}{2}H$ connects P and R. Show that the time that elapses before the string becomes taut is

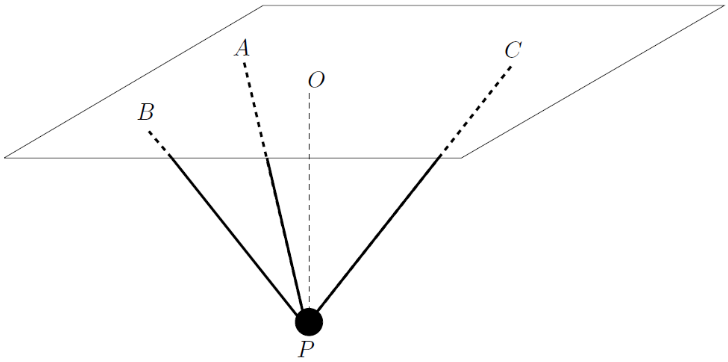
$$(\sqrt{2}-1)\sqrt{\frac{H}{g}}.$$

When the string becomes taut, R leaves the plane, the string remaining taut. Given that P and R have equal masses, determine the total horizontal distance, D, travelled by R from the moment its motion begins to the moment it lands on the plane again, giving your answer in terms of u, g and θ.

Given that $D = H$, find the value of $\tan\theta$.

[STEP 2, 2011Q11]

Three non-collinear points A, B and C lie in a horizontal ceiling. A particle P of weight W is suspended from this ceiling by means of three light inextensible strings AP, BP and CP, as shown in the diagram. The point O lies vertically above P in the ceiling.



The angles AOB and AOC are $90° + \theta$ and $90° + \varphi$, respectively, where θ and φ are acute angles such that $\tan\theta = \sqrt{2}$ and $\tan\varphi = \frac{1}{4}\sqrt{2}$.

The strings AP, BP and CP make angles $30°$, $90° - \theta$ and $60°$, respectively, with the vertical, and the tensions in these strings have magnitudes T, U and V respectively.

(i) Show that the unit vector in the direction PB can be written in the form

$$-\frac{1}{3}\hat{\mathbf{i}} - \frac{\sqrt{2}}{3}\hat{\mathbf{j}} + \frac{\sqrt{2}}{\sqrt{3}}\hat{\mathbf{k}},$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the usual mutually perpendicular unit vectors with $\hat{\mathbf{j}}$ parallel to OA and $\hat{\mathbf{k}}$ vertically upwards.

(ii) Find expressions in vector form for the forces acting on P.

(iii) Show that $U = \sqrt{6}V$ and find T, U and V in terms of W.

Section C: Probability and Statistics

[STEP 2, 2011Q12]

Xavier and Younis are playing a match. The match consists of a series of games and each game consists of three points.

Xavier has probability p and Younis has probability $1 - p$ of winning the first point of any game. In the second and third points of each game, the player who won the previous point has probability p and the player who lost the previous point has probability $1 - p$ of winning the point. If a player wins two consecutive points in a single game, the match ends and that player has won; otherwise the match continues with another game.

(i) Let w be the probability that Younis wins the match. Show that, for $p \neq 0$,

$$w = \frac{1 - p^2}{2 - p}.$$

Show that $w > \frac{1}{2}$ if $p < \frac{1}{2}$, and $w < \frac{1}{2}$ if $p > \frac{1}{2}$. Does w increase whenever p decreases?

(ii) If Xavier wins the match, Younis gives him £1; if Younis wins the match, Xavier gives him £k. Find the value of k for which the game is fair in the case when $p = \frac{2}{3}$.

(iii) What happens when $p = 0$?

[STEP 2, 2011Q13]

What property of a distribution is measured by its *skewness*?

(i) One measure of skewness, γ, is given by

$$\gamma = \frac{\mathrm{E}((X-\mu)^3)}{\sigma^3},$$

where μ and σ^2 are the mean and variance of the random variable X. Show that

$$\gamma = \frac{\mathrm{E}(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}.$$

The continuous random variable X has probability density function f where

$$f(x) = \begin{cases} 2x, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that for this distribution $\gamma = -\frac{2\sqrt{2}}{5}$.

(ii) The *decile skewness*, D, of a distribution is defined by

$$D = \frac{F^{-1}\left(\frac{9}{10}\right) - 2F^{-1}\left(\frac{1}{2}\right) + F^{-1}\left(\frac{1}{10}\right)}{F^{-1}\left(\frac{9}{10}\right) - F^{-1}\left(\frac{1}{10}\right)},$$

where F^{-1} is the inverse of the cumulative distribution function. Show that, for the above distribution, $D = 2 - \sqrt{5}$.

The *Pearson skewness*, P, of a distribution is defined by

$$P = \frac{3(\mu - M)}{\sigma},$$

where M is the median. Find P for the above distribution and show that $D > P > \gamma$.

STEP 2 2012

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Section A: Pure Mathematics

[STEP 2, 2012Q1]

Write down the general term in the expansion in powers of x of $(1-x^6)^{-2}$.

(i) Find the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}.$$

Obtain also, and simplify, formulae for the coefficient of x^n in the different cases that arise.

(ii) Show that the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}(1-x)^{-1}$$

is 55, and find the coefficients of x^{25} and x^{66}.

[STEP 2, 2012Q2]

If $p(x)$ and $q(x)$ are polynomials of degree m and n, respectively, what is the degree of $p\big(q(x)\big)$?

(i) The polynomial $p(x)$ satisfies

$$p\Big(p\big(p(x)\big)\Big) - 3p(x) = -2x$$

for all x. Explain carefully why $p(x)$ must be of degree 1, and find all polynomials that satisfy this equation.

(ii) Find all polynomials that satisfy

$$2p\big(p(x)\big) + 3[p(x)]^2 - 4p(x) = x^4$$

for all x.

[STEP 2, 2012Q3]

Show that, for any function f (for which the integrals exist),

$$\int_0^\infty f\left(x+\sqrt{1+x^2}\right)\mathrm{d}x = \frac{1}{2}\int_1^\infty \left(1+\frac{1}{t^2}\right) f(t)\,\mathrm{d}t.$$

Hence evaluate

$$\int_0^\infty \frac{1}{2x^2+1+2x\sqrt{x^2+1}}\,\mathrm{d}x,$$

and, using the substitution $x=\tan\theta$,

$$\int_0^{\frac{1}{2}\pi} \frac{1}{(1+\sin\theta)^3}\,\mathrm{d}\theta.$$

[STEP 2, 2012Q4]

In this question, you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

is valid for $|x| < 1$.

(i) Let n be an integer greater than 1. Show that, for any positive integer k,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}.$$

Hence show that $\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$. Deduce that

$$\left(1+\frac{1}{n}\right)^n < \mathrm{e}.$$

(ii) Show, using an expansion in powers of $\frac{1}{y}$, that $\ln\left(\frac{2y+1}{2y-1}\right) > \frac{1}{y}$ for $y > \frac{1}{2}$.

Deduce that, for any positive integer n,

$$\mathrm{e} < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}.$$

(iii) Use parts (i) and (ii) to show that as $n \to \infty$

$$\left(1+\frac{1}{n}\right)^n \to \mathrm{e}.$$

[STEP 2, 2012Q5]

(i) Sketch the curve $y = f(x)$, where

$$f(x) = \frac{1}{(x-a)^2 - 1} \qquad (x \neq a \pm 1),$$

and a is a constant.

(ii) The function $g(x)$ is defined by

$$g(x) = \frac{1}{((x-a)^2 - 1)((x-b)^2 - 1)} \qquad (x \neq a \pm 1, \qquad x \neq b \pm 1),$$

where a and b are constants, and $b > a$. Sketch the curves $y = g(x)$ in the two cases $b > a + 2$ and $b = a + 2$, finding the values of x at the stationary points.

[STEP 2, 2012Q6]

A cyclic quadrilateral $ABCD$ has sides AB, BC, CD and DA of lengths a, b, c and d, respectively. The area of the quadrilateral is Q, and angle DAB is θ.

Find an expression for $\cos\theta$ in terms of a, b, c and d, and an expression for $\sin\theta$ in terms of a, b, c, d and Q. Hence show that

$$16Q^2 = 4(ad+bc)^2 - (a^2+d^2-b^2-c^2)^2,$$

and deduce that

$$Q^2 = (s-a)(s-b)(s-c)(s-d),$$

where $s = \frac{1}{2}(a+b+c+d)$.

Deduce a formula for the area of a triangle with sides of length a, b and c.

[STEP 2, 2012Q7]

Three distinct points, X_1, X_2 and X_3, with position vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ respectively, lie on a circle of radius 1 with its centre at the origin O. The point G has position vector $\frac{1}{3}(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3)$. The line through X_1 and G meets the circle again at the point Y_1 and the points Y_2 and Y_3 are defined correspondingly.

Given that $\overrightarrow{GY_1} = -\lambda_1\overrightarrow{GX_1}$, where λ_1 is a positive scalar, show that

$$\overrightarrow{OY_1} = \frac{1}{3}\big((1-2\lambda_1)\mathbf{x}_1 + (1+\lambda_1)(\mathbf{x}_2+\mathbf{x}_3)\big)$$

and hence that

$$\lambda_1 = \frac{3-\alpha-\beta-\gamma}{3+\alpha-2\beta-2\gamma},$$

where $\alpha = \mathbf{x}_2.\mathbf{x}_3$, $\beta = \mathbf{x}_3.\mathbf{x}_1$ and $\gamma = \mathbf{x}_1.\mathbf{x}_2$.

Deduce that $\frac{GX_1}{GY_1} + \frac{GX_2}{GY_2} + \frac{GX_3}{GY_3} = 3$.

[STEP 2, 2012Q8]

The positive numbers α, β and q satisfy $\beta - \alpha > q$. Show that

$$\frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} - 2 > 0.$$

The sequence $u_0, u_1, \ldots$ is defined by $u_0 = \alpha$, $u_1 = \beta$ and

$$u_{n+1} = \frac{u_n^2 - q^2}{u_{n-1}} \qquad (n \geq 1),$$

where α, β and q are given positive numbers (and α and β are such that no term in the sequence is zero). Prove that $u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$. Prove also that

$$u_{n+1} - pu_n + u_{n-1} = 0$$

for some number p which you should express in terms of α, β and q.

Hence, or otherwise, show that if $\beta > \alpha + q$, the sequence is strictly increasing (that is, $u_{n+1} - u_n > 0$ for all n). Comment on the case $\beta = \alpha + q$.

Section B: Mechanics

[STEP 2, 2012Q9]

A tennis ball is projected from a height of $2h$ above horizontal ground with speed u and at an angle of α below the horizontal. It travels in a plane perpendicular to a vertical net of height h which is a horizontal distance of a from the point of projection. Given that the ball passes over the net, show that

$$\frac{1}{u^2} < \frac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}.$$

The ball lands before it has travelled a horizontal distance of b from the point of projection. Show that

$$\sqrt{u^2\sin^2\alpha + 4gh} < \frac{bg}{u\cos\alpha} + u\sin\alpha.$$

Hence show that

$$\tan\alpha < \frac{h(b^2 - 2a^2)}{ab(b-a)}.$$

[STEP 2, 2012Q10]

A hollow circular cylinder of internal radius r is held fixed with its axis horizontal. A uniform rod of length $2a$ (where $a < r$) rests in equilibrium inside the cylinder inclined at an angle of θ to the horizontal, where $\theta \neq 0$. The vertical plane containing the rod is perpendicular to the axis of the cylinder. The coefficient of friction between the cylinder and each end of the rod is μ, where $\mu > 0$.

Show that, if the rod is on the point of slipping, then the normal reactions R_1 and R_2 of the lower and higher ends of the rod, respectively, on the cylinder are related by

$$\mu(R_1 + R_2) = (R_1 - R_2)\tan\varphi$$

where φ is the angle between the rod and the radius to an end of the rod.

Show further that

$$\tan\theta = \frac{\mu r^2}{r^2 - a^2(1+\mu^2)}.$$

Deduce that $\lambda < \varphi$, where $\tan\lambda = \mu$.

[STEP 2, 2012Q11]

A small block of mass km is initially at rest on a smooth horizontal surface. Particles $P_1, P_2, P_3, \ldots$, are fired, in order, along the surface from a fixed point towards the block. The mass of the ith particle is im $(i = 1, 2, \ldots)$ and the speed at which it is fired is $\frac{u}{i}$. Each particle that collides with the block is embedded in it. Show that, if the nth particle collides with the block, the speed of the block after the collision is

$$\frac{2nu}{2k + n(n+1)}.$$

In the case $2k = N(N+1)$, where N is a positive integer, determine the number of collisions that occur. Show that the total kinetic energy lost in all the collisions is

$$\frac{1}{2}mu^2\left(\sum_{n=2}^{N+1}\frac{1}{n}\right).$$

Section C: Probability and Statistics

[STEP 2, 2012Q12]

A modern villa has complicated lighting controls. In order for the light in the swimming pool to be on, a particular switch in the hallway must be on and a particular switch in the kitchen must be on. There are four identical switches in the hallway and four identical switches in the kitchen. Guests cannot tell whether the switches are on or off, or what they control.

Each Monday morning a guest arrives, and the switches in the hallway are either all on or all off. The probability that they are all on is p and the probability that they are all off is $1-p$. The switches in the kitchen are each on or off, independently, with probability $\frac{1}{2}$.

(i) On the first Monday, a guest presses one switch in the hallway at random and one switch in the kitchen at random. Find the probability that the swimming pool light is on at the end of this process. Show that the probability that the guest has pressed the swimming pool light switch in the hallway, given that the light is on at the end of the process, is $\frac{1-p}{1+2p}$.

(ii) On each of seven Mondays, guests go through the above process independently of each other, and each time the swimming pool light is found to be on at the end of the process. Given that the most likely number of days on which the swimming pool light switch in the hallway was pressed is 3, show that $\frac{1}{4} < p < \frac{5}{14}$.

[STEP 2, 2012Q13]

In this question, you may assume that

$$\int_0^\infty e^{-\frac{x^2}{2}}\,\mathrm{d}x = \sqrt{\frac{1}{2}\pi}.$$

The number of supermarkets situated in any given region can be modelled by a Poisson random variable, where the mean is k times the area of the given region. Find the probability that there are no supermarkets within a circle of radius y.

The random variable Y denotes the distance between a randomly chosen point in the region and the nearest supermarket. Write down $\mathrm{P}(Y < y)$ and hence show that the probability density function of Y is $2\pi yk\mathrm{e}^{-\pi ky^2}$ for $y \geq 0$.

Find $\mathrm{E}(Y)$ and show that $\mathrm{Var}(Y) = \frac{4-\pi}{4\pi k}$.

STEP 2 2013

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Section A: Pure Mathematics

[STEP 2, 2013Q1]

(i) Find the value of m for which the line $y = mx$ touches the curve $y = \ln x$.

If instead the line intersects the curve when $x = a$ and $x = b$, where $a < b$, show that $a^b = b^a$. Show by means of a sketch that $a < \mathrm{e} < b$.

(ii) The line $y = mx + c$, where $c > 0$, intersects the curve $y = \ln x$ when $x = p$ and $x = q$, where $p < q$. Show by means of a sketch, or otherwise, that $p^q > q^p$.

(iii) Show by means of a sketch that the straight line through the points $(p, \ln p)$ and $(q, \ln q)$, where $\mathrm{e} \le p < q$, intersects the y-axis at a positive value of y. Which is greater, π^{e} or e^{π}?

(iv) Show, using a sketch or otherwise, that if $0 < p < q$ and $\frac{\ln q - \ln p}{q - p} = \mathrm{e}^{-1}$, then $q^p > p^q$.

[STEP 2, 2013Q2]

For $n \ge 0$, let

$$I_n = \int_0^1 x^n (1-x)^n \, \mathrm{d}x.$$

(i) For $n \ge 1$, show by means of a substitution that

$$\int_0^1 x^{n-1}(1-x)^n \, \mathrm{d}x = \int_0^1 x^n (1-x)^{n-1} \, \mathrm{d}x$$

and deduce that

$$2\int_0^1 x^{n-1}(1-x)^n \, \mathrm{d}x = I_{n-1}.$$

Show also, for $n \ge 1$, that

$$I_n = \frac{n}{n+1}\int_0^1 x^{n-1}(1-x)^{n+1} \, \mathrm{d}x$$

and hence that $I_n = \frac{n}{2(2n+1)} I_{n-1}$.

(ii) When n is a positive integer, show that

$$I_n = \frac{(n!)^2}{(2n+1)!}.$$

(iii) Use the substitution $x = \sin^2\theta$ to show that $I_{\frac{1}{2}} = \frac{\pi}{8}$, and evaluate $I_{\frac{3}{2}}$.

[STEP 2, 2013Q3]

(i) Given that the cubic equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real roots and $c < 0$, show with the help of sketches that either exactly one of the roots is positive or all three of the roots are positive.

(ii) Given that the equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real positive roots show that

$$a^2 > b > 0, \qquad a < 0, \qquad c < 0. \qquad\qquad (*)$$

[**Hint**: Consider the turning points.]

(iii) Given that the equation $x^3 + 3ax^2 + 3bx + c = 0$ has three distinct real roots and that

$$ab < 0, \qquad c > 0,$$

determine, with the help of sketches, the signs of the roots.

(iv) Show by means of an explicit example (giving values for a, b and c) that it is possible for the conditions $(*)$ to be satisfied even though the corresponding cubic equation has only one real root.

[STEP 2, 2013Q4]

The line passing through the point $(a, 0)$ with gradient b intersects the circle of unit radius centred at the origin at P and Q, and M is the midpoint of the chord PQ. Find the coordinates of M in terms of a and b.

(i) Suppose b is fixed and positive. As a varies, M traces out a curve (the *locus* of M). Show that $x = -by$ on this curve. Given that a varies with $-1 \le a \le 1$, show that the locus is a line segment of length $\frac{2b}{(1+b^2)^{\frac{1}{2}}}$. Give a sketch showing the locus and the unit circle.

(ii) Find the locus of M in the following cases, giving in each case its cartesian equation, describing it geometrically and sketching it in relation to the unit circle:

(a) a is fixed with $0 < a < 1$, and b varies with $-\infty < b < \infty$.

(b) $ab = 1$, and b varies with $0 < b \le 1$.

[STEP 2, 2013Q5]

(i) A function $f(x)$ satisfies $f(x) = f(1-x)$ for all x. Show, by differentiating with respect to x, that $f'\left(\frac{1}{2}\right) = 0$. If, in addition, $f(x) = f\left(\frac{1}{x}\right)$ for all (non-zero) x, show that $f'(-1) = 0$ and that $f'(2) = 0$.

(ii) The function f is defined, for $x \neq 0$ and $x \neq 1$, by

$$f(x) = \frac{(x^2 - x + 1)^3}{(x^2 - x)^2}.$$

Show that $f(x) = f\left(\frac{1}{x}\right)$ and $f(x) = f(1-x)$.

Given that it has exactly three stationary points, sketch the curve $y = f(x)$.

(iii) Hence, or otherwise, find all the roots of the equation $f(x) = \frac{27}{4}$ and state the ranges of values of x for which $f(x) > \frac{27}{4}$.

Find also all the roots of the equation $f(x) = \frac{343}{36}$ and state the ranges of values of x for which $f(x) > \frac{343}{36}$.

[STEP 2, 2013Q6]

In this question, the following theorem may be used.

Let $u_1, u_2, \ldots$ be a sequence of (real) numbers. If the sequence is bounded above (that is, $u_n \leq b$ for all n, where b is some fixed number) and increasing (that is, $u_n \geq u_{n-1}$ for all n), then the sequence tends to a limit (that is, converges).

The sequence $u_1, u_2, \ldots$ is defined by $u_1 = 1$ and

$$u_{n+1} = 1 + \frac{1}{u_n} \qquad (n \geq 1). \qquad (*)$$

(i) Show that, for $n \geq 3$,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1+u_n)(1+u_{n-2})}.$$

(ii) Prove, by induction or otherwise, that $1 \leq u_n \leq 2$ for all n.

(iii) Show that the sequence $u_1, u_3, u_5, \ldots$ tends to a limit, and that the sequence $u_2, u_4, u_6, \ldots$ tends to a limit. Find these limits and deduce that the sequence $u_1, u_2, u_3, \ldots$. tends to a limit.

Would this conclusion change if the sequence were defined by $(*)$ and $u_1 = 3$?

[STEP 2, 2013Q7]

(i) Write down a solution of the equation

$$x^2 - 2y^2 = 1, \qquad (*)$$

for which x and y are non-negative integers.

Show that, if $x = p$, $y = q$ is a solution of $(*)$, then so also is $x = 3p + 4q$, $y = 2p + 3q$.

Hence find two solutions of $(*)$ for which x is a positive odd integer and y is a positive even integer.

(ii) Show that, if x is an odd integer and y is an even integer, $(*)$ can be written in the form

$$n^2 = \frac{1}{2}m(m+1),$$

where m and n are integers.

(iii) The positive integers a, b and c satisfy

$$b^3 = c^4 - a^2,$$

where b is a prime number. Express a and c^2 in terms of b in the two cases that arise.

Find a solution of $a^2 + b^3 = c^4$, where a, b and c are positive integers but b is not prime.

[STEP 2, 2013Q8]

The function f satisfies $f(x) > 0$ for $x \geq 0$ and is strictly decreasing (which means that $f(b) < f(a)$ for $b > a$).

(i) For $t \geq 0$, let $A_0(t)$ be the area of the largest rectangle with sides parallel to the coordinate axes that can fit in the region bounded by the curve $y = f(x)$, the y-axis and the line $y = f(t)$. Show that $A_0(t)$ can be written in the form

$$A_0(t) = x_0\big(f(x_0) - f(t)\big),$$

where x_0 satisfies $x_0 f'(x_0) + f(x_0) = f(t)$.

(ii) The function g is defined, for $t > 0$, by

$$g(t) = \frac{1}{t}\int_0^t f(x)\,\mathrm{d}x.$$

Show that $tg'(t) = f(t) - g(t)$.

Making use of a sketch show that, for $t > 0$,

$$\int_0^t \big(f(x) - f(t)\big)\,\mathrm{d}x > A_0(t)$$

and deduce that $-t^2 g'(t) > A_0(t)$.

(iii) In the case $f(x) = \frac{1}{1+x}$, use the above to establish the inequality

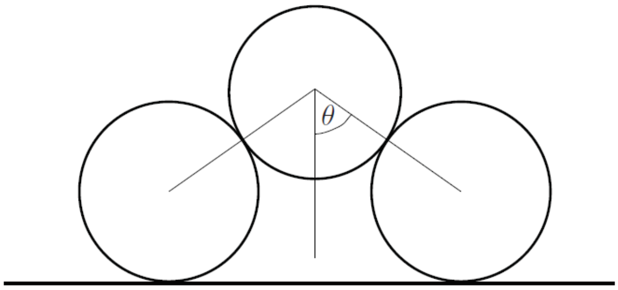
$$\ln\sqrt{1+t} > 1 - \frac{1}{\sqrt{1+t}},$$

for $t > 0$.

Section B: Mechanics

[STEP 2, 2013Q9]

The diagram shows three identical discs in equilibrium in a vertical plane. Two discs rest, not in contact with each other, on a horizontal surface and the third disc rests on the other two. The angle at the upper vertex of the triangle joining the centres of the discs is 2θ.



The weight of each disc is W. The coefficient of friction between a disc and the horizontal surface is μ and the coefficient of friction between the discs is also μ.

(i) Show that the normal reaction between the horizontal surface and a disc in contact with the surface is $\frac{3}{2}W$.

(ii) Find the normal reaction between two discs in contact and show that the magnitude of the frictional force between two discs in contact is $\frac{W\sin\theta}{2(1+\cos\theta)}$.

(iii) Show that if $\mu < 2-\sqrt{3}$ there is no value of θ for which equilibrium is possible.

[STEP 2, 2013Q10]

A particle is projected at an angle of elevation α (where $\alpha > 0$) from a point A on horizontal ground. At a general point in its trajectory the angle of elevation of the particle from A is θ and its direction of motion is at an angle φ above the horizontal (with $\varphi \geq 0$ for the first half of the trajectory and $\varphi \leq 0$ for the second half).

Let B denote the point on the trajectory at which $\theta = \frac{1}{2}\alpha$ and let C denote the point on the trajectory at which $\varphi = -\frac{1}{2}\alpha$.

(i) Show that, at a general point on the trajectory, $2\tan\theta = \tan\alpha + \tan\varphi$.

(ii) Show that, if B and C are the same point, then $\alpha = 60°$.

(iii) Given that $\alpha < 60°$, determine whether the particle reaches the point B first or the point C first.

[STEP 2, 2013Q11]

Three identical particles lie, not touching one another, in a straight line on a smooth horizontal surface. One particle is projected with speed u directly towards the other two which are at rest. The coefficient of restitution in all collisions is e, where $0 < e < 1$.

(i) Show that, after the second collision, the speeds of the particles are $\frac{1}{2}u(1-e)$, $\frac{1}{4}u(1-e^2)$ and $\frac{1}{4}u(1+e)^2$. Deduce that there will be a third collision whatever the value of e.

(ii) Show that there will be a fourth collision if and only if e is less than a particular value which you should determine.

Section C: Probability and Statistics

[STEP 2, 2013Q12]

The random variable U has a Poisson distribution with parameter λ. The random variables X and Y are defined as follows.

$$X = \begin{cases} U & \text{if } U \text{ is } 1, 3, 5, 7, \ldots \\ 0 & \text{otherwise} \end{cases}$$
$$Y = \begin{cases} U & \text{if } U \text{ is } 2, 4, 6, 8, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(i) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ in terms of λ, α and β, where

$$\alpha = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots$$

and

$$\beta = \frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots.$$

(ii) Show that

$$\mathrm{Var}(X) = \frac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$$

and obtain the corresponding expression for $\mathrm{Var}(Y)$. Are there any non-zero values of λ for which $\mathrm{Var}(X) + \mathrm{Var}(Y) = \mathrm{Var}(X + Y)$?

[STEP 2, 2013Q13]

A biased coin has probability p of showing a head and probability q of showing a tail, where $p \neq 0$, $q \neq 0$ and $p \neq q$. When the coin is tossed repeatedly, runs occur. A *straight run* of length n is a sequence of n consecutive heads or n consecutive tails. An *alternating run* of length n is a sequence of length n alternating between heads and tails. An alternating run can start with either a head or a tail.

Let S be the length of the longest straight run beginning with the first toss and let A be the length of the longest alternating run beginning with the first toss.

(i) Explain why $\mathrm{P}(A = 1) = p^2 + q^2$ and find $\mathrm{P}(S = 1)$. Show that $\mathrm{P}(S = 1) < \mathrm{P}(A = 1)$.

(ii) Show that $\mathrm{P}(S = 2) = \mathrm{P}(A = 2)$ and determine the relationship between $\mathrm{P}(S = 3)$ and $\mathrm{P}(A = 3)$.

(iii) Show that, for $n > 1$, $\mathrm{P}(S = 2n) > \mathrm{P}(A = 2n)$ and determine the corresponding relationship between $\mathrm{P}(S = 2n + 1)$ and $\mathrm{P}(A = 2n + 1)$. [You are advised *not* to use $p + q = 1$ in this part.]

STEP 2 2014

STEP 2 2014
On-line Exam

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Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 2014Q1]

In the triangle ABC, the base AB is of length 1 unit and the angles at A and B are α and β respectively, where $0 < \alpha \leq \beta$. The points P and Q lie on the sides AC and BC respectively, with $AP = PQ = QB = x$. The line PQ makes an angle of θ with the line through P parallel to AB.

(i) Show that $x\cos\theta = 1 - x\cos\alpha - x\cos\beta$, and obtain an expression for $x\sin\theta$ in terms of x, α and β. Hence show that

$$(1 + 2\cos(\alpha+\beta))x^2 - 2(\cos\alpha + \cos\beta)x + 1 = 0. \qquad (*)$$

Show that $(*)$ is also satisfied if P and Q lie on AC produced and BC produced, respectively. [By definition, P lies on AC produced if P lies on the line through A and C and the points are in the order A, C, P.]

(ii) State the condition on α and β for $(*)$ to be linear in x. If this condition does not hold (but the condition $0 < \alpha \leq \beta$ still holds), show that $(*)$ has distinct real roots.

(iii) Find the possible values of x in the two cases (a) $\alpha = \beta = 45°$ and (b) $\alpha = 30°, \beta = 90°$, and illustrate each case with a sketch.

[STEP 2, 2014Q2]

This question concerns the inequality

$$\int_0^{\pi} \big(f(x)\big)^2\,\mathrm{d}x \leq \int_0^{\pi} \big(f'(x)\big)^2\,\mathrm{d}x. \qquad (*)$$

(i) Show that $(*)$ is satisfied in the case $f(x) = \sin nx$, where n is a positive integer.

Show by means of counterexamples that $(*)$ is not necessarily satisfied if either $f(0) \neq 0$ or $f(\pi) \neq 0$.

(ii) You may now assume that $(*)$ is satisfied for any (differentiable) function f for which $f(0) = f(\pi) = 0$.

By setting $f(x) = ax^2 + bx + c$, where a, b and c are suitably chosen, show that $\pi^2 \leq 10$.

By setting $f(x) = p\sin\frac{1}{2}x + q\cos\frac{1}{2}x + r$, where p, q and r are suitably chosen, obtain another inequality for π.

Which of these inequalities leads to a better estimate for π^2?

[STEP 2, 2014Q3]

(i) Show, geometrically or otherwise, that the shortest distance between the origin and the line $y = mx + c$, where $c \geq 0$, is $c(m^2+1)^{-\frac{1}{2}}$.

(ii) The curve C lies in the x-y plane. Let the line L be tangent to C at a point P on C, and let a be the shortest distance between the origin and L. The curve C has the property that the distance a is the same for all points P on C.

Let P be the point on C with coordinates $\big(x, y(x)\big)$. Given that the tangent to C at P is not vertical, show that

$$(y - xy')^2 = a^2(1 + (y')^2). \qquad (*)$$

By first differentiating $(*)$ with respect to x, show that either $y = mx \pm a(1+m^2)^{\frac{1}{2}}$ for some m or $x^2 + y^2 = a^2$.

(iii) Now suppose that C (as defined above) is a continuous curve for $-\infty < x < \infty$, consisting of the arc of a circle and two straight lines. Sketch an example of such a curve which has a non-vertical tangent at each point.

[STEP 2, 2014Q4]

(i) By using the substitution $u = \frac{1}{x}$, show that for $b > 0$

$$\int_{\frac{1}{b}}^{b} \frac{x \ln x}{(a^2+x^2)(a^2x^2+1)}\,\mathrm{d}x = 0.$$

(ii) By using the substitution $u = \frac{1}{x}$, show that for $b > 0$,

$$\int_{\frac{1}{b}}^{b} \frac{\arctan x}{x}\,\mathrm{d}x = \frac{\pi \ln b}{2}.$$

(iii) By using the result $\int_0^\infty \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{\pi}{2a}$ (where $a > 0$), and a substitution of the form $u = \frac{k}{x}$, for suitable k, show that

$$\int_0^\infty \frac{1}{(a^2+x^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3} \qquad (a > 0).$$

[STEP 2, 2014Q5]

Given that $y = xu$, where u is a function of x, write down an expression for $\frac{dy}{dx}$.

(i) Use the substitution $y = xu$ to solve

$$\frac{dy}{dx} = \frac{2y + x}{y - 2x}$$

given that the solution curve passes through the point $(1, 1)$.

Give your answer in the form of a quadratic in x and y.

(ii) Using the substitutions $x = X + a$ and $y = Y + b$ for appropriate values of a and b, or otherwise, solve

$$\frac{dy}{dx} = \frac{x - 2y - 4}{2x + y - 3},$$

given that the solution curve passes through the point $(1, 1)$.

[STEP 2, 2014Q6]

By simplifying $\sin\left(r + \frac{1}{2}\right)x - \sin\left(r - \frac{1}{2}\right)x$ or otherwise show that, for $\sin\frac{1}{2}x \neq 0$,

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(n + \frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}.$$

The functions S_n, for $n = 1, 2, \ldots$, are defined by

$$S_n(x) = \sum_{r=1}^{n} \frac{1}{r}\sin rx \qquad (0 \leq x \leq \pi).$$

(i) Find the stationary points of $S_2(x)$ for $0 \leq x \leq \pi$, and sketch this function.

(ii) Show that if $S_n(x)$ has a stationary point at $x = x_0$, where $0 < x_0 < \pi$, then

$$\sin nx_0 = (1 - \cos nx_0)\tan\frac{1}{2}x_0$$

and hence that $S_n(x_0) \geq S_{n-1}(x_0)$. Deduce that if $S_{n-1}(x) > 0$ for all x in the interval $0 < x < \pi$, then $S_n(x) > 0$ for all x in this interval.

(iii) Prove that $S_n(x) \geq 0$ for $n \geq 1$ and $0 \leq x \leq \pi$.

[STEP 2, 2014Q7]

(i) The function f is defined by $f(x) = |x-a| + |x-b|$, where $a < b$. Sketch the graph of $f(x)$, giving the gradient in each of the regions $x < a$, $a < x < b$ and $x > b$. Sketch on the same diagram the graph of $g(x)$, where $g(x) = |2x - a - b|$.

What shape is the quadrilateral with vertices $(a, 0)$, $(b, 0)$, $\big(b, f(b)\big)$ and $\big(a, f(a)\big)$?

(ii) Show graphically that the equation

$$|x-a| + |x-b| = |x-c|,$$

where $a < b$, has 0, 1 or 2 solutions, stating the relationship of c to a and b in each case.

(iii) For the equation

$$|x-a| + |x-b| = |x-c| + |x-d|,$$

where $a < b$, $c < d$ and $d - c < b - a$, determine the number of solutions in the various cases that arise, stating the relationship between a, b, c and d in each case.

[STEP 2, 2014Q8]

For positive integers n, a and b, the integer c_r $(0 \le r \le n)$ is defined to be the coefficient of x^r in the expansion in powers of x of $(a + bx)^n$. Write down an expression for c_r in terms of r, n, a and b.

For given n, a and b, let m denote a value of r for which c_r is greatest (that is, $c_m \ge c_r$ for $0 \le r \le n$).

Show that

$$\frac{b(n+1)}{a+b} - 1 \le m \le \frac{b(n+1)}{a+b}.$$

Deduce that m is either a unique integer or one of two consecutive integers.

Let $G(n, a, b)$ denote the unique value of m (if there is one) or the larger of the two possible values of m.

(i) Evaluate $G(9, 1, 3)$ and $G(9, 2, 3)$.

(ii) For any positive integer k, find $G(2k, a, a)$ and $G(2k - 1, a, a)$ in terms of k.

(iii) For fixed n and b, determine a value of a for which $G(n, a, b)$ is greatest.

(iv) For fixed n, find the greatest possible value of $G(n, 1, b)$. For which values of b is this greatest value achieved?

Section B: Mechanics

[STEP 2, 2014Q9]

A uniform rectangular lamina $ABCD$ rests in equilibrium in a vertical plane with the corner A in contact with a rough vertical wall. The plane of the lamina is perpendicular to the wall. It is supported by a light inextensible string attached to the side AB at a distance d from A. The other end of the string is attached to a point on the wall above A where it makes an acute angle θ with the downwards vertical. The side AB makes an acute angle φ with the upwards vertical at A. The sides BC and AB have lengths $2a$ and $2b$ respectively. The coefficient of friction between the lamina and the wall is μ.

(i) Show that, when the lamina is in limiting equilibrium with the frictional force acting upwards,

$$d\sin(\theta+\varphi) = (\cos\theta + \mu\sin\theta)(a\cos\varphi + b\sin\varphi). \qquad (*)$$

(ii) How should $(*)$ be modified if the lamina is in limiting equilibrium with the frictional force acting downwards?

(iii) Find a condition on d, in terms of a, b, $\tan\theta$ and $\tan\varphi$, which is necessary and sufficient for the frictional force to act upwards. Show that this condition cannot be satisfied if $b(2\tan\theta + \tan\varphi) < a$.

[STEP 2, 2014Q10]

A particle is projected from a point O on horizontal ground with initial speed u and at an angle of θ above the ground. The motion takes place in the x-y plane, where the x-axis is horizontal, the y-axis is vertical and the origin is O. Obtain the Cartesian equation of the particle's trajectory in terms of u, g and λ, where $\lambda = \tan\theta$.

Now consider the trajectories for different values of θ with u fixed. Show that for a given value of x, the coordinate y can take all values up to a maximum value, Y, which you should determine as a function of x, u and g.

Sketch a graph of Y against x and indicate on your graph the set of points that can be reached by a particle projected from O with speed u.

Hence find the furthest distance from O that can be achieved by such a projectile.

[STEP 2, 2014Q11]

A small smooth ring R of mass m is free to slide on a fixed smooth horizontal rail. A light inextensible string of length L is attached to one end, O, of the rail. The string passes through the ring, and a particle P of mass km (where $k > 0$) is attached to its other end; this part of the string hangs at an acute angle α to the vertical and it is given that α is constant in the motion.

Let x be the distance between O and the ring. Taking the y-axis to be vertically upwards, write down the Cartesian coordinates of P relative to O in terms of x, L and α.

(i) By considering the vertical component of the equation of motion of P, show that

$$km\ddot{x}\cos\alpha = T\cos\alpha - kmg,$$

where T is the tension in the string. Obtain two similar equations relating to the horizontal components of the equations of motion of P and R.

(ii) Show that $\frac{\sin\alpha}{(1-\sin\alpha)^2} = k$, and deduce, by means of a sketch or otherwise, that motion with α constant is possible for all values of k.

(iii) Show that $\ddot{x} = -g\tan\alpha$.

Section C: Probability and Statistics

[STEP 2, 2014Q12]

The lifetime of a fly (measured in hours) is given by the continuous random variable T with probability density function $f(t)$ and cumulative distribution function $F(t)$. The *hazard function*, $h(t)$, is defined, for $F(t) < 1$, by

$$h(t) = \frac{f(t)}{1 - F(t)}.$$

(i) Given that the fly lives to at least time t, show that the probability of its dying within the following δt is approximately $h(t)\delta t$ for small values of δt.

(ii) Find the hazard function in the case $F(t) = \frac{t}{a}$ for $0 < t < a$. Sketch $f(t)$ and $h(t)$ in this case.

(iii) The random variable T is distributed on the interval $t > a$, where $a > 0$, and its hazard function is t^{-1}. Determine the probability density function for T.

(iv) Show that $h(t)$ is constant for $t > b$ and zero otherwise if and only if $f(t) = ke^{-k(t-b)}$ for $t > b$, where k is a positive constant.

(v) The random variable T is distributed on the interval $t > 0$ and its hazard function is given by

$$h(t) = \left(\frac{\lambda}{\theta^{\lambda}}\right) t^{\lambda - 1},$$

where λ and θ are positive constants. Find the probability density function for T.

[STEP 2, 2014Q13]

A random number generator prints out a sequence of integers $I_1, I_2, I_3, \dots$. Each integer is independently equally likely to be any one of $1, 2, \dots, n$, where n is fixed. The random variable X takes the value r, where I_r is the first integer which is a repeat of some earlier integer.

Write down an expression for $\mathrm{P}(X = 4)$.

(i) Find an expression for $\mathrm{P}(X = r)$, where $2 \le r \le n + 1$. Hence show that, for any positive integer n,

$$\frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{2}{n} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots = 1.$$

(ii) Write down an expression for $\mathrm{E}(X)$. (You do not need to simplify it.)

(iii) Write down an expression for $\mathrm{P}(X \ge k)$.

(iv) Show that, for any discrete random variable Y taking the values $1, 2, \dots, N$,

$$\mathrm{E}(Y) = \sum_{k=1}^{N} \mathrm{P}(Y \ge k).$$

Hence show that, for any positive integer n,

$$\left(1 - \frac{1^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3^2}{n}\right) + \cdots = 0$$

STEP 2 2015

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Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 2015Q1]

(i) By use of calculus, show that $x - \ln(1+x)$ is positive for all positive x. Use this result to show that

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1).$$

(ii) By considering $x + \ln(1-x)$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2.$$

[STEP 2, 2015Q2]

In the triangle ABC, angle $BAC = \alpha$ and angle $CBA = 2\alpha$, where 2α is acute, and $BC = x$. Show that $AB = (3 - 4\sin^2\alpha)x$.

The point D is the midpoint of AB and the point E is the foot of the perpendicular from C to AB. Find an expression for DE in terms of x.

The point F lies on the perpendicular bisector of AB and is a distance x from C. The points F and B lie on the same side of the line through A and C. Show that the line FC trisects the angle ACB.

[STEP 2, 2015Q3]

Three rods have lengths a, b and c, where $a < b < c$. The three rods can be made into a triangle (possibly of zero area) if $a + b \geq c$.

Let T_n be the number of triangles that can be made with three rods chosen from n rods of lengths $1, 2, 3, \ldots, n$ (where $n \geq 3$). Show that $T_8 - T_7 = 2 + 4 + 6$ and evaluate $T_8 - T_6$. Write down expressions for $T_{2m} - T_{2m-1}$ and $T_{2m} - T_{2m-2}$.

Prove by induction that $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$, and find the corresponding result for an odd number of rods.

[STEP 2, 2015Q4]

(i) The continuous function f is defined by

$$\tan f(x) = x \qquad (-\infty < x < \infty)$$

and $f(0) = \pi$. Sketch the curve $y = f(x)$.

(ii) The continuous function g is defined by

$$\tan g(x) = \frac{x}{1+x^2} \qquad (-\infty < x < \infty)$$

and $g(0) = \pi$. Sketch the curves $y = \frac{x}{1+x^2}$ and $y = g(x)$.

(iii) The continuous function h is defined by $h(0) = \pi$ and

$$\tan h(x) = \frac{x}{1-x^2} \qquad (x \neq \pm 1).$$

(The values of $h(x)$ at $x = \pm 1$ are such that $h(x)$ is continuous at these points.) Sketch the curves $y = \frac{x}{1-x^2}$ and $y = h(x)$.

[STEP 2, 2015Q5]

In this question, the arctan function satisfies $0 \leq \arctan x < \frac{1}{2}\pi$ for $x \geq 0$.

(i) Let

$$S_n = \sum_{m=1}^{n} \arctan\left(\frac{1}{2m^2}\right),$$

for $n = 1, 2, 3, \ldots$. Prove by induction that

$$\tan S_n = \frac{n}{n+1}.$$

Prove also that

$$S_n = \arctan\frac{n}{n+1}.$$

(ii) In a triangle ABC, the lengths of the sides AB and BC are $4n^2$ and $4n^4 - 1$, respectively, and the angle at B is a right angle. Let angle $BCA = 2\alpha_n$. Show that

$$\sum_{n=1}^{\infty} \alpha_n = \frac{1}{4}\pi.$$

[STEP 2, 2015Q6]

(i) Show that

$$\sec^2\left(\frac{1}{4}\pi - \frac{1}{2}x\right) = \frac{2}{1+\sin x}.$$

Hence integrate $\frac{1}{1+\sin x}$ with respect to x.

(ii) By means of the substitution $y = \pi - x$, show that

$$\int_0^\pi x f(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi f(\sin x)\,\mathrm{d}x,$$

where f is any function for which these integrals exist.

Hence evaluate

$$\int_0^\pi \frac{x}{1+\sin x}\,\mathrm{d}x.$$

(iii) Evaluate

$$\int_0^\pi \frac{2x^3 - 3\pi x^2}{(1+\sin x)^2}\,\mathrm{d}x.$$

[STEP 2, 2015Q7]

A circle C is said to be *bisected* by a curve X if X meets C in exactly two points and these points are diametrically opposite each other on C.

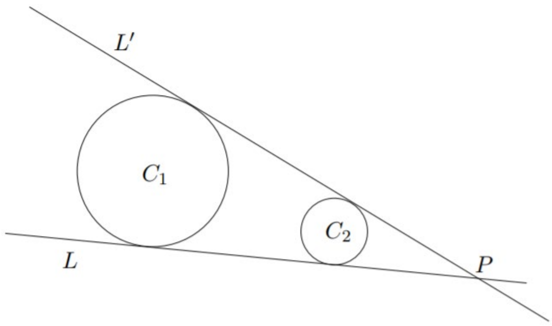
(i) Let C be the circle of radius a in the x-y plane with centre at the origin. Show, by giving its equation, that it is possible to find a circle of given radius r that bisects C provided $r > a$. Show that no circle of radius r bisects C if $r \le a$.

(ii) Let C_1 and C_2 be circles with centres at $(-d, 0)$ and $(d, 0)$ and radii a_1 and a_2, respectively, where $d > a_1$ and $d > a_2$. Let D be a circle of radius r that bisects both C_1 and C_2. Show that the x-coordinate of the centre of D is $\frac{a_2^2 - a_1^2}{4d}$.

Obtain an expression in terms of d, r, a_1 and a_2 for the y-coordinate of the centre of D, and deduce that r must satisfy

$$16r^2d^2 \ge (4d^2 + (a_2 - a_1)^2)(4d^2 + (a_2 + a_1)^2).$$

[STEP 2, 2015Q8]



The diagram above shows two non-overlapping circles C_1 and C_2 of different sizes. The lines L and L' are the two common tangents to C_1 and C_2 such that the two circles lie on the same side of each of the tangents. The lines L and L' intersect at the point P which is called the *focus* of C_1 and C_2.

(i) Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the position vectors of the centres of C_1 and C_2, respectively. Show that the position vector of P is

$$\frac{r_1\mathbf{x}_2 - r_2\mathbf{x}_1}{r_1 - r_2},$$

where r_1 and r_2 are the radii of C_1 and C_2, respectively.

(ii) The circle C_3 does not overlap either C_1 or C_2 and its radius, r_3, satisfies $r_1 \neq r_3 \neq r_2$. The focus of C_1 and C_3 is Q, and the focus of C_2 and C_3 is R. Show that P, Q and R lie on the same straight line.

(iii) Find a condition on r_1, r_2 and r_3 for Q to lie half-way between P and R.

Section B: Mechanics

[STEP 2, 2015Q9]

An equilateral triangle ABC is made of three light rods each of length a. It is free to rotate in a vertical plane about a horizontal axis through A. Particles of mass $3m$ and $5m$ are attached to B and C respectively. Initially, the system hangs in equilibrium with BC below A.

(i) Show that, initially, the angle θ that BC makes with the horizontal is given by $\sin\theta = \frac{1}{7}$.

(ii) The triangle receives an impulse that imparts a speed v to the particle B. Find the minimum speed v_0 such that the system will perform complete rotations if $v > v_0$.

[STEP 2, 2015Q10]

A particle of mass m is pulled along the floor of a room in a straight line by a light string which is pulled at constant speed V through a hole in the ceiling. The floor is smooth and horizontal, and the height of the room is h. Find, in terms of V and θ, the speed of the particle when the string makes an angle of θ with the vertical (and the particle is still in contact with the floor). Find also the acceleration, in terms of V, h and θ.

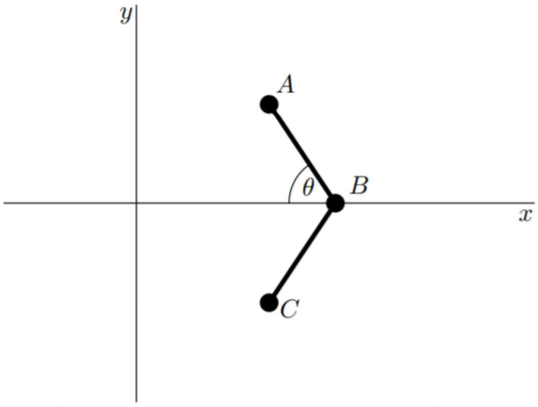
Find the tension in the string and hence show that the particle will leave the floor when

$$\tan^4\theta = \frac{V^2}{gh}.$$

[STEP 2, 2015Q11]

Three particles, A, B and C, each of mass m, lie on a smooth horizontal table. Particles A and C are attached to the two ends of a light inextensible string of length $2a$ and particle B is attached to the midpoint of the string. Initially, A, B and C are at rest at points $(0, a)$, $(0, 0)$ and $(0, -a)$, respectively.

An impulse is delivered to B, imparting to it a speed u in the positive x direction. The string remains taut throughout the subsequent motion.



(i) At time t, the angle between the x-axis and the string joining A and B is θ, as shown in the diagram, and B is at $(x, 0)$. Write down the coordinates of A in terms of x, a and θ at this time. Given that the velocity of B is $(v, 0)$, show that the velocity of A is $\left(v + a\dot{\theta}\sin\theta\,, a\dot{\theta}\cos\theta\right)$, where the dot denotes differentiation with respect to time.

(ii) Show that, before A and C first collide,

$$3v + 2a\dot{\theta}\sin\theta = u$$

and

$$\dot{\theta}^2 = \frac{u^2}{a^2(3 - 2\sin^2\theta)}.$$

(iii) When A and C collide, the collision is elastic (no energy is lost). At what value of θ does the second collision between particles A and C occur? (You should justify your answer.)

(iv) When $v = 0$, what are the possible values of θ? Is $v = 0$ whenever θ takes these values?

Section C: Probability and Statistics

[STEP 2, 2015Q12]

Four players A, B, C and D play a coin-tossing game with a fair coin. Each player chooses a sequence of heads and tails, as follows:

Player A: HHT; Player B: THH; Player C: TTH; Player D: HTT.

The coin is then tossed until one of these sequences occurs, in which case the corresponding player is the winner.

(i) Show that, if only A and B play, then A has a probability of $\frac{1}{4}$ of winning.

(ii) If all four players play together, find the probabilities of each one winning.

(iii) Only B and C play. What is the probability of C winning if the first two tosses are TT? Let the probabilities of C winning if the first two tosses are HT, TH and HH be p, q and r, respectively. Show that $p = \frac{1}{2} + \frac{1}{2}q$.

Find the probability that C wins.

[STEP 2, 2015Q13]

The maximum height X of flood water each year on a certain river is a random variable with probability density function f given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant.

It costs ky pounds each year to prepare for flood water of height y or less, where k is a positive constant and $y \geq 0$. If $X \leq y$ no further costs are incurred but if $X > y$ the additional cost of flood damage is $a(X - y)$ pounds where a is a positive constant.

(i) Let C be the total cost of dealing with the floods in the year. Show that the expectation of C is given by

$$\mathrm{E}(C) = ky + \frac{a}{\lambda} e^{-\lambda y}.$$

How should y be chosen in order to minimise $\mathrm{E}(C)$, in the different cases that arise according to the value of $\frac{a}{k}$?

(ii) Find the variance of C, and show that the more that is spent on preparing for flood water in advance the smaller this variance.

STEP 2 2016

STEP 2 2016
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Section A: Pure Mathematics

[STEP 2, 2016Q1]

The curve C_1 has parametric equations $x = t^2$, $y = t^3$, where $-\infty < t < \infty$. Let O denote the point $(0,0)$. The points P and Q on C_1 are such that $\angle POQ$ is a right angle. Show that the tangents to C_1 at P and Q intersect on the curve C_2 with equation $4y^2 = 3x - 1$.

Determine whether C_1 and C_2 meet, and sketch the two curves on the same axes.

[STEP 2, 2016Q2]

Use the factor theorem to show that $a + b - c$ is a factor of

$$(a+b+c)^3 - 6(a+b+c)(a^2+b^2+c^2) + 8(a^3+b^3+c^3). \qquad (*)$$

Hence factorise $(*)$ completely.

(i) Use the result above to solve the equation

$$(x+1)^3 - 3(x+1)(2x^2+5) + 2(4x^3+13) = 0.$$

(ii) By setting $d + e = c$, or otherwise, show that $(a + b - d - e)$ is a factor of

$$(a+b+d+e)^3 - 6(a+b+d+e)(a^2+b^2+d^2+e^2) + 8(a^3+b^3+d^3+e^3)$$

and factorise this expression completely.

Hence solve the equation

$$(x+6)^3 - 6(x+6)(x^2+14) + 8(x^3+36) = 0.$$

[STEP 2, 2016Q3]

For each non-negative integer n, the polynomial f_n is defined by

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

(i) Show that $f_n'(x) = f_{n-1}(x)$ (for $n \geq 1$).

(ii) Show that, if a is a real root of the equation

$$f_n(x) = 0, \qquad (*)$$

then $a < 0$.

(iii) Let a and b be distinct real roots of $(*)$, for $n \geq 2$. Show that $f_n'(a)f_n'(b) > 0$ and use a sketch to deduce that $f_n(c) = 0$ for some number c between a and b.

Deduce that $(*)$ has at most one real root. How many real roots does $(*)$ have if n is odd? How many real roots does $(*)$ have if n is even?

[STEP 2, 2016Q4]

Let

$$y = \frac{x^2 + x\sin\theta + 1}{x^2 + x\cos\theta + 1}.$$

(i) Given that x is real, show that

$$(y\cos\theta - \sin\theta)^2 \geq 4(y-1)^2.$$

Deduce that

$$y^2 + 1 \geq 4(y-1)^2,$$

and hence that

$$\frac{4-\sqrt{7}}{3} \leq y \leq \frac{4+\sqrt{7}}{3}.$$

(ii) In the case $y = \frac{4+\sqrt{7}}{3}$, show that

$$\sqrt{y^2+1} = 2(y-1)$$

and find the corresponding values of x and $\tan\theta$.

[STEP 2, 2016Q5]

In this question, the definition of $\binom{p}{q}$ is taken to be

$$\binom{p}{q} = \begin{cases} \dfrac{p!}{q!\,(p-q)!} & \text{if } p \geq q \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Write down the coefficient of x^n in the binomial expansion for $(1-x)^{-N}$, where N is a positive integer, and write down the expansion using the Σ summation notation.

By considering $(1-x)^{-1}(1-x)^{-N}$, where N is a positive integer, show that

$$\sum_{j=0}^{n} \binom{N+j-1}{j} = \binom{N+n}{n}.$$

(ii) Show that, for any positive integers m, n and r with $r \leq m+n$,

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j}.$$

(iii) Show that, for any positive integers m and N,

$$\sum_{j=0}^{n} (-1)^j \binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n}.$$

[STEP 2, 2016Q6]

This question concerns solutions of the differential equation

$$(1-x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + k^2y^2 = k^2 \qquad (*)$$

where k is a positive integer.

For each value of k, let $y_k(x)$ be the solution of $(*)$ that satisfies $y_k(1) = 1$; you may assume that there is only one such solution for each value of k.

(i) Write down the differential equation satisfied by $y_1(x)$ and verify that $y_1(x) = x$.

(ii) Write down the differential equation satisfied by $y_2(x)$ and verify that $y_2(x) = 2x^2 - 1$.

(iii) Let $z(x) = 2\big(y_n(x)\big)^2 - 1$. Show that

$$(1-x^2)\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 + 4n^2z^2 = 4n^2$$

and hence obtain an expression for $y_{2n}(x)$ in terms of $y_n(x)$.

(iv) Let $v(x) = y_n\big(y_m(x)\big)$. Show that $v(x) = y_{mn}(x)$.

[STEP 2, 2016Q7]

Show that

$$\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x, \qquad (*)$$

where f is any function for which the integrals exist.

(i) Use $(*)$ to evaluate

$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x.$$

(ii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x.$$

(iii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \ln(1 + \tan x)\,\mathrm{d}x.$$

(iv) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{x}{\cos x\,(\cos x + \sin x)}\,\mathrm{d}x.$$

[STEP 2, 2016Q8]

Evaluate the integral

$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \qquad \left(m > \frac{1}{2}\right).$$

Show by means of a sketch that

$$\sum_{r=m}^{n} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2}\,\mathrm{d}x, \qquad (*)$$

where m and n are positive integers with $m < n$.

(i) You are given that the infinite series $\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges to a value denoted by E. Use $(*)$ to obtain the following approximations for E:

$$E \approx 2; \quad E \approx \frac{5}{3}; \quad E \approx \frac{33}{20}.$$

(ii) Show that, when r is large, the error in approximating $\frac{1}{r^2}$ by $\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}\,\mathrm{d}x$ is approximately $\frac{1}{4r^4}$.

Given that $E \approx 1.645$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1.08.$$

Section B: Mechanics

[STEP 2, 2016Q9]

A small bullet of mass m is fired into a block of wood of mass M which is at rest. The speed of the bullet on entering the block is u. Its trajectory within the block is a horizontal straight line and the resistance to the bullet's motion is R, which is constant.

(i) The block is fixed. The bullet travels a distance a inside the block before coming to rest. Find an expression for a in terms of m, u and R.

(ii) Instead, the block is free to move on a smooth horizontal table. The bullet travels a distance b inside the block before coming to rest relative to the block, at which time the block has moved a distance c on the table. Find expressions for b and c in terms of M, m and a.

[STEP 2, 2016Q10]

A thin uniform wire is bent into the shape of an isosceles triangle ABC, where AB and AC are of equal length and the angle at A is 2θ. The triangle ABC hangs on a small rough horizontal peg with the side BC resting on the peg. The coefficient of friction between the wire and the peg is μ. The plane containing ABC is vertical. Show that the triangle can rest in equilibrium with the peg in contact with any point on BC provided

$$\mu \geq 2\tan\theta\,(1+\sin\theta).$$

[STEP 2, 2016Q11]

(i) Two particles move on a smooth horizontal surface. The positions, in Cartesian coordinates, of the particles at time t are $(a+ut\cos\alpha\,, ut\sin\alpha)$ and $(vt\cos\beta\,, b+vt\sin\beta)$, where a, b, u and v are positive constants, α and β are constant acute angles, and $t \geq 0$.

Given that the two particles collide, show that

$$u\sin(\theta+\alpha) = v\sin(\theta+\beta)\,,$$

where θ is the acute angle satisfying $\tan\theta = \frac{b}{a}$.

(ii) A gun is placed on the top of a vertical tower of height b which stands on horizontal ground. The gun fires a bullet with speed v and (acute) angle of elevation β. Simultaneously, a target is projected from a point on the ground a horizontal distance a from the foot of the tower. The target is projected with speed u and (acute) angle of elevation α, in a direction directly away from the tower.

Given that the target is hit before it reaches the ground, show that

$$2u\sin\alpha\,(u\sin\alpha - v\sin\beta) > bg.$$

Explain, with reference to part (i), why the target can only be hit if $\alpha > \beta$.

Section C: Probability and Statistics

[STEP 2, 2016Q12]

Starting with the result $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$, prove that

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A) + \mathrm{P}(A \cap B \cap C).$$

Write down, without proof, the corresponding result for four events A, B, C and D.

A pack of n cards, numbered $1, 2, \dots, n$, is shuffled and laid out in a row. The result of the shuffle is that each card is equally likely to be in any position in the row. Let E_i be the event that the card bearing the number i is in the ith position in the row. Write down the following probabilities:

(i) $\mathrm{P}(E_i)$.

(ii) $\mathrm{P}(E_i \cap E_j)$, where $i \neq j$.

(iii) $\mathrm{P}(E_i \cap E_j \cap E_k)$, where $i \neq j$, $j \neq k$ and k$\neq i$.

Hence show that the probability that at least one card is in the same position as the number it bears is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}.$$

Find the probability that exactly one card is in the same position as the number it bears.

[STEP 2, 2016Q13]

(i) The random variable X has a binomial distribution with parameters n and p, where $n = 16$ and $p = \frac{1}{2}$. Show, using an approximation in terms of the standard normal density function $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}$, that

$$\mathrm{P}(X = 8) \approx \frac{1}{2\sqrt{2\pi}}.$$

(ii) By considering a binomial distribution with parameters $2n$ and $\frac{1}{2}$, show that

$$(2n)! \approx \frac{2^{2n}(n!)^2}{\sqrt{n\pi}}.$$

(iii) By considering a Poisson distribution with parameter n, show that

$$n! \approx \sqrt{2\pi n}\mathrm{e}^{-n}n^n.$$

STEP 2 2017

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Section A: Pure Mathematics

[STEP 2, 2017Q1]

Note: In this question you may use without proof the result $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$.

Let

$$I_n = \int_0^1 x^n \arctan x \,\mathrm{d}x,$$

where $n = 0, 1, 2, 3, \ldots$.

(i) Show that, for $n \geq 0$,

$$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,\mathrm{d}x$$

and evaluate I_0.

(ii) Find an expression, in terms of n, for $(n+3)I_{n+2} + (n+1)I_n$.

Use this result to evaluate I_4.

(iii) Prove by induction that, for $n \geq 1$,

$$(4n+1)I_{4n} = A - \frac{1}{2}\sum_{r=1}^{2n}(-1)^r \frac{1}{r},$$

where A is a constant to be determined.

[STEP 2, 2017Q2]

The sequence of numbers $x_0, x_1, x_2, \ldots$ satisfies

$$x_{n+1} = \frac{ax_n - 1}{x_n + b}.$$

(You may assume that a, b and x_0 are such that $x_n + b \neq 0$.)

Find an expression for x_{n+2} in terms of a, b and x_n.

(i) Show that $a + b = 0$ is a necessary condition for the sequence to be periodic with period 2.

Note: The sequence is said to be periodic with period k if $x_{n+k} = x_n$ for all n, and there is no integer m with $0 < m < k$ such that $x_{n+m} = x_n$ for all n.

(ii) Find necessary and sufficient conditions for the sequence to have period 4.

[STEP 2, 2017Q3]

(i) Sketch, on x-y axes, the set of all points satisfying $\sin y = \sin x$, for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$. You should give the equations of all the lines on your sketch.

(ii) Given that

$$\sin y = \frac{1}{2}\sin x$$

obtain an expression, in terms of x, for y' when $0 \le x \le \frac{1}{2}\pi$ and $0 \le y \le \frac{1}{2}\pi$, and show that

$$y'' = -\frac{3\sin x}{(4 - \sin^2 x)^{\frac{3}{2}}}.$$

Use these results to sketch the set of all points satisfying $\sin y = \frac{1}{2}\sin x$ for $0 \le x \le \frac{1}{2}\pi$ and $0 \le y \le \frac{1}{2}\pi$.

Hence sketch the set of all points satisfying $\sin y = \frac{1}{2}\sin x$ for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$.

(iii) Without further calculation, sketch the set of all points satisfying $\cos y = \frac{1}{2}\sin x$ for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$.

[STEP 2, 2017Q4]

The Schwarz inequality is

$$\left(\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2 \le \left(\int_a^b \big(f(x)\big)^2 \mathrm{d}x\right)\left(\int_a^b \big(g(x)\big)^2\,\mathrm{d}x\right). \qquad (*)$$

(i) By setting $f(x) = 1$ in $(*)$, and choosing $g(x)$, a and b suitably, show that for $t > 0$,

$$\frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} \le \frac{t}{2}.$$

(ii) By setting $f(x) = x$ in $(*)$, and choosing $g(x)$ suitably, show that

$$\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \ge 12\left(1 - \mathrm{e}^{-\frac{1}{4}}\right)^2.$$

(iii) Use $(*)$ to show that

$$\frac{64}{25\pi} \le \int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,\mathrm{d}x \le \sqrt{\frac{\pi}{2}}.$$

[STEP 2, 2017Q5]

A curve C is determined by the parametric equations

$$x = at^2, \qquad y = 2at,$$

where $a > 0$.

(i) Show that the normal to C at a point P, with non-zero parameter p, meets C again at a point N, with parameter n, where

$$n = -\left(p + \frac{2}{p}\right).$$

(ii) Show that the distance $|PN|$ is given by

$$|PN|^2 = 16a^2 \frac{(p^2+1)^3}{p^4}$$

and that this is minimised when $p^2 = 2$.

(iii) The point Q, with parameter q, is the point at which the circle with diameter PN cuts C again. By considering the gradients of QP and QN, show that

$$2 = p^2 - q^2 + \frac{2q}{p}.$$

Deduce that $|PN|$ is at its minimum when Q is at the origin.

[STEP 2, 2017Q6]

Let

$$S_n = \sum_{r=1}^{n} \frac{1}{\sqrt{r}},$$

where n is a positive integer.

(i) Prove by induction that

$$S_n \leq 2\sqrt{n} - 1.$$

(ii) Show that $(4k+1)\sqrt{k+1} > (4k+3)\sqrt{k}$ for $k \geq 0$.

Determine the smallest number C such that

$$S_n \geq 2\sqrt{n} + \frac{1}{2\sqrt{n}} - C.$$

[STEP 2, 2017Q7]

The functions f and g are defined, for $x > 0$, by

$$f(x) = x^x, \qquad g(x) = x^{f(x)}.$$

(i) By taking logarithms, or otherwise, show that $f(x) > x$ for $0 < x < 1$. Show further that $x < g(x) < f(x)$ for $0 < x < 1$.

Write down the corresponding results for $x > 1$.

(ii) Find the value of x for which $f'(x) = 0$.

(iii) Use the result $x \ln x \to 0$ as $x \to 0$ to find $\lim_{x\to 0} f(x)$, and write down $\lim_{x\to 0} g(x)$.

(iv) Show that $x^{-1} + \ln x \geq 1$ for $x > 0$.

Using this result, or otherwise, show that $g'(x) > 0$.

Sketch the graphs, for $x > 0$, of $y = x$, $y = f(x)$ and $y = g(x)$ on the same axes.

[STEP 2, 2017Q8]

All vectors in this question lie in the same plane.

The vertices of the non-right-angled triangle ABC have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to BC and CA, respectively.

Write down the vector equation of the line through A perpendicular to BC, in terms of $\mathbf{u}$, $\mathbf{a}$ and a parameter λ.

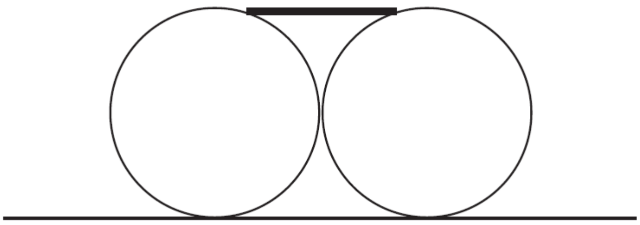
The line through A perpendicular to BC intersects the line through B perpendicular to CA at P. Find the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{u}$.

Hence show that the line CP is perpendicular to the line AB.

Section B: Mechanics

[STEP 2, 2017Q9]

Two identical rough cylinders of radius r and weight W rest, not touching each other but a negligible distance apart, on a horizontal floor. A thin flat rough plank of width $2a$, where $a < r$, and weight kW rests symmetrically and horizontally on the cylinders, with its length parallel to the axes of the cylinders and its faces horizontal. A vertical cross-section is shown in the diagram below.



The coefficient of friction at all four contacts is $\frac{1}{2}$. The system is in equilibrium.

(i) Let F be the frictional force between one cylinder and the floor, and let R be the normal reaction between the plank and one cylinder. Show that

$$R \sin\theta = F(1 + \cos\theta),$$

where θ is the acute angle between the plank and the tangent to the cylinder at the point of contact.

Deduce that $2\sin\theta \leq 1 + \cos\theta$.

(ii) Show that

$$N = \left(1 + \frac{2}{k}\right)\left(1 + \frac{\cos\theta}{\sin\theta}\right)F,$$

where N is the normal reaction between the floor and one cylinder.

Write down the condition that the cylinder does not slip on the floor and show that it is satisfied with no extra restrictions on θ.

(iii) Show that $\sin\theta \leq \frac{4}{5}$ and hence that $r \leq 5a$.

[STEP 2, 2017Q10]

A car of mass m makes a journey of distance $2d$ in a straight line. It experiences air resistance and rolling resistance so that the total resistance to motion when it is moving with speed v is $Av^2 + R$, where A and R are constants.

The car starts from rest and moves with constant acceleration a for a distance d. Show that the work done by the engine for this half of the journey is

$$\int_0^d (ma + R + Av^2)\,\mathrm{d}x$$

and that it can be written in the form

$$\int_0^w \frac{(ma + R + Av^2)v}{a}\,\mathrm{d}v,$$

where $w = \sqrt{2ad}$.

For the second half of the journey, the acceleration of the car is $-a$.

(i) In the case $R > ma$, show that the work done by the engine for the whole journey is

$$2Aad^2 + 2Rd.$$

(ii) In the case $ma - 2Aad < R < ma$, show that at a certain speed the driving force required to maintain the constant acceleration falls to zero.

Thereafter, the engine does not work (and the driver applies the brakes to maintain the constant acceleration). Show that the work done by the engine for the whole journey is

$$2Aad^2 + 2Rd + \frac{(ma - R)^2}{4Aa}.$$

[STEP 2, 2017Q11]

Two thin vertical parallel walls, each of height $2a$, stand a distance a apart on horizontal ground. The projectiles in this question move in a plane perpendicular to the walls.

(i) A particle is projected with speed $\sqrt{5ag}$ towards the two walls from a point A at ground level. It just clears the first wall. By considering the energy of the particle, find its speed when it passes over the first wall.

Given that it just clears the second wall, show that the angle its trajectory makes with the horizontal when it passes over the first wall is 45°.

Find the distance of A from the foot of the first wall.

(ii) A second particle is projected with speed $\sqrt{5ag}$ from a point B at ground level towards the two walls. It passes a distance h above the first wall, where $h > 0$. Show that it does not clear the second wall.

Section C: Probability and Statistics

[STEP 2, 2017Q12]

Adam and Eve are catching fish. The number of fish, X, that Adam catches in a fixed time interval T has a Poisson distribution with parameter λ. The number of fish, Y, that Eve catches in the same time interval has a Poisson distribution with parameter μ. The two Poisson variables are independent.

(i) By considering $\mathrm{P}(X + Y = r)$, show that the total number of fish caught by Adam and Eve in time T also has a Poisson distribution.

(ii) Given that Adam and Eve catch a total of k fish in time T, where k is fixed, show that the number caught by Adam has a binomial distribution.

(iii) Given that Adam and Eve start fishing at the same time, find the probability that the first fish is caught by Adam.

(iv) You are now given that, for a Poisson distribution with parameter θ, the expected time from any starting point until the next event is θ^{-1}.

Find the expected time from the moment Adam and Eve start fishing until they have each caught at least one fish.

[STEP 2, 2017Q13]

In a television game show, a contestant has to open a door using a key. The contestant is given a bag containing n keys, where $n \geq 2$. Only one key in the bag will open the door. There are three versions of the game. In each version, the contestant starts by choosing a key at random from the bag.

(i) In version 1, after each failed attempt at opening the door the key that has been tried is put back into the bag and the contestant again selects a key at random from the bag. By considering the binomial expansion of $(1-q)^{-2}$, or otherwise, find the expected number of attempts required to open the door.

(ii) In version 2, after each failed attempt at opening the door the key that has been tried is put aside and the contestant selects another key at random from the bag. Find the expected number of attempts required to open the door.

(iii) In version 3, after each failed attempt at opening the door the key that has been tried is put back into the bag and another incorrect key is added to the bag. The contestant then selects a key at random from the bag. Show that the probability that the contestant draws the correct key at the kth attempt is

$$\frac{n-1}{(n+k-1)(n+k-2)}.$$

Show also, using partial fractions, that the expected number of attempts required to open the door is infinite.

You may use without proof the result that $\sum_{m=1}^{N} \frac{1}{m} \to \infty$ as $N \to \infty$.

STEP 2 2018

STEP 2 2018
On-line Exam

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TIME ALLOWED: 180 MINUTES

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INFORMATION FOR CANDIDATES

This paper contains **three** sections: **A**, **B**, and **C**.

Section A	Pure Mathematics
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Section C	Probability and Statistics

There are 13 questions in this paper.

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Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 2018Q1]

Show that, if k is a root of the quartic equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0\,, \qquad (*)$$

then k^{-1} is a root.

You are now given that a and b in $(*)$ are both real and are such that the roots are all real.

(i) Write down all the values of a and b for which $(*)$ has only one distinct root.

(ii) Given that $(*)$ has exactly three distinct roots, show that either $b = 2a - 2$ or $b = -2a - 2$.

(iii) Solve $(*)$ in the case $b = 2a - 2$, giving your solutions in terms of a.

Given that a and b are both real and that the roots of $(*)$ are all real, find necessary and sufficient conditions, in terms of a and b, for $(*)$ to have exactly three distinct real roots.

[STEP 2, 2018Q2]

A function $f(x)$ is said to be *concave* for $a < x < b$ if

$$tf(x_1) + (1-t)f(x_2) \leq f(tx_1 + (1-t)x_2),$$

for $a < x_1 < b$, $a < x_2 < b$ and $0 \leq t \leq 1$.

Illustrate this definition by means of a sketch, showing the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, in the case $x_1 < x_2$ and $f(x_1) < f(x_2)$.

Explain why a function $f(x)$ satisfying $f''(x) < 0$ for $a < x < b$ is concave for $a < x < b$.

(i) By choosing t, x_1 and x_2 suitably, show that, if $f(x)$ is concave for $a < x < b$, then

$$f\left(\frac{u+v+w}{3}\right) \geq \frac{f(u)+f(v)+f(w)}{3},$$

for $a < u < b$, $a < v < b$ and $a < w < b$.

(ii) Show that, if A, B and C are the angles of a triangle, then

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

(iii) By considering $\ln(\sin x)$, show that, if A, B and C are the angles of a triangle, then

$$\sin A \times \sin B \times \sin C \leq \frac{3\sqrt{3}}{8}.$$

[STEP 2, 2018Q3]

(i) Let

$$f(x) = \frac{1}{1 + \tan x}$$

for $0 \le x < \frac{1}{2}\pi$.

Show that $f'(x) = -\frac{1}{1+\sin 2x}$ and hence find the range of $f'(x)$.

Sketch the curve $y = f(x)$.

(ii) The function $g(x)$ is continuous for $-1 \le x \le 1$.

Show that the curve $y = g(x)$ has rotational symmetry of order 2 about the point (a, b) on the curve if and only if

$$g(x) + g(2a - x) = 2b.$$

Given that the curve $y = g(x)$ passes through the origin and has rotational symmetry of order 2 about the origin, write down the value of

$$\int_{-1}^{1} g(x)\,\mathrm{d}x.$$

(iii) Show that the curve $y = \frac{1}{1+\tan^k x}$, where k is a positive constant and $0 < x < \frac{1}{2}\pi$, has rotational symmetry of order 2 about a certain point (which you should specify) and evaluate

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1 + \tan^k x}\,\mathrm{d}x.$$

[STEP 2, 2018Q4]

In this question, you may use the following identity without proof:

$$\cos A + \cos B = 2\cos\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B).$$

(i) Given that $0 \le x \le 2\pi$, find all the values of x that satisfy the equation

$$\cos x + 3\cos 2x + 3\cos 3x + \cos 4x = 0.$$

(ii) Given that $0 \le x \le \pi$ and $0 \le y \le \pi$ and that

$$\cos(x + y) + \cos(x - y) - \cos 2x = 1,$$

show that either $x = y$ or x takes one specific value which you should find.

(iii) Given that $0 \le x \le \pi$ and $0 \le y \le \pi$, find the values of x and y that satisfy the equation

$$\cos x + \cos y - \cos(x + y) = \frac{3}{2}.$$

[STEP 2, 2018Q5]

In this question, you should ignore issues of convergence.

(i) Write down the binomial expansion, for $|x| < 1$, of $\frac{1}{1+x}$ and deduce that

$$\ln(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$$

for $|x| < 1$.

(ii) Write down the series expansion in powers of x of e^{-ax}. Use this expansion to show that

$$\int_0^{\infty} \frac{(1-\mathrm{e}^{-ax})\mathrm{e}^{-x}}{x}\,\mathrm{d}x = \ln(1+a) \qquad (|a| < 1).$$

(iii) Deduce the value of

$$\int_0^1 \frac{x^p - x^q}{\ln x}\,\mathrm{d}x \qquad (|p| < 1, \quad |q| < 1)\,.$$

[STEP 2, 2018Q6]

(i) Find all pairs of positive integers (n, p), where p is a prime number, that satisfy

$$n! + 5 = p.$$

(ii) In this part of the question you may use the following two theorems:

1. For $n \geq 7$, $1! \times 3! \times \cdots \times (2n-1)! > (4n)!$.
2. For every positive integer n, there is a prime number between $2n$ and $4n$.

Find all pairs of positive integers (n, m) that satisfy

$$1! \times 3! \times \cdots \times (2n-1)! = m!.$$

[STEP 2, 2018Q7]

The points O, A and B are the vertices of an acute-angled triangle. The points M and N lie on the sides OA and OB respectively, and the lines AN and BM intersect at Q. The position vector of A with respect to O is $\mathbf{a}$, and the position vectors of the other points are labelled similarly.

Given that $|MQ| = \mu|QB|$, and that $|NQ| = \nu|QA|$, where μ and ν are positive and $\mu\nu < 1$, show that

$$\mathbf{m} = \frac{(1+\mu)\nu}{1+\nu}\mathbf{a}.$$

The point L lies on the side OB, and $|OL| = \lambda|OB|$. Given that ML is parallel to AN, express λ in terms of μ and ν.

What is the geometrical significance of the condition $\mu\nu < 1$?

[STEP 2, 2018Q8]

(i) Use the substitution $v = \sqrt{y}$ to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha y^{\frac{1}{2}} - \beta y \qquad\qquad (y \geq 0,\ \ t \geq 0),$$

where α and β are positive constants. Find the non-constant solution $y_1(x)$ that satisfies $y_1(0) = 0$.

(ii) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha y^{\frac{2}{3}} - \beta y \qquad\qquad (y \geq 0,\ \ t \geq 0),$$

where α and β are positive constants. Find the non-constant solution $y_2(x)$ that satisfies $y_2(0) = 0$.

(iii) In the case $\alpha = \beta$, sketch $y_1(x)$ and $y_2(x)$ on the same axes, indicating clearly which is $y_1(x)$ and which is $y_2(x)$. You should explain how you determined the positions of the curves relative to each other.

Section B: Mechanics

[STEP 2, 2018Q9]

Two small beads, A and B, of the same mass, are threaded onto a vertical wire on which they slide without friction, and which is fixed to the ground at P. They are released simultaneously from rest, A from a height of $8h$ above P and B from a height of $17h$ above P.

When A reaches the ground for the first time, it is moving with speed V. It then rebounds with coefficient of restitution $\frac{1}{2}$ and subsequently collides with B at height H above P.

Show that $H = \frac{15}{8}h$ and find, in terms of g and h, the speeds u_A and u_B of the two beads just before the collision.

When A reaches the ground for the second time, it is again moving with speed V. Determine the coefficient of restitution between the two beads.

[STEP 2, 2018Q10]

A uniform elastic string lies on a smooth horizontal table. One end of the string is attached to a fixed peg, and the other end is pulled at constant speed u. At time $t = 0$, the string is taut and its length is a. Obtain an expression for the speed, at time t, of the point on the string which is a distance x from the peg at time t.

An ant walks along the string starting at $t = 0$ at the peg. The ant walks at constant speed v along the string (so that its speed relative to the peg is the sum of v and the speed of the point on the string beneath the ant). At time t, the ant is a distance x from the peg. Write down a first order differential equation for x, and verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{a+ut}\right) = \frac{v}{a+ut}.$$

Show that the time T taken for the ant to reach the end of the string is given by

$$uT = a\left(\mathrm{e}^k - 1\right),$$

where $k = \frac{u}{v}$.

On reaching the end of the string, the ant turns round and walks back to the peg. Find in terms of T and k the time taken for the journey back.

[STEP 2, 2018Q11]

The axles of the wheels of a motorbike of mass m are a distance b apart. Its centre of mass is a horizontal distance of d from the front axle, where $d < b$, and a vertical distance h above the road, which is horizontal and straight. The engine is connected to the rear wheel. The coefficient of friction between the ground and the rear wheel is μ, where $\mu < \frac{b}{h}$, and the front wheel is smooth.

You may assume that the sum of the moments of the forces acting on the motorbike about the centre of mass is zero. By taking moments about the centre of mass show that, as the acceleration of the motorbike increases from zero, the rear wheel will slip before the front wheel loses contact with the road if

$$\mu < \frac{b-d}{h} \qquad (*).$$

If the inequality $(*)$ holds and the rear wheel does not slip, show that the maximum acceleration is

$$\frac{\mu d g}{b - \mu h}.$$

If the inequality $(*)$ does not hold, find the maximum acceleration given that the front wheel remains in contact with the road.

Section C: Probability and Statistics

[STEP 2, 2018Q12]

In a game, I toss a coin repeatedly. The probability, p, that the coin shows Heads on any given toss is given by

$$p = \frac{N}{N+1},$$

where N is a positive integer. The outcomes of any two tosses are independent.

The game has two versions. In each version, I can choose to stop playing after any number of tosses, in which case I win £H, where H is the number of Heads I have tossed. However, the game may end before that, in which case I win nothing.

(i) In version 1, the game ends when the coin first shows Tails (if I haven't stopped playing before that).

I decide from the start to toss the coin until a total of h Heads have been shown, unless the game ends before then. Find, in terms of h and p, an expression for my expected winnings and show that I can maximise my expected winnings by choosing $h = N$.

(ii) In version 2, the game ends when the coin shows Tails on two *consecutive* tosses (if I haven't stopped playing before that).

I decide from the start to toss the coin until a total of h Heads have been shown, unless the game ends before then. Show that my expected winnings are

$$\frac{hN^h(N+2)^h}{(N+1)^{2h}}.$$

In the case $N = 2$, use the approximation $\log_3 2 \approx 0.63$ to show that the maximum value of my expected winnings is approximately £3.

[STEP 2, 2018Q13]

Four children, A, B, C and D, are playing a version of the game 'pass the parcel'. They stand in a circle, so that $ABCDA$ is the clockwise order. Each time a whistle is blown, the child holding the parcel is supposed to pass the parcel immediately exactly one place clockwise. In fact each child, independently of any other past event, passes the parcel clockwise with probability $\frac{1}{4}$, passes it anticlockwise with probability $\frac{1}{4}$ and fails to pass it at all with probability $\frac{1}{2}$. At the start of the game, child A is holding the parcel.

The probability that child A is holding the parcel just after the whistle has been blown for the nth time is A_n, and B_n, C_n and D_n are defined similarly.

(i) Find A_1, B_1, C_1 and D_1. Find also A_2, B_2, C_2 and D_2.

(ii) By first considering $B_{n+1} + D_{n+1}$, or otherwise, find B_n and D_n.

Find also expressions for A_n and C_n in terms of n.

STEP 2 2019

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There is NO Mathematical Formulae booklet.

Calculators are not permitted.

Section A: Pure Mathematics

[STEP 2, 2019Q1]

Let $f(x) = (x - p)g(x)$, where g is a polynomial. Show that the tangent to the curve $y = f(x)$ at the point with $x = a$, where $a \neq p$, passes through the point $(p, 0)$ if and only if $g'(a) = 0$.

The curve C has equation

$$y = A(x - p)(x - q)(x - r),$$

where p, q and r are constants with $p < q < r$, and A is a non-zero constant.

(i) The tangent to C at the point with $x = a$, where $a \neq p$, passes through the point $(p, 0)$. Show that $2a = q + r$ and find an expression for the gradient of this tangent in terms of A, q and r.

(ii) The tangent to C at the point with $x = c$, where $c \neq r$, passes through the point $(r, 0)$. Show that this tangent is parallel to the tangent in part (i) if and only if the tangent to C at the point with $x = q$ does not meet the curve again.

[STEP 2, 2019Q2]

The function f satisfies $f(0) = 0$ and $f'(t) > 0$ for $t > 0$. Show by means of a sketch that, for $x > 0$,

$$\int_0^x f(t)\,\mathrm{d}t + \int_0^{f(x)} f^{-1}(y)\mathrm{d}y = xf(x).$$

(i) The (real) function g is defined, for all t, by

$$\big(g(t)\big)^3 + g(t) = t.$$

Prove that $g(0) = 0$, and that $g'(t) > 0$ for all t.

Evaluate $\int_0^2 g(t)\ \mathrm{d}t$.

(ii) The (real) function h is defined, for all t, by

$$\big(h(t)\big)^3 + h(t) = t + 2.$$

Evaluate $\int_0^8 h(t)\ \mathrm{d}t$.

[STEP 2, 2019Q3]

For any two real numbers x_1 and x_2, show that

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

Show further that, for any real numbers $x_1, x_2, \dots, x_n$,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

(i) The polynomial f is defined by

$$f(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + x^n$$

where the coefficients are real and satisfy $|a_i| \leq A$ for $i = 1, 2, \dots, n-1$, where $A \geq 1$.

(a) If $|x| < 1$, show that

$$|f(x) - 1| \leq \frac{A|x|}{1 - |x|}.$$

(b) Let ω be a real root of f, so that $f(\omega) = 0$. In the case $|\omega| < 1$, show that

$$\frac{1}{1+A} \leq |\omega| \leq 1 + A. \qquad (*)$$

(c) Show further that the inequalities $(*)$ also hold if $|\omega| \geq 1$.

(ii) Find the integer root or roots of the quintic equation

$$135x^5 - 135x^4 - 100x^3 - 91x^2 - 126x + 135 = 0.$$

[STEP 2, 2019Q4]

You are not required to consider issues of convergence in this question.

For any sequence of numbers $a_1, a_2, \dots, a_m, \dots, a_n$, the notation $\prod_{i=m}^{n} a_i$ denotes the product $a_m a_{m+1} \cdots a_n$.

(i) Use the identity $2\cos x \sin x = \sin(2x)$ to evaluate the product $\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)$.

(ii) Simplify the expression

$$\prod_{k=0}^{n} \cos\left(\frac{x}{2^k}\right) \qquad \left(0 < x < \frac{1}{2}\pi\right).$$

Using differentiation, or otherwise, show that, for $0 < x < \frac{1}{2}\pi$,

$$\sum_{k=0}^{n} \frac{1}{2^k} \tan\left(\frac{x}{2^k}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - 2\cot(2x).$$

(iii) Using the results $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$, show that

$$\prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) = \frac{\sin x}{x}$$

and evaluate

$$\sum_{j=2}^{\infty} \frac{1}{2^{j-2}} \tan\left(\frac{\pi}{2^j}\right).$$

[STEP 2, 2019Q5]

The sequence $u_0, u_1, \ldots$ is said to be a *constant sequence* if $u_n = u_{n+1}$ for $n = 0, 1, 2, \ldots$.

The sequence is said to be a *sequence of period* 2 if $u_n = u_{n+2}$ for $n = 0, 1, 2, \ldots$ and the sequence is not constant.

(i) A sequence of real numbers is defined by $u_0 = a$ and $u_{n+1} = f(u_n)$ for $n = 0, 1, 2, \ldots$, where

$$f(x) = p + (x - p)x,$$

and p is a given real number.

Find the values of a for which the sequence is constant.

Show that the sequence has period 2 for some value of a if and only if $p > 3$ or $p < -1$.

(ii) A sequence of real numbers is defined by $u_0 = a$ and $u_{n+1} = f(u_n)$ for $n = 0, 1, 2, \ldots$, where

$$f(x) = q + (x - p)x,$$

and p and q are given real numbers.

Show that there is no value of a for which the sequence is constant if and only if $f(x) > x$ for all x.

Deduce that, if there is no value of a for which the sequence is constant, then there is no value of a for which the sequence has period 2.

Is it true that, if there is no value of a for which the sequence has period 2, then there is no value of a for which the sequence is constant?

[STEP 2, 2019Q6]

Note: You may assume that if the functions $y_1(x)$ and $y_2(x)$ both satisfy one of the differential equations in this question, then the curves $y = y_1(x)$ and $y = y_2(x)$ do not intersect.

(i) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + x + 1$$

that has the form $y = mx + c$, where m and c are constants.

Let $y_3(x)$ be the solution of this differential equation with $y_3(0) = k$. Show that any stationary point on the curve $y = y_3(x)$ lies on the line $y = -x - 1$. Deduce that solution curves with $k < -2$ cannot have any stationary points.

Show further that any stationary point on the solution curve is a local minimum.

Use the substitution $Y = y + x$ to solve the differential equation, and sketch, on the same axes, the solutions with $k = 0$, $k = -2$ and $k = -3$.

(ii) Find the two solutions of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2 - 2xy - 4x + 4y + 3$$

that have the form $y = mx + c$.

Let $y_4(x)$ be the solution of this differential equation with $y_4(0) = -2$. (Do not attempt to find this solution.)

Show that any stationary point on the curve $y = y_4(x)$ lies on one of two lines that you should identify. What can be said about the gradient of the curve at points between these lines?

Sketch the curve $y = y_4(x)$. You should include on your sketch the two straight line solutions and the two lines of stationary points.

[STEP 2, 2019Q7]

(i) The points A, B and C have position vectors a, b and c, respectively. Each of these vectors is a unit vector (so $\mathbf{a} \,.\, \mathbf{a} = 1$, for example) and

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}.$$

Show that $\mathbf{a} \,.\, \mathbf{b} = -\frac{1}{2}$. What can be said about the triangle ABC? You should justify your answer.

(ii) The four distinct points A_i $(i = 1, 2, 3, 4)$ have unit position vectors $\mathbf{a}_i$ and

$$\sum_{i=1}^{4} \mathbf{a}_i = \mathbf{0}.$$

Show that $\mathbf{a}_1 \,.\, \mathbf{a}_2 = \mathbf{a}_3 \,.\, \mathbf{a}_4$.

(a) Given that the four points lie in a plane, determine the shape of the quadrilateral with vertices A_1, A_2, A_3 and A_4.

(b) Given instead that the four points are the vertices of a regular tetrahedron, find the length of the sides of this tetrahedron.

[STEP 2, 2019Q8]

The domain of the function f is the set of all 2×2 matrices and its range is the set of real numbers. Thus, if $\mathbf{M}$ is a 2×2 matrix, then $f(\mathbf{M}) \in \mathbb{R}$.

The function f has the property that $f(\mathbf{MN}) = f(\mathbf{M})f(\mathbf{N})$ for any 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$.

(i) You are given that there is a matrix $\mathbf{M}$ such that $f(\mathbf{M}) \neq 0$. Let $\mathbf{I}$ be the 2×2 identity matrix. By considering $f(\mathbf{MI})$, show that $f(\mathbf{I}) = 1$.

(ii) Let $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. You are given that $f(\mathbf{J}) \neq 1$. By considering $\mathbf{J}^2$, evaluate $f(\mathbf{J})$.

Using $\mathbf{J}$, show that, for any real numbers a, b, c and d,

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = -f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = f\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right).$$

(iii) Let $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ where $k \in \mathbb{R}$. Use $\mathbf{K}$ to show that, if the second row of the matrix $\mathbf{A}$ is a multiple of the first row, then $f(\mathbf{A}) = 0$.

(iv) Let $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By considering the matrices $\mathbf{P}^2$, $\mathbf{P}^{-1}$, and $\mathbf{K}^{-1}\mathbf{PK}$ for suitable values of k, evaluate $f(\mathbf{P})$.

Section B: Mechanics

[STEP 2, 2019Q9]

A particle P is projected from a point O on horizontal ground with speed u and angle of projection α, where $0 < \alpha < \frac{1}{2}\pi$.

(i) Show that if $\sin\alpha < \frac{2\sqrt{2}}{3}$, then the distance OP is increasing throughout the flight.

Show also that if $\sin\alpha > \frac{2\sqrt{2}}{3}$, then OP will be decreasing at some time before the particle lands.

(ii) At the same time as P is projected, a particle Q is projected horizontally from O with speed v along the ground in the opposite direction from the trajectory of P. The ground is smooth. Show that if

$$2\sqrt{2}v > \left(\sin\alpha - 2\sqrt{2}\cos\alpha\right)u,$$

then QP is increasing throughout the flight of P.

[STEP 2, 2019Q10]

A small light ring is attached to the end A of a uniform rod AB of weight W and length $2a$. The ring can slide on a rough horizontal rail.

One end of a light inextensible string of length $2a$ is attached to the rod at B and the other end is attached to a point C on the rail so that the rod makes an angle of θ with the rail, where $0 < \theta < 90°$. The rod hangs in the same vertical plane as the rail.

A force of kW acts vertically downwards on the rod at B and the rod is in equilibrium.

(i) You are given that the string will break if the tension T is greater than W. Show that (assuming that the ring does not slip) the string will break if

$$2k + 1 > 4\sin\theta.$$

(ii) Show that (assuming that the string does not break) the ring will slip if

$$2k + 1 > (2k + 3)\mu\tan\theta,$$

where μ is the coefficient of friction between the rail and the ring.

(iii) You are now given that $\mu\tan\theta < 1$.

Show that, when k is increased gradually from zero, the ring will slip before the string breaks if

$$\mu < \frac{2\cos\theta}{1 + 2\sin\theta}.$$

Section C: Probability and Statistics

[STEP 2, 2019Q11]

(i) The three integers n_1, n_2 and n_3 satisfy $0 < n_1 < n_2 < n_3$ and $n_1 + n_2 > n_3$. Find the number of ways of choosing the pair of numbers n_1 and n_2 in the cases $n_3 = 9$ and $n_3 = 10$.

Given that $n_3 = 2n + 1$, where n is a positive integer, write down an expression (which you need not prove is correct) for the number of ways of choosing the pair of numbers n_1 and n_2. Simplify your expression.

Write down and simplify the corresponding expression when $n_3 = 2n$, where n is a positive integer.

(ii) You have N rods, of lengths $1, 2, 3, \dots, N$ (one rod of each length). You take the rod of length N, and choose two more rods at random from the remainder, each choice of two being equally likely. Show that, in the case $N = 2n + 1$ where n is a positive integer, the probability that these three rods can form a triangle (of non-zero area) is

$$\frac{n-1}{2n-1}.$$

Find the corresponding probability in the case $N = 2n$, where n is a positive integer.

(iii) You have $2M + 1$ rods, of lengths $1, 2, 3, \dots, 2M + 1$ (one rod of each length), where M is a positive integer. You choose three at random, each choice of three being equally likely. Show that the probability that the rods can form a triangle (of non-zero area) is

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}.$$

Note: $\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1)$.

[STEP 2, 2019Q12]

The random variable X has the probability density function on the interval $[0, 1]$:

$$f(x) = \begin{cases} nx^{n-1}, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where n is an integer greater than 1.

(i) Let $\mu = \mathrm{E}(X)$. Find an expression for μ in terms of n, and show that the variance, σ^2, of X is given by

$$\sigma^2 = \frac{n}{(n+1)^2(n+2)}.$$

(ii) In the case $n = 2$, show without using decimal approximations that the interquartile range is less than 2σ.

(iii) Write down the first three terms and the $(k+1)$th term (where $0 \le k \le n$) of the binomial expansion of $(1+x)^n$ in ascending powers of x.

By setting $x = \frac{1}{n}$, show that μ is less than the median and greater than the lower quartile.

Note: You may assume that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots < 4$$

STEP 2 2020

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Section A: Pure Mathematics

[STEP 2, 2020Q1]

(i) Use the substitution $x = \frac{1}{1-u}$, where $0 < u < 1$, to find in terms of x the integral

$$\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}\,\mathrm{d}x \qquad\qquad \text{(where } x > 1\text{).}$$

(ii) Find in terms of x the integral

$$\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\,\mathrm{d}x \qquad\qquad \text{(where } x > 2\text{).}$$

(iii) show that

$$\int_2^{\infty} \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}\,\mathrm{d}x = \frac{1}{3}\pi.$$

[STEP 2, 2020Q2]

The curves C_1 and C_2 both satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kxy - y}{x - kxy},$$

where $k = \ln 2$.

All points on C_1 have positive x and y co-ordinates and C_1 passes through $(1, 1)$. All points on C_2 have negative x and y co-ordinates and C_2 passes through $(-1, -1)$.

(i) Show that the equation of C_1 can be written as $(x-y)^2 = (x+y)^2 - 2^{x+y}$.

Determine a similar result for curve C_2.

Hence show that $y = x$ is a line of symmetry of each curve.

(ii) Sketch on the same axes the curves $y = x^2$ and $y = 2^x$, for $x \geq 0$. Hence show that C_1 lies between the lines $x + y = 2$ and $x + y = 4$.

Sketch curve C_1.

(iii) Sketch curve C_2.

[STEP 2, 2020Q3]

A sequence $u_1, u_2, \dots, u_n$ of positive real numbers is said to be unimodal if there is a value k such that

$$u_1 \leq u_2 \leq \cdots \leq u_k$$

and

$$u_k \geq u_{k+1} \geq \cdots \geq u_n.$$

So the sequences $1, 2, 3, 2, 1$; $1, 2, 3, 4, 5$; $1, 1, 3, 3, 2$ and $2, 2, 2, 2, 2$ are all unimodal, but $1, 2, 1, 3, 1$ is not.

A sequence $u_1, u_2, \dots, u_n$ of positive real numbers is said to have property L if $u_{r-1}u_{r+1} \leq u_r^2$ for all r with $2 \leq r \leq n-1$.

(i) Show that, in any sequence of positive real numbers with property L,

$$u_{r-1} \geq u_r \Longrightarrow ur \geq u_{r+1}.$$

Prove that any sequence of positive real numbers with property L is unimodal.

(ii) A sequence $u_1, u_2, \dots, u_n$ of real numbers satisfies $u_r = 2\alpha u_{r-1} - \alpha^2 u_{r-2}$ for $3 \leq r \leq n$, where α is a positive real constant. Prove that, for $2 \leq r \leq n$,

$$ur - \alpha u_{r-1} = \alpha^{r-2}(u_2 - \alpha u_1)$$

and, for $2 \leq r \leq n-1$,

$$u_r^2 - u_{r-1}u_{r+1} = (u_r - \alpha u_{r-1})^2.$$

Hence show that the sequence consists of positive terms and is unimodal, provided $u_2 > \alpha u_1 > 0$.

In the case $u_1 = 1$ and $u_2 = 2$, prove by induction that $u_r = (2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}$.

Let $\alpha = 1 - \frac{1}{N}$, where N is an integer with $2 \leq N \leq n$.

In the case $u_1 = 1$ and $u_2 = 2$, prove that u_r is largest when $r = N$.

[STEP 2, 2020Q4]

(i) Given that a, b and c are the lengths of the sides of a triangle, explain why $c < a + b$, $a < b + c$ and $b < a + c$.

(ii) Use a diagram to show that the converse of the result in part (i) also holds: if a, b and c are positive numbers such that $c < a + b$, $a < b + c$ and $b < c + a$ then it is possible to construct a triangle with sides of length a, b and c.

(iii) When a, b and c are the lengths of the sides of a triangle, determine in each case whether the following sets of three lengths can

- always
- sometimes but not always
- never

form the sides of a triangle. Prove your claims.

(A) $a + 1, b + 1, c + 1$.

(B) $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$.

(C) $|a - b|, |b - c|, |c - a|$.

(D) $a^2 + bc, b^2 + ca, c^2 + ab$.

(iv) Let f be a function defined on the positive real numbers and such that, whenever $x > y > 0$,

$$f(x) > f(y) > 0 \text{ but } \frac{f(x)}{x} < \frac{f(y)}{y}.$$

Show that, whenever a, b and c are the lengths of the sides of a triangle, then $f(a)$, $f(b)$ and $f(c)$ can also be the lengths of the sides of a triangle.

[STEP 2, 2020Q5]

If x is a positive integer, the value of the function $d(x)$ is the sum of the digits of x in base 10. For example, $d(249) = 2 + 4 + 9 = 15$.

An n-digit positive integer x is written in the form $\sum_{r=0}^{n-1} a_r \times 10^r$, where $0 \le a_r \le 9$ for all $0 \le r \le n - 1$ and $a_{n-1} > 0$.

(i) Prove that $x - d(x)$ is non-negative and divisible by 9.

(ii) Prove that $x - 44d(x)$ is a multiple of 9 if and only if x is a multiple of 9.

Suppose that $x = 44d(x)$. Show that if x has n digits, then $x \le 396n$ and $x \ge 10^{n-1}$, and hence that $n \le 4$.

Find a value of x for which $x = 44d(x)$. Show that there are no further values of x satisfying this equation.

(iii) Find a value of x for which $x = 107d(d(x))$. Show that there are no further values of x satisfying this equation.

[STEP 2, 2020Q6]

A 2×2 matrix $\mathbf{M}$ is real if it can be written as $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are real.

In this case, the *trace* of matrix $\mathbf{M}$ is defined to be $\text{tr}(\mathbf{M}) = a + d$ and $\det(\mathbf{M})$ is the determinant of matrix $\mathbf{M}$. In this question, $\mathbf{M}$ is a real 2×2 matrix.

(i) Prove that

$$\text{tr}(\mathbf{M}^2) = \text{tr}(\mathbf{M})^2 - 2\det(\mathbf{M}).$$

(ii) Prove that

$$\mathbf{M}^2 = \mathbf{I} \text{ but } \mathbf{M} \neq \pm\mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = -1,$$

and that

$$\mathbf{M}^2 = -\mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = 1.$$

(iii) Use part (ii) to prove that

$$\mathbf{M}^4 = \mathbf{I} \iff \mathbf{M}^2 = \pm\mathbf{I}.$$

Find a necessary and sufficient condition on $\det(\mathbf{M})$ and $\text{tr}(\mathbf{M})$ so that $\mathbf{M}^4 = -\mathbf{I}$.

(iv) Give an example of a matrix $\mathbf{M}$ for which $\mathbf{M}^8 = \mathbf{I}$, but which does not represent a rotation or reflection. [Note that the matrices $\pm\mathbf{I}$ are both rotations.]

[STEP 2, 2020Q7]

In this question, $w = \frac{2}{z-2}$.

(i) Let z be the complex number $3 + t\mathrm{i}$, where $t \in \mathbb{R}$. Show that $|w - 1|$ is independent of t. Hence show that, if z is a complex number on the line $\text{Re}(z) = 3$ in the Argand diagram, then w lies on a circle in the Argand diagram with centre 1.

Let V be the line $\text{Re}(z) = p$, where p is a real constant not equal to 2. Show that, if z lies on V, then w lies on a circle whose centre and radius you should give in terms of p. For which z on V is $\text{Im}(w) > 0$?

(ii) Let H be the line $\text{Im}(z) = q$, where q is a non-zero real constant. Show that, if z lies on H, then w lies on a circle whose centre and radius you should give in terms of q. For which z on H is $\text{Re}(w) > 0$?

[STEP 2, 2020Q8]

In this question, $f(x)$ is a quartic polynomial where the coefficient of x^4 is equal to 1, and which has four real roots, 0, a, b and c, where $0 < a < b < c$.

$F(x)$ is defined by $F(x) = \int_0^x f(t)\mathrm{d}t$.

The area enclosed by the curve $y = f(x)$ and the x-axis between 0 and a is equal to that between b and c, and half that between a and b.

(i) Sketch the curve $y = F(x)$, showing the x-co-ordinates of its turning points.

Explain why $F(x)$ must have the form $F(x) = \frac{1}{5}x^2(x-c)^2(x-h)$, where $0 < h < c$.

Find, in factorised form, an expression for $F(x) + F(c - x)$ in terms of c, h and x.

(ii) If $0 \le x \le c$, explain why $F(b) + F(x) \ge 0$ and why $F(b) + F(x) > 0$ if $x \ne a$.

Hence show that $c - b = a$ or $c > 2h$.

By considering also $F(a) + F(x)$, show that $c = a + b$ and that $c = 2h$.

(iii) Find an expression for $f(x)$ in terms of c and x only.

Show that the points of inflection on $y = f(x)$ lie on the x-axis.

Section B: Mechanics

[STEP 2, 2020Q9]

Point A is a distance h above ground level and point N is directly below A at ground level. Point B is also at ground level, a distance d horizontally from N. The angle of elevation of A from B is β. A particle is projected horizontally from A, with initial speed V. A second particle is projected from B with speed U at an acute angle θ above the horizontal. The horizontal components of the velocities of the two particles are in opposite directions. The two particles are projected simultaneously, in the vertical plane through A, N and B.

Given that the two particles collide, show that

$$d\sin\theta - h\cos\theta = \frac{Vh}{U}$$

and also that

(i) $\theta > \beta$.

(ii) $U\sin\theta \geq \sqrt{\frac{gh}{2}}$.

(iii) $\frac{U}{V} > \sin\beta$.

Show that the particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.

[STEP 2, 2020Q10]

A particle P of mass m moves freely and without friction on a wire circle of radius a, whose axis is horizontal. The highest point of the circle is H, the lowest point of the circle is L and angle $PHL = \theta$. A light spring of modulus of elasticity λ is attached to P and to H. The natural length of the spring is l, which is less than the diameter of the circle.

(i) Show that, if there is an equilibrium position of the particle at $\theta = \alpha$, where $\alpha > 0$, then $\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}$.

Show also that there will only be such an equilibrium position if $\lambda > \frac{2mgl}{2a-l}$.

When the particle is at the lowest point L of the circular wire, it has speed u.

(ii) Show that, if the particle comes to rest before reaching H, it does so when $\theta = \beta$, where $\cos\beta$ satisfies

$$(\cos\alpha - \cos\beta)^2 = (1-\cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha\,,$$

where $\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}$.

Show also that this will only occur if $u^2 < \frac{2a\lambda}{m}(2-\sec\alpha)$.

Section C: Probability and Statistics

[STEP 2, 2020Q11]

A coin is tossed repeatedly. The probability that a head appears is p and the probability that a tail appears is $q = 1 - p$.

(i) A and B play a game. The game ends if two successive heads appear, in which case A wins, or if two successive tails appear, in which case B wins.

Show that the probability that the game never ends is 0.

Given that the first toss is a head, show that the probability that A wins is $\frac{p}{1-pq}$.

Find and simplify an expression for the probability that A wins.

(ii) A and B play another game. The game ends if three successive heads appear, in which case A wins, or if three successive tails appear, in which case B wins.

Show that

$$\mathrm{P}(A \text{ wins} \mid \text{the first toss is a head}) = p^2 + (q + pq)\,\mathrm{P}(A \text{ wins} \mid \text{the first toss is a tail})$$

and give a similar result for $\mathrm{P}(A \text{ wins} \mid \text{the first toss is a tail})$.

Show that

$$\mathrm{P}(A \text{ wins}) = \frac{p^2(1-q^3)}{1-(1-p^2)(1-q^2)}.$$

(iii) A and B play a third game. The game ends if a successive heads appear, in which case A wins, or if b successive tails appear, in which case B wins, where a and b are integers greater than 1.

Find the probability that A wins this game.

Verify that your result agrees with part (i) when $a = b = 2$.

[STEP 2, 2020Q12]

The score shown on a biased n-sided die is represented by the random variable X which has distribution $\mathrm{P}(X = i) = \frac{1}{n} + \varepsilon_i$ for $i = 1, 2, \dots, n$, where not all the ε_i are equal to 0.

(i) Find the probability that, when the die is rolled twice, the same score is shown on both rolls. Hence determine whether it is more likely for a fair die or a biased die to show the same score on two successive rolls.

(ii) Use part (i) to prove that, for any set of n positive numbers x_i ($i = 1, 2, \dots, n$),

$$\sum_{i=2}^{n}\sum_{j=1}^{i-1} x_i x_j \le \frac{n-1}{2n}\left(\sum_{i=1}^{n} x_i\right)^2.$$

(iii) Determine, with justification, whether it is more likely for a fair die or a biased die to show the same score on three successive rolls.

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Section A: Pure Mathematics

[STEP 2, 2021Q1]

Prove, from the identities for $\cos(A \pm B)$, that $\cos a \cos 3a \equiv \frac{1}{2}(\cos 4a + \cos 2a)$.

Find a similar identity for $\sin a \cos 3a$.

(i) Solve the equation

$$4 \cos x \cos 2x \cos 3x = 1$$

for $0 \leq x \leq \pi$.

(ii) Prove that if

$$\tan x = \tan 2x \tan 3x \tan 4x \qquad (*)$$

then $\cos 6x = \frac{1}{2}$ or $\sin 4x = 0$.

Hence determine the solutions of equation $(*)$ with $0 \leq x \leq \pi$.

[STEP 2, 2021Q2]

In this question, the numbers a, b and c may be complex.

(i) Let p, q and r be real numbers. Given that there are numbers a and b such that

$$a + b = p, a^2 + b^2 = q \text{ and } a^3 + b^3 = r. \qquad (*)$$

show that $3pq - p^3 = 2r$.

(ii) Conversely, you are given that the real numbers p, q and r satisfy $3pq - p^3 = 2r$. By considering the equation $2x^2 - 2px + (p^2 - q) = 0$, show that there exist numbers a and b such that the three equations $(*)$ hold.

(iii) Let s, t, u and v be real numbers. Given that there are distinct numbers a, b and c such that

$$a + b + c = s, a^2 + b^2 + c^2 = t, a^3 + b^3 + c^3 = u \text{ and } abc = v,$$

show, using part (i), that c is a root of the equation

$$6x^3 - 6sx^2 + 3(s^2 - t)x + 3st - s^3 - 2u = 0$$

and write down the other two roots.

Deduce that $s^3 - 3st + 2u = 6v$.

(iv) Find numbers a, b and c such that

$$a + b + c = 3, a^2 + b^2 + c^2 = 1, a^3 + b^3 + c^3 = -3 \text{ and } abc = 2, \qquad (**)$$

and verify that your solution satisfies the four equations $(**)$.

[STEP 2, 2021Q3]

In this question, x, y and z are real numbers.

Let $\lfloor x \rfloor$ denote the largest integer that satisfies $\lfloor x \rfloor \le x$ and let $\{x\}$ denote the fractional part of x, so that $x = \lfloor x \rfloor + \{x\}$ and $0 \le \{x\} < 1$. For example, if $x = 4.2$, then $\lfloor x \rfloor = 4$ and $\{x\} = 0.2$ and if $x = -4.2$, then $\lfloor x \rfloor = -5$ and $\{x\} = 0.8$.

(i) Solve the simultaneous equations

$$\begin{aligned} \lfloor x \rfloor + \{y\} &= 4.9, \\ \{x\} + \lfloor y \rfloor &= -1.4. \end{aligned}$$

(ii) Given that x, y and z satisfy the simultaneous equations

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 3.9, \\ \{x\} + y + \lfloor z \rfloor &= 5.3, \\ \lfloor x \rfloor + \{y\} + z &= 5. \end{aligned}$$

show that $\{y\} + \lfloor z \rfloor = 3.2$ and solve the equations.

(iii) Solve the simultaneous equations

$$\begin{aligned} x + 2\lfloor y \rfloor + \{z\} &= 3.9, \\ \{x\} + 2y + \lfloor z \rfloor &= 5.3, \\ \lfloor x \rfloor + 2\{y\} + z &= 5. \end{aligned}$$

[STEP 2, 2021Q4]

(i) Sketch the curve $y = xe^x$, giving the coordinates of any stationary points.

(ii) The function f is defined by $f(x) = xe^x$ for $x \ge a$, where a is the minimum possible value such that f has an inverse function. What is the value of a?

Let g be the inverse of f. Sketch the curve $y = g(x)$.

(iii) For each of the following equations, find a real root in terms of a value of the function g, or demonstrate that the equation has no real root. If the equation has two real roots, determine whether the root you have found is greater than or less than the other root.

(a) $e^{-x} = 5x$ (b) $2x \ln x + 1 = 0$ (c) $3x \ln x + 1 = 0$ (d) $x = 3 \ln x$

(iv) Given that the equation $x^x = 10$ has a unique positive root, find this root in terms of a value of the function g.

[STEP 2, 2021Q5]

(i) Use the substitution $y = (x - a)u$, where u is a function of x, to solve the differential equation

$$(x-a)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x,$$

where a is a constant.

(ii) The curve C with equation $y = f(x)$ has the property that, for all values of t except $t = 1$, the tangent at the point $\big(t, f(t)\big)$ passes through the point $(1, t)$.

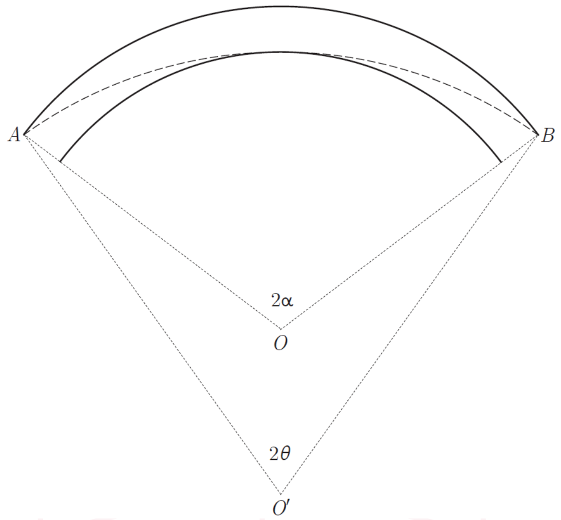
(a) Given that $f(0) = 0$, find $f(x)$ for $x < 1$.

Sketch C for $x < 1$. You should find the co-ordinates of any stationary points and consider the gradient of C as $x \longrightarrow 1$. You may assume that $z\ln|z| \to 0$ as $z \to 0$.

(b) Given that $f(2) = 2$, sketch C for $x > 1$, giving the co-ordinates of any stationary points.

[STEP 2, 2021Q6]

A plane circular road is bounded by two concentric circles with centres at point O. The inner circle has radius R and the outer circle has radius $R + w$. The points A and B lie on the outer circle, as shown in the diagram, with $\angle AOB = 2\alpha$, $\frac{1}{3}\pi \leq \alpha \leq \frac{1}{2}\pi$ and $0 < w < R$.



(i) Show that I cannot cycle from A to B in a straight line, while remaining on the road.

(ii) I take a path from A to B that is an arc of a circle. This circle is tangent to the inner edge of the road, and has radius $R + d$ (where $d > w$) and centre O'.

My path is represented by the dashed arc in the above diagram.

Let $\angle AO'B = 2\theta$.

(a) Use the cosine rule to find d in terms of w, R and $\cos\alpha$.

(b) Find also an expression for $\sin(\alpha - \theta)$ in terms of R, d and $\sin\alpha$.

You are now given that $\frac{w}{R}$ is much less than 1.

(iii) Show that $\frac{d}{R}$ and $\alpha - \theta$ are also both much less than 1.

(iv) My friend cycles from A to B along the outer edge of the road.

Let my path be shorter than my friend's path by distance S. Show that

$$S = 2(R + d)(\alpha - \theta) + 2\alpha(w - d).$$

Hence show that S is approximately a fraction

$$\left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha(1 - \cos\alpha)}\right)\frac{w}{R}$$

of the length of my friend's path.

[STEP 2, 2021Q7]

(i) The matrix $\mathbf{R}$ represents an anticlockwise rotation through angle ϕ ($0° \leq \phi < 360°$) in two dimensions, and the matrix $\mathbf{R} + \mathbf{I}$ also represents a rotation in two dimensions. Determine the possible values of ϕ and deduce that $\mathbf{R}^3 = \mathbf{I}$.

(ii) Let S be a real matrix with $\mathbf{S}^3 = \mathbf{I}$, but $\mathbf{S} \neq \mathbf{I}$.

Show that $\det(\mathbf{S}) = 1$.

Given that

$$\mathbf{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

show that $\mathbf{S}^2 = (a + d)\mathbf{S} - \mathbf{I}$.

Hence prove that $a + d = -1$.

(iii) Let $\mathbf{S}$ be a real 2×2 matrix.

Show that if $\mathbf{S}^3 = \mathbf{I}$ and $\mathbf{S} + \mathbf{I}$ represents a rotation, then $\mathbf{S}$ also represents a rotation. What are the possible angles of the rotation represented by $\mathbf{S}$?

[STEP 2, 2021Q8]

(i) Show that, for $n = 2, 3, 4, \ldots$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(t^n(1-t)^n) = nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)].$$

(ii) The sequence $T_0, T_1, \ldots$ is defined by

$$T_n = \int_0^1 \frac{t^n(1-t)^n}{n!}\mathrm{e}^t\,\mathrm{d}t$$

Show that, for $n \geq 2$,

$$T_n = T_{n-2} - 2(2n-1)T_{n-1}.$$

(iii) Evaluate T_0 and T_1 and deduce that, for $n \geq 0$, T_n can be written in the form

$$T_n = a_n + b_n\mathrm{e},$$

where a_n and b_n are integers (which you should not attempt to evaluate).

(iv) Show that $0 < T_n < \frac{\mathrm{e}}{n!}$ for $n \geq 0$. Given that b_n is non-zero for all n, deduce that $\frac{-a_n}{b_n}$ tends to e as n tends to infinity.

Section B: Mechanics

[STEP 2, 2021Q9]

Two particles, of masses m_1 and m_2 where $m_1 > m_2$, are attached to the ends of a light, inextensible string. A particle of mass M is fixed to a point P on the string. The string passes over two small, smooth pulleys at Q and R, where QR is horizontal, so that the particle of mass m_1 hangs vertically below Q and the particle of mass m_2 hangs vertically below R. The particle of mass M hangs between the two pulleys with the section of the string PQ making an acute angle of θ_1 with the upward vertical and the section of the string PR making an acute angle of θ_2 with the upward vertical. S is the point on QR vertically above P. The system is in equilibrium.

(i) Using a triangle of forces, or otherwise, show that:

(a)

$$\sqrt{m_1^2 - m_2^2} < M < m_1 + m_2;$$

(b) S divides QR in the ratio $r : 1$, where

$$r = \frac{M^2 - m_1^2 + m_2^2}{M^2 - m_2^2 + m_1^2}.$$

(ii) You are now given that $M^2 = m_1^2 + m_2^2$.

Show that $\theta_1 + \theta_2 = 90°$ and determine the ratio of QR to SP in terms of the masses only.

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[STEP 2, 2021Q10]

A train moves westwards on a straight horizontal track with constant acceleration a, where $a > 0$. Axes are chosen as follows: the origin is fixed in the train; the x-axis is in the direction of the track with the positive x-axis pointing to the East; and the positive y-axis points vertically upwards.

A smooth wire is fixed in the train. It lies in the x-y plane and is bent in the shape given by $ky = x^2$, where k is a positive constant. A small bead is threaded onto the wire. Initially, the bead is held at the origin. It is then released.

(i) Explain why the bead cannot remain stationary relative to the train at the origin.

(ii) Show that, in the subsequent motion, the coordinates (x, y) of the bead satisfy

$$\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = 0$$

and deduce that $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ is constant during the motion.

(iii) Find an expression for the maximum vertical displacement, b, of the bead from its initial position in terms of a, k and g.

(iv) Find the value of x for which the speed of the bead relative to the train is greatest and give this maximum speed in terms of a, k and g.

Section C: Probability and Statistics

[STEP 2, 2021Q11]

A train has n seats, where $n \geq 2$. For a particular journey, all n seats have been sold, and each of the n passengers has been allocated a seat.

The passengers arrive one at a time and are labelled $T_1, \dots, T_n$ according to the order in which they arrive: T_1 arrives first and T_n arrives last. The seat allocated to T_r $(r = 1, \dots, n)$ is labelled S_r.

Passenger T_1 ignores their allocation and decides to choose a seat at random (each of the n seats being equally likely). However, for each $r \geq 2$, passenger T_r sits in S_r if it is available or, if S_r is not available, chooses from the available seats at random.

(i) Let P_n be the probability that, in a train with n seats, T_n sits in S_n. Write down the value of P_2 and find the value of P_3.

(ii) Explain why, for $k = 2, 3, \dots, n-1$,

$$\mathrm{P}(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_k) = P_{n-k+1},$$

and deduce that, for $n \geq 3$,

$$P_n = \frac{1}{n}\left(1 + \sum_{r=2}^{n-1} P_r\right).$$

(iii) Give the value of P_n in its simplest form and prove your result by induction.

(iv) Let Q_n be the probability that, in a train with n seats, T_{n-1} sits in S_{n-1}. Determine Q_n for $n \geq 2$.

[STEP 2, 2021Q12]

(i) A game for two players, A and B, can be won by player A, with probability p_A, won by player B, with probability p_B, where $0 < p_A + p_B < 1$, or drawn. A match consists of a series of games and is won by the first player to win a game. Show that the probability that A wins the match is

$$\frac{p_A}{p_A + p_B}.$$

(ii) A second game for two players, A and B, can be won by player A, with probability p, or won by player B, with probability $q = 1 - p$. A match consists of a series of games and is won by the first player to have won two more games than the other. Show that the match is won after an even number of games, and that the probability that A wins the match is

$$\frac{p^2}{p^2 + q^2}.$$

(iii) A third game, for only one player, consists of a series of rounds. The player starts the game with one token, wins the game if they have four tokens at the end of a round and loses the game if they have no tokens at the end of a round. There are two versions of the game. In the cautious version, in each round where the player has any tokens, the player wins one token with probability p and loses one token with probability $q = 1 - p$. In the bold version, in each round where the player has any tokens, the player's tokens are doubled in number with probability p and all lost with probability $q = 1 - p$.

In each of the two versions of the game, find the probability that the player wins.

Hence show that the player is more likely to win in the cautious version if $1 > p > \frac{1}{2}$ and more likely to win in the bold version if $0 < p < \frac{1}{2}$.

STEP 2 2022

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Section A: Pure Mathematics

[STEP 2, 2022Q1]

(i) By integrating one of the two terms in the integrand by parts, or otherwise, find

$$\int \left(2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}} \right) \mathrm{d}x.$$

(ii) Find

$$\int (x^2+2)\frac{\sin x}{x^3}\,\mathrm{d}x.$$

(iii) (a) Sketch the graph with equation $y = \frac{\mathrm{e}^x}{x}$, giving the coordinates of any stationary points.

(b) Find a if

$$\int_a^{2a} \frac{\mathrm{e}^x}{x}\,\mathrm{d}x = \int_a^{2a} \frac{\mathrm{e}^x}{x^2}\,\mathrm{d}x.$$

(c) Show that it is not possible to find distinct integers m and n such that

$$\int_m^{n} \frac{\mathrm{e}^x}{x}\,\mathrm{d}x = \int_m^{n} \frac{\mathrm{e}^x}{x^2}\,\mathrm{d}x.$$

[STEP 2, 2022Q2]

A sequence u_n, where $n = 1, 2, \ldots$, is said to have *degree* d if u_n, as a function of n, is a polynomial of degree d.

(i) Show that, in any sequence u_n $(n = 1, 2, \ldots)$ that satisfies $u_{n+1} = \frac{1}{2}(u_{n+2} + u_n)$ for all $n \geq 1$, there is a constant difference between successive terms.

Deduce that any sequence u_n for which $u_{n+1} = \frac{1}{2}(u_{n+2} + u_n)$, for all $n \geq 1$, has degree at most 1.

(ii) The sequence v_n $(n = 1, 2, \ldots)$ satisfies $v_{n+1} = \frac{1}{2}(v_{n+2} + v_n) - p$ for all $n \geq 1$, where p is a non-zero constant. By writing $v_n = t_n + pn^2$, show that the sequence v_n has degree 2.

Given that $v_1 = v_2 = 0$, find v_n in terms of n and p.

(iii) The sequence w_n $(n = 1, 2, \ldots)$ satisfies $w_{n+1} = \frac{1}{2}(w_{n+2} + w_n) - an - b$ for all $n \geq 1$, where a and b are constants with $a \neq 0$. Show that the sequence w_n has degree 3.

Given that $w_1 = w_2 = 0$, find w_n in terms of n, a and b.

[STEP 2, 2022Q3]

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and, for $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$.

(i) Prove that $F_r \leq 2^{r-n} F_n$ for all $n \geq 1$ and all $r \geq n$.

(ii) Let $S_n = \sum_{r=1}^{n} \frac{F_r}{10^r}$.

Show that

$$\sum_{r=1}^{n} \frac{F_{r+1}}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_r}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_{r-1}}{10^{r-1}} = 89S_n - 10F_1 - F_0 + \frac{F_n}{10^n} + \frac{F_{n+1}}{10^{n-1}}.$$

(iii) Show that $\sum_{r=1}^{\infty} \frac{F_r}{10^r} = \frac{10}{89}$ and that $\sum_{r=7}^{\infty} \frac{F_r}{10^r} < 2 \times 10^{-6}$. Hence find, with justification, the first six digits after the decimal point in the decimal expansion of $\frac{1}{89}$.

(iv) Find, with justification, a number of the form $\frac{r}{s}$ with r and s both positive integers less than 10000 whose decimal expansion starts

$$0.0001010203050813213455\ldots$$

[STEP 2, 2022Q4]

(i) Show that the function f, given by the single formula $f(x) = |x| - |x-5| + 1$, can be written without using modulus signs as

$$f(x) = \begin{cases} -4 & x \leq 0, \\ 2x - 4 & 0 \leq x \leq 5, \\ 6 & 5 \leq x. \end{cases}$$

Sketch the graph with equation $y = f(x)$.

(ii) The function g is given by:

$$g(x) = \begin{cases} -x & x \leq 0, \\ 3x & 0 \leq x \leq 5, \\ x + 10 & 5 \leq x. \end{cases}$$

Use modulus signs to write $g(x)$ as a single formula.

(iii) Sketch the graph with equation $y = h(x)$, where $h(x) = x^2 - x - 4|x| + |x(x-5)|$.

(iv) The function k is given by:

$$k(x) = \begin{cases} 10x & x \leq 0, \\ 2x^2 & 0 \leq x \leq 5, \\ 50 & 5 \leq x. \end{cases}$$

Use modulus signs to write $k(x)$ as a single formula, explicitly verifying that your formula is correct.

[STEP 2, 2022Q5]

(i) Given that $a > b > c > 0$ are constants, and that x, y, z are non-negative variables, show that

$$ax + by + cz \leq a(x + y + z).$$

In the acute-angled triangle ABC, a, b and c are the lengths of sides BC, CA and AB, respectively, with $a > b > c$. P is a point inside, or on the sides of, the triangle, and x, y and z are the perpendicular distances from P to BC, CA and AB, respectively. The area of the triangle is Δ.

(ii) (a) Find Δ in terms of a, b, c, x, y and z.

(b) Find both the minimum value of the sum of the perpendicular distances from P to the three sides of the triangle and the values of x, y and z which give this minimum sum, expressing your answers in terms of some or all of a, b, c and Δ.

(iii) (a) Show that, for all real a, b, c, x, y and z,

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 + (ax + by + cz)^2.$$

(b) Find both the minimum value of the sum of the squares of the perpendicular distances from P to the three sides of the triangle and the values of x, y and z which give this minimum sum, expressing your answers in terms of some or all of a, b, c and Δ.

(iv) Find both the maximum value of the sum of the squares of the perpendicular distances from P to the three sides of the triangle and the values of x, y and z which give this maximum sum, expressing your answers in terms of some or all of a, b, c and Δ.

[STEP 2, 2022Q6]

In this question, you should consider only points lying in the first quadrant, that is with $x > 0$ and $y > 0$.

(i) The equation $x^2 + y^2 = 2ax$ defines a *family* of curves in the first quadrant, one curve for each positive value of a. A second family of curves in the first quadrant is defined by the equation $x^2 + y^2 = 2by$, where $b > 0$.

(a) Differentiate the equation $x^2 + y^2 = 2ax$ implicitly with respect to x, and hence show that every curve in the first family satisfies the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2.$$

Find similarly a differential equation, independent of b, for the second family of curves.

(b) Hence, or otherwise, show that, at every point with $y \neq x$ where a curve in the first family meets a curve in the second family, the tangents to the two curves are perpendicular.

A curve in the first family meets a curve in the second family at (c, c), where $c > 0$. Find the equations of the tangents to the two curves at this point. Is it true that where a curve in the first family meets a curve in the second family on the line $y = x$, the tangents to the two curves are perpendicular?

(ii) Given the family of curves in the first quadrant $y = c\ln x$, where c takes any non-zero value, find, by solving an appropriate differential equation, a second family of curves with the property that at every point where a curve in the first family meets a curve in the second family, the tangents to the two curves are perpendicular.

(iii) A family of curves in the first quadrant is defined by the equation $y^2 = 4k(x + k)$, where k takes any non-zero value.

Show that, at every point where one curve in this family meets a second curve in the family, the tangents to the two curves are perpendicular.

[STEP 2, 2022Q7]

Let $h(z) = nz^6 + z^5 + z + n$, where z is a complex number and $n \geq 2$ is an integer.

(i) Let w be a root of the equation $h(z) = 0$.

(a) Show that $|w^5| = \sqrt{\frac{f(w)}{g(w)}}$, where

$$f(z) = n^2 + 2n\text{Re}(z) + |z|^2 \text{ and } g(z) = n^2|z|^2 + 2n\text{Re}(z) + 1.$$

(b) By considering $f(w) - g(w)$, prove by contradiction that $|w| \geq 1$.

(c) Show that $|w| = 1$.

(ii) It is given that the equation $h(z) = 0$ has six distinct roots, none of which is purely real.

(a) Show that $h(z)$ can be written in the form

$$h(z) = n(z^2 - a_1 z + 1)(z^2 - a_2 z + 1)(z^2 - a_3 z + 1),$$

where a_1, a_2 and a_3 are real constants.

(b) Find $a_1 + a_2 + a_3$ in terms of n.

(c) By considering the coefficient of z^3 in $h(z)$, find $a_1 a_2 a_3$ in terms of n.

(d) How many of the six roots of the equation $h(z) = 0$ have a negative real part? Justify your answer.

[STEP 2, 2022Q8]

Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $a \neq d$. The transformation represented by $\mathbf{M}$ has exactly two distinct invariant lines through the origin.

(i) Show that, if neither invariant line is the y-axis, then the gradients of the invariant lines are the roots of the equation

$$bm^2 + (a - d)m - c = 0.$$

If one invariant line is the y-axis, what is the gradient of the other?

(ii) Show that, if the angle between the two invariant lines is 45°, then

$$(a - d)^2 = (b - c)^2 - 4bc.$$

(iii) Find a necessary and sufficient condition, on some or all of a, b, c and d, for the two invariant lines to make equal angles with the line $y = x$.

(iv) Give an example of a matrix which satisfies both the conditions in parts (ii) and (iii).

Section B: Mechanics

[STEP 2, 2022Q9]

A rectangular prism is fixed on a horizontal surface. A vertical wall, parallel to a vertical face of the prism, stands at a distance d from it. A light plank, making an acute angle θ with the horizontal, rests on an upper edge of the prism and is in contact with the wall below the level of that edge of the prism and above the level of the horizontal plane. You may assume that the plank is long enough and the prism high enough to make this possible.

The contact between the plank and the prism is smooth, and the coefficient of friction at the contact between the plank and the wall is μ. When a heavy point mass is fixed to the plank at a distance x, along the plank, from its point of contact with the wall, the system is in equilibrium.

(i) Show that, if $x = d\sec^3\theta$, then there is no frictional force acting between the plank and the wall.

(ii) Show that, if $x > d\sec^3\theta$, it is necessary that

$$\mu \geq \frac{x - d\sec^3\theta}{x\tan\theta}$$

and give the corresponding inequality if $x < d\sec^3\theta$.

(iii) Show that

$$\frac{x}{d} \geq \frac{\sec^3\theta}{1 + \mu\tan\theta}.$$

Show also that, if $\mu < \cot\theta$, then

$$\frac{x}{d} \leq \frac{\sec^3\theta}{1 - \mu\tan\theta}.$$

(iv) Show that if x is such that the point mass is fixed to the plank somewhere between the edge of the prism and the wall, then $\tan\theta < \mu$.

[STEP 2, 2022Q10]

(i) Show that, if a particle is projected at an angle α above the horizontal with speed u, it will reach height h at a horizontal distance s from the point of projection where

$$h = s\tan\alpha - \frac{gs^2}{2u^2\cos^2\alpha}.$$

The remainder of this question uses axes with the x- and y-axes horizontal and the z-axis vertically upwards. The ground is a sloping plane with equation $z = y\tan\theta$ and a road runs along the x-axis. A cannon, which may have any angle of inclination and be pointed in any direction, fires projectiles from ground level with speed u. Initially, the cannon is placed at the origin.

(ii) Let a point P on the plane have coordinates $(x, y, y\tan\theta)$. Show that the condition for it to be possible for a projectile from the cannon to land at point P is

$$x^2 + \left(y + \frac{u^2\tan\theta}{g}\right)^2 \leq \frac{u^4\sec^2\theta}{g^2}.$$

(iii) Show that the furthest point directly up the plane that can be reached by a projectile from the cannon is a distance

$$\frac{u^2}{g(1+\sin\theta)}$$

from the cannon.

How far from the cannon is the furthest point directly down the plane that can be reached by a projectile from it?

(iv) Find the length of road which can be reached by projectiles from the cannon.

The cannon is now moved to a point on the plane vertically above the y-axis, and a distance r from the road. Find the value of r which maximises the length of road which can be reached by projectiles from the cannon. What is this maximum length?

Section C: Probability and Statistics

[STEP 2, 2022Q11]

A batch of N USB sticks is to be used on a network. Each stick has the same unknown probability p of being infected with a virus. Each stick is infected, or not, independently of the others.

The network manager decides on an integer value of T with $0 \leq T < N$. If $T = 0$ no testing takes place and the N sticks are used on the network, but if $T > 0$, the batch is subject to the following procedure.

- Each of T sticks, chosen at random from the batch, undergoes a test during which it is destroyed.
- If any of these T sticks is infected, all the remaining $N - T$ sticks are destroyed.
- If none of the T sticks is infected, the remaining $N - T$ sticks are used on the network.

If any stick used on the network is infected, the network has to be disinfected at a cost of £D, where $D > 0$. If no stick used on the network is infected, there is a gain of £1 for each of the $N - T$ sticks. There is no cost to testing or destroying a stick.

(i) Find an expression in terms of N, T, D and q, where $q = 1 - p$, for the expected net loss.

(ii) Let $\alpha = \frac{DT}{N(N-T+D)}$. Show that $0 \leq \alpha < 1$.

Show that, for fixed values of N, D and T, the greatest value of the expected net loss occurs when q satisfies the equation $q^{N-T} = \alpha$.

Show further that this greatest value is £$\frac{D(N-T)\alpha^k}{N}$, where $k = \frac{T}{N-T}$.

(iii) For fixed values of N and D, show that there is some $\beta > 0$ so that for all $p < \beta$, the expression for the expected loss found in part (i) is an increasing function of T. Deduce that, for small enough values of p, testing no sticks minimises the expected net loss.

[STEP 2, 2022Q12]

The random variable X has probability density function

$$f(x) = \begin{cases} kx^n(1-x) & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where n is an integer greater than 1.

(i) Show that $k = (n+1)(n+2)$ and find μ, where $\mu = \mathrm{E}(X)$.

(ii) Show that μ is less than the median of X if

$$6 - \frac{8}{n+3} < \left(1 + \frac{2}{n+1}\right)^{n+1}.$$

By considering the first four terms of the expansion of the right-hand side of this inequality, or otherwise, show that the median of X is greater than μ.

(iii) You are given that, for positive x, $\left(1 + \frac{1}{x}\right)^{x+1}$ is a decreasing function of x.

Show that the mode of X is greater than its median.

STEP 2 2023

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Section A: Pure Mathematics

[STEP 2, 2023Q1]

(i) Show that making the substitution $x = \frac{1}{t}$ in the integral

$$\int_a^b \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x$$

where $b > a > 0$, gives the integral

$$\int_{a^{-1}}^{b^{-1}} \frac{-t}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t.$$

(ii) Evaluate:

(a)

$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x.$$

(b)

$$\int_{-2}^{2} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x.$$

(iii) (a) Show that

$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \int_{\frac{1}{2}}^{2} \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{1+x^2}\,\mathrm{d}x$$

and hence evaluate

$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^2}\,\mathrm{d}x.$$

(b) Evaluate

$$\int_{\frac{1}{2}}^{2} \frac{1-x}{x(1+x^2)^{\frac{1}{2}}}\,\mathrm{d}x.$$

[STEP 2, 2023Q2]

(i) The real numbers x, y and z satisfy the equations

$$y = \frac{2x}{1-x^2}\,,$$
$$z = \frac{2y}{1-y^2}\,,$$
$$x = \frac{2z}{1-z^2}\,.$$

Let $x = \tan\alpha$. Deduce that $y = \tan 2\alpha$ and show that $\tan\alpha = \tan 8\alpha$.

Find all solutions of the equations, giving each value of x, y and z in the form $\tan\theta$ where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(ii) Determine the number of real solutions of the simultaneous equations

$$y = \frac{3x-x^3}{1-3x^2}\,,$$
$$z = \frac{3y-y^3}{1-3y^2}\,,$$
$$x = \frac{3z-z^3}{1-3z^2}\,.$$

(iii) Consider the simultaneous equations

$$y = 2x^2 - 1$$
$$z = 2y^2 - 1$$
$$x = 2z^2 - 1$$

(a) Determine the number of real solutions of these simultaneous equations with $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$.

(b) By finding the degree of a single polynomial equation which is satisfied by x, show that all solutions of these simultaneous equations have $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$.

[STEP 2, 2023Q3]

Let $p(x)$ be a polynomial of degree n with $p(x) > 0$ for all x and let

$$q(x) = \sum_{k=0}^{n} p^{(k)}(x),$$

where $p^{(k)}(x) \equiv \frac{\mathrm{d}^k p(x)}{\mathrm{d}x^k}$ for $k \geq 1$ and $p^{(0)}(x) \equiv p(x)$.

(i) (a) Explain why n must be even and show that $q(x)$ takes positive values for some values of x.

(b) Show that $q'(x) = q(x) - p(x)$.

(ii) In this part you will be asked to show the same result in three different ways.

(a) Show that the curves $y = p(x)$ and $y = q(x)$ meet at every stationary point of $y = q(x)$.

Hence show that $q(x) > 0$ for all x.

(b) Show that $\mathrm{e}^{-x}q(x)$ is a decreasing function.

Hence show that $q(x) > 0$ for all x.

(c) Show that

$$\int_0^\infty p(x+t)\mathrm{e}^{-t}\,\mathrm{d}t = p(x) + \int_0^\infty p^{(1)}(x+t)\mathrm{e}^{-t}\,\mathrm{d}t.$$

Show further that

$$\int_0^\infty p(x+t)\mathrm{e}^{-t}\,\mathrm{d}t = q(x).$$

Hence show that $q(x) > 0$ for all x.

[STEP 2, 2023Q4]

(i) Show that, if $\left(x - \sqrt{2}\right)^2 = 3$, then $x^4 - 10x^2 + 1 = 0$.

Deduce that, if $f(x) = x^4 - 10x^2 + 1$, then $f\left(\sqrt{2} + \sqrt{3}\right) = 0$.

(ii) Find a polynomial g of degree 8 with integer coefficients such that $g\left(\sqrt{2} + \sqrt{3} + \sqrt{5}\right) = 0$. Write your answer in a form without brackets.

(iii) Let a, b and c be the three roots of $t^3 - 3t + 1 = 0$.

Find a polynomial h of degree 6 with integer coefficients such that $h\left(a + \sqrt{2}\right) = 0$, $h\left(b + \sqrt{2}\right) = 0$ and $h\left(c + \sqrt{2}\right) = 0$. Write your answer in a form without brackets.

(iv) Find a polynomial k with integer coefficients such that $k\left(\sqrt[3]{2} + \sqrt[3]{3}\right) = 0$. Write your answer in a form without brackets.

[STEP 2, 2023Q5]

(i) The sequence x_n for $n = 0,1,2,\ldots$ is defined by $x_0 = 1$ and by

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}$$

for $n \geq 0$.

(a) Explain briefly why $x_n \geq 1$ for all n.

(b) Show that $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite sign, and that

$$\left|x_{n+1}^2 - 2\right| \leq \frac{1}{4}\left|x_n^2 - 2\right|.$$

(c) Show that

$$2 - 10^{-6} \leq x_{10}^2 \leq 2.$$

(ii) The sequence y_n for $n = 0,1,2,\ldots$ is defined by $y_0 = 1$ and by

$$y_{n+1} = \frac{y_n^2 + 2}{2y_n}$$

for $n \geq 0$.

(a) Show that, for $n \geq 0$,

$$y_{n+1} - \sqrt{2} = \frac{\left(y_n - \sqrt{2}\right)^2}{2y_n}$$

and deduce that $y_n \geq 1$ for $n \geq 0$.

(b) Show that

$$y_n - \sqrt{2} \leq 2\left(\frac{\sqrt{2} - 1}{2}\right)^{2^n}$$

for $n \geq 1$.

(c) Using the fact that

$$\sqrt{2} - 1 < \frac{1}{2},$$

or otherwise, show that

$$\sqrt{2} \leq y_{10} \leq \sqrt{2} + 10^{-600}.$$

[STEP 2, 2023Q6]

The sequence F_n, for $n = 0,1,2, \ldots$, is defined by $F_0 = 0$, $F_1 = 1$ and by $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

Prove by induction that, for all positive integers n,

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \mathbf{Q}^n,$$

where the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) By considering the matrix $\mathbf{Q}^n$, show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all positive integers n.

(ii) By considering the matrix $\mathbf{Q^{m+n}}$, show that $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$ for all positive integers m and n.

(iii) Show that $\mathbf{Q}^2 = \mathbf{I} + \mathbf{Q}$.

In the following parts, you may use without proof the Binomial Theorem for matrices:

$$(\mathbf{I} + \mathbf{A})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{A}^k.$$

(a) Show that, for all positive integers n,

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k.$$

(b) Show that, for all positive integers n,

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k.$$

and also that

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}.$$

(c) Show that, for all positive integers n,

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} F_{n+k} = 0.$$

[STEP 2, 2023Q7]

(i) The complex numbers z and w have real and imaginary parts given by $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$. Prove that $|zw| = |z||w|$.

(ii) By considering the complex numbers $2 + \mathrm{i}$ and $10 + 11\mathrm{i}$, find positive integers h and k such that $h^2 + k^2 = 5 \times 221$.

(iii) Find positive integers m and n such that $m^2 + n^2 = 8045$.

(iv) You are given that $102^2 + 201^2 = 50805$.

Find positive integers p and q such that $p^2 + q^2 = 36 \times 50805$.

(v) Find three distinct pairs of positive integers r and s such that $r^2 + s^2 = 25 \times 1002082$ and $r < s$.

(vi) You are given that $109 \times 9193 = 1002037$.

Find positive integers t and u such that $t^2 + u^2 = 9193$.

[STEP 2, 2023Q8]

A tetrahedron is called isosceles if each pair of edges which do not share a vertex have equal length.

(i) Prove that a tetrahedron is isosceles if and only if all four faces have the same perimeter.

Let $OABC$ be an isosceles tetrahedron and let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(ii) By considering the lengths of OA and BC, show that

$$2\mathbf{b}.\mathbf{c} = |b|^2 + |\mathbf{c}|^2 - |\mathbf{a}|^2.$$

Show that

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = |\mathbf{a}|^2.$$

(iii) Let G be the centroid of the tetrahedron, defined by $\overrightarrow{OG} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

Show that G is equidistant from all four vertices of the tetrahedron.

(iv) By considering the length of the vector $\mathbf{a} - \mathbf{b} - \mathbf{c}$, or otherwise, show that, in an isosceles tetrahedron, none of the angles between pairs of edges which share a vertex can be obtuse. Can any of them be right angles?

Section B: Mechanics

[STEP 2, 2023Q9]

A truck of mass M is connected by a light, rigid tow-bar, which is parallel to the ground, to a trailer of mass kM. A constant driving force D which is parallel to the ground acts on the truck, and the only resistance to motion is a frictional force acting on the trailer, with coefficient of friction μ.

- When the truck pulls the trailer up a slope which makes an angle α to the horizontal, the acceleration is a_1 and there is a tension T_1 in the tow-bar.
- When the truck pulls the trailer on horizontal ground, the acceleration is a_2 and there is a tension T_2 in the tow-bar.
- When the truck pulls the trailer down a slope which makes an angle α to the horizontal, the acceleration is a_3 and there is a tension T_3 in the tow-bar.

All accelerations are taken to be positive when in the direction of motion of the truck.

(i) Show that $T_1 = T_3$ and that $M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$.

(ii) It is given that $\mu < 1$.

(a) Show that

$$a_2 < \frac{1}{2}(a_1 + a_3) < a_3.$$

(b) Show further that

$$a_1 < a_2.$$

[STEP 2, 2023Q10]

In this question, the x- and y-axes are horizontal and the z-axis is vertically upwards.

(i) A particle P_α is projected from the origin with speed u at an acute angle α above the positive x-axis.

The curve E is given by $z = A - Bx^2$ and $y = 0$. If E and the trajectory of P_α touch exactly once, show that

$$u^2 - 2gA = u^2(1 - 4AB)\cos^2\alpha.$$

E and the trajectory of P_α touch exactly once for all α with $0 < \alpha < \frac{1}{2}\pi$. Write down the values of A and B in terms of u and g.

An explosion takes place at the origin and results in a large number of particles being simultaneously projected with speed u in different directions. You may assume that all the particles move freely under gravity for $t \geq 0$.

(ii) Describe the set of points which can be hit by particles from the explosion, explaining your answer.

(iii) Show that, at a time t after the explosion, the particles lie on a sphere whose centre and radius you should find.

(iv) Another particle Q is projected horizontally from the point $(0,0,A)$ with speed u in the positive x direction.

Show that, at all times, Q lies on the curve E.

(v) Show that for particles Q and P_α to collide, Q must be projected a time $\frac{u(1-\cos\alpha)}{g\sin\alpha}$ after the explosion.

Section C: Probability and Statistics

[STEP 2, 2023Q11]

(i) X_1 and X_2 are both random variables which take values $x_1, x_2, \dots, x_n$, with probabilities $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ respectively.

The value of random variable Y is defined to be that of X_1 with probability p and that of X_2 with probability $q = 1 - p$.

If X_1 has mean μ_1 and variance σ_1^2, and X_2 has mean μ_2 and variance σ_2^2, find the mean of Y and show that the variance of Y is $p\sigma_1^2 + q\sigma_2^2 + pq(\mu_1 - \mu_2)^2$.

(ii) To find the value of random variable B, a fair coin is tossed and a fair six-sided die is rolled. If the coin shows heads, then $B = 1$ if the die shows a six and $B = 0$ otherwise; if the coin shows tails, then $B = 1$ if the die does **not** show a six and $B = 0$ if it does. The value of Z_1 is the sum of n independent values of B, where n is large.

Show that Z_1 is a Binomial random variable with probability of success $\frac{1}{2}$.

Using a Normal approximation, show that the probability that Z_1 is within 10% of its mean tends to 1 as $n \longrightarrow \infty$.

(iii) To find the value of random variable Z_2, a fair coin is tossed and n fair six-sided dice are rolled, where n is large. If the coin shows heads, then the value of Z_2 is the number of dice showing a six; if the coin shows tails, then the value of Z_2 is the number of dice **not** showing a six.

Use part (i) to write down the mean and variance of Z_2.

Explain why a Normal distribution with this mean and variance will not be a good approximation to the distribution of Z_2.

Show that the probability that Z_2 is within 10% of its mean tends to 0 as $n \longrightarrow \infty$.

[STEP 2, 2023Q12]

Each of the independent random variables $X_1, X_2, \dots, X_n$ has the probability density function $f(x) = \frac{1}{2}\sin x$ for $0 \le x \le \pi$ (and zero otherwise). Let Y be the random variable whose value is the maximum of the values of $X_1, X_2, \dots, X_n$.

(i) Explain why $\mathrm{P}(Y \le t) = [\mathrm{P}(X_1 \le t)]^n$ and hence, or otherwise, find the probability density function of Y.

Let $m(n)$ be the median of Y and $\mu(n)$ be the mean of Y.

(ii) Find an expression for $m(n)$ in terms of n. How does $m(n)$ change as n increases?

(iii) Show that

$$\mu(n) = \pi - \frac{1}{2^n}\int_0^{\pi} (1 - \cos x)^n \, \mathrm{d}x.$$

(a) Show that $\mu(n)$ increases with n.

(b) Show that $\mu(2) < m(2)$.