UE OXBRIDGE-PREP

OXFORD MATHEMATICS ADMISSIONS TEST

MAT

PAST PAPERS

2001-2023

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On-line Complete Solutions

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Introduction

"MAT Past Papers" is presented by UE International Education (ueie.com), which is designed as a companion to the MAT Standard Course and the MAT Question Practice. It aims to help students to prepare the Oxford Mathematics Admissions Test. It is also a useful reference for teachers who are teaching MAT.

All questions in this collection are reproduced from the official past papers released by the University of Oxford, with a few typos from the source files corrected. The 2024 Edition collects a total of 357 MAT questions from 2001 to 2023.

In addition, subscribed users can access three more on-line MAT mock papers, which are made up by our professional teachers based on the latest research on MAT questions.

How to Access Full Solutions

Although this document is free for everyone to use, the detailed solutions to all questions are **only available for subscribed users** who have purchased one of the following products of the UE Oxbridge-Prep series (click on the link to learn more):

MAT Standard Course

MAT Question Practice

At least one of the official solution, hand-written solution or video solution is provided for each question. Hand-written solutions are provided if official solutions are unavailable. There are video solutions for some questions.

All solutions can be accessed **ON-LINE ONLY**.

MAT Past Scores

You may look up MAT past scores through the following page:

Oxford MAT Scores

Statistics of Solutions

Year	Number of Questions	Official Solutions	Handwritten Solutions	Video Solutions
2001	**14**	14	7	2
2002	**14**	0	14	5
2003	**14**	1	13	4
2004	**14**	6	8	3
2005	**14**	4	10	5
2006	**16**	0	16	1
2007	**16**	16	4	3
2008	**16**	16	7	3
2009	**16**	16	11	6
2010	**16**	16	6	4
2011	**16**	16	9	7
2012	**16**	16	9	6
2013	**16**	16	13	6
2014	**16**	16	11	10
2015	**16**	16	9	4
2016	**16**	16	9	6
2017	**16**	16	2	2
2018	**16**	16	5	5
2019	**16**	16	1	1
2020	**16**	16	0	0
2021	**16**	16	0	0
2022	**16**	16	0	0
2023	**15**	15	0	0
Total	**357**	296 (83%)	164 (46%)	83 (23%)

简介

《MAT 历年真题集》由优易国际教育（ueie.com）出品，是 MAT 标准课程和 MAT 刷题训练的配套资料之一。其主要用途是帮助学生提高备考牛津 MAT 数学考试的效率，以及为教授 MAT 考试的同行老师提供参考。

真题集中的所有真题均由牛津大学官方发布的真题重新排版制作而成，并修订了源文件中的若干印刷错误。2024 版收录了 2001 年至 2023 年共 357 道 MAT 真题。

此外，我们还为付费订阅用户提供三套线上 MAT 模考题。这些模考题是由我们的专业教师团队依据近几年 MAT 考试命题趋势而命制的。

真题解析在哪里可以看到

所有用户均可免费使用真题集，但所有题目的解析仅向购买以下任意优易牛剑备考系列产品之一的**付费用户开放**：

MAT 标准课

MAT 刷题训练

所有真题都有详细解析，解析形式为官方解析、手写解析或视频讲解中的一种或多种。如果没有官方解析，则提供手写解析。部分题目提供视频讲解。

所有解析均**只能在线查看**。

MAT 历年分数线

你可以通过下方页面查询 MAT 历年分数线：

牛津 MAT 分数

解析数量统计

年份	真题数量	官方解析题量	手写解析题量	视频講解题量
2001	**14**	14	7	2
2002	**14**	0	14	5
2003	**14**	1	13	4
2004	**14**	6	8	3
2005	**14**	4	10	5
2006	**16**	0	16	1
2007	**16**	16	4	3
2008	**16**	16	7	3
2009	**16**	16	11	6
2010	**16**	16	6	4
2011	**16**	16	9	7
2012	**16**	16	9	6
2013	**16**	16	13	6
2014	**16**	16	11	10
2015	**16**	16	9	4
2016	**16**	16	9	6
2017	**16**	16	2	2
2018	**16**	16	5	5
2019	**16**	16	1	1
2020	**16**	16	0	0
2021	**16**	16	0	0
2022	**16**	16	0	0
2023	**15**	15	0	0
总计	**357**	296 (83%)	164 (46%)	83 (23%)

簡介

《MAT 歷年真題集》由優易國際教育（ueie.com）出品，是 MAT 標準課程和 MAT 刷題訓練的配套資料之一。其主要用途是幫助學生提高備考牛津 MAT 數學考試的效率，以及為教授 MAT 考試的同儕老師提供參考。

真題集中的所有真題均由牛津大學官方發布的真題重新排版製作而成，並修訂了源文檔中的若干印刷錯誤。2024 版收錄了 2001 年至 2023 年共 357 道 MAT 真題。

此外，我們還為付費訂閱用戶提供三套在線 MAT 模擬題。這些模擬題系我們的專業教師團隊依據近幾年 MAT 考試命題趨勢而命製的。

真題解析在哪裡可以看到

所有用戶均可免費使用真題集，但所有題目的解析僅向購買以下任意優易牛劍備考系列產品之一的**付費用戶開放**：

MAT 標準課

MAT 刷題訓練

所有真題都有詳細解析，解析形式為官方解析、手寫解析或視訊講解中的一種或多種。如果沒有官方解析，則提供手寫解析。部分題目提供影片講解。

所有解析均只能線上查看。

MAT 歷年分數線

你可以透過下方頁面查詢 MAT 歷年分數線：

牛津 MAT 分數

解析數量統計

年份	真題數量	官方解析題量	手寫解析題量	影片講解題量
2001	**14**	14	7	2
2002	**14**	0	14	5
2003	**14**	1	13	4
2004	**14**	6	8	3
2005	**14**	4	10	5
2006	**16**	0	16	1
2007	**16**	16	4	3
2008	**16**	16	7	3
2009	**16**	16	11	6
2010	**16**	16	6	4
2011	**16**	16	9	7
2012	**16**	16	9	6
2013	**16**	16	13	6
2014	**16**	16	11	10
2015	**16**	16	9	4
2016	**16**	16	9	6
2017	**16**	16	2	2
2018	**16**	16	5	5
2019	**16**	16	1	1
2020	**16**	16	0	0
2021	**16**	16	0	0
2022	**16**	16	0	0
2023	**15**	15	0	0
總計	**357**	296 (83%)	164 (46%)	83 (23%)

Answer Keys

MAT 2007-2023

Only parts of the keys to multiple-choice questions (2007-2023) are provided here.
Full solutions can be accessed on-line by the links or scanning the QR codes provided.

2007 Answer Keys	
Q1(A)	B
Q1(B)	C
Q1(C)	B
Q1(D)	A
Q1(E)	B
Q1(F)	C
Q1(G)	A
Q1(H)	D
Q1(I)	C
Q1(J)	D

2008 Answer Keys	
Q1(A)	C
Q1(B)	A
Q1(C)	A
Q1(D)	B
Q1(E)	D
Q1(F)	D
Q1(G)	C
Q1(H)	A
Q1(I)	C
Q1(J)	B

2009 Answer Keys	
Q1(A)	B
Q1(B)	C
Q1(C)	B
Q1(D)	C
Q1(E)	A
Q1(F)	D
Q1(G)	C
Q1(H)	B
Q1(I)	B
Q1(J)	C

2010 Answer Keys	
Q1(A)	C
Q1(B)	A
Q1(C)	D
Q1(D)	B
Q1(E)	A
Q1(F)	D
Q1(G)	C
Q1(H)	C
Q1(I)	B
Q1(J)	D

2011 Answer Keys	
Q1(A)	C
Q1(B)	C
Q1(C)	A
Q1(D)	B
Q1(E)	B
Q1(F)	B
Q1(G)	D
Q1(H)	C
Q1(I)	B
Q1(J)	A

2012 Answer Keys	
Q1(A)	B
Q1(B)	D
Q1(C)	D
Q1(D)	A
Q1(E)	D
Q1(F)	B
Q1(G)	C
Q1(H)	B
Q1(I)	A
Q1(J)	B

2013 Answer Keys	
Q1(A)	A
Q1(B)	C
Q1(C)	C
Q1(D)	B
Q1(E)	D
Q1(F)	A
Q1(G)	D
Q1(H)	B
Q1(I)	B
Q1(J)	B

2014 Answer Keys	
Q1(A)	A
Q1(B)	E
Q1(C)	C
Q1(D)	D
Q1(E)	B
Q1(F)	C
Q1(G)	D
Q1(H)	C
Q1(I)	B
Q1(J)	A

2015 Answer Keys	
Q1(A)	E
Q1(B)	B
Q1(C)	C
Q1(D)	B
Q1(E)	D
Q1(F)	B
Q1(G)	C
Q1(H)	C
Q1(I)	D
Q1(J)	A

2016 Answer Keys	
Q1(A)	D
Q1(B)	B
Q1(C)	A
Q1(D)	D
Q1(E)	A
Q1(F)	B
Q1(G)	D
Q1(H)	E
Q1(I)	C
Q1(J)	B

2017 Answer Keys	
Q1(A)	B
Q1(B)	A
Q1(C)	D
Q1(D)	C
Q1(E)	D
Q1(F)	C
Q1(G)	B
Q1(H)	B
Q1(I)	D
Q1(J)	D

2018 Answer Keys	
Q1(A)	D
Q1(B)	D
Q1(C)	E
Q1(D)	B
Q1(E)	A
Q1(F)	B
Q1(G)	B
Q1(H)	C
Q1(I)	C
Q1(J)	C

2019 Answer Keys	
Q1(A)	B
Q1(B)	C
Q1(C)	D
Q1(D)	B
Q1(E)	E
Q1(F)	C
Q1(G)	A
Q1(H)	C
Q1(I)	A
Q1(J)	D

2020 Answer Keys	
Q1(A)	D
Q1(B)	D
Q1(C)	E
Q1(D)	B
Q1(E)	C
Q1(F)	C
Q1(G)	B
Q1(H)	D
Q1(I)	B
Q1(J)	D

2021 Answer Keys	
Q1(A)	E
Q1(B)	C
Q1(C)	C
Q1(D)	B
Q1(E)	C
Q1(F)	D
Q1(G)	D
Q1(H)	A
Q1(I)	B
Q1(J)	D

2022 Answer Keys	
Q1(A)	D
Q1(B)	D
Q1(C)	B
Q1(D)	E
Q1(E)	C
Q1(F)	A
Q1(G)	B
Q1(H)	C
Q1(I)	A
Q1(J)	E

2023 Answer Keys	
Q1(A)	C
Q1(B)	C
Q1(C)	B
Q1(D)	C
Q1(E)	A
Q1(F)	A
Q1(G)	A
Q1(H)	D
Q1(I)	E
Q1(J)	C

MAT 2001 (On-line only)

MAT 2001
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and Questions 2, 3, 4, 5 are worth 15 marks each, giving a total of 100.

Question 1 is a multiple choice question for which marks are given solely for correct answers. Answer Question 1 on the grid on Page 2.

Answers to questions 2-5 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Note that in any of the Questions 2-5 you may use any earlier parts (even those you do not attempt) in your solutions to later parts.

THE USE OF CALCULATORS OR FORMULA SHEET IS PROHIBITED.

MAT 2002 (On-line only)

MAT 2002
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and Questions 2, 3, 4, 5 are worth 15 marks each, giving a total of 100.

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MAT 2003 (On-line only)

MAT 2003
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and Questions 2, 3, 4, 5 are worth 15 marks each, giving a total of 100.

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MAT 2004 (On-line only)

MAT 2004
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and Questions 2, 3, 4, 5 are worth 15 marks each, giving a total of 100.

Question 1 is a multiple choice question for which marks are given solely for correct answers. Answer Question 1 on the grid on Page 2.

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MAT 2005

MAT 2005
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and Questions 2, 3, 4, 5 are worth 15 marks each, giving a total of 100.

Question 1 is a multiple choice question for which marks are given solely for correct answers. Answer Question 1 on the grid on Page 2.

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THE USE OF CALCULATORS OR FORMULA SHEET IS PROHIBITED.

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), or (D) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

[MAT, 2005Q1(A)]

The area of the region bounded by the curves $y = x^2$ and $y = x + 2$ equals

(A) $\frac{9}{2}$

(B) $\frac{7}{3}$

(C) $\frac{7}{2}$

(D) $\frac{11}{2}$

[MAT, 2005Q1(B)]

The equation

$$(x^2 + 1)^{10} = 2x - x^2 - 2$$

(A) has $x = 2$ as a solution.

(B) has no real solutions.

(C) has an odd number of real solutions.

(D) has twenty real solutions.

[MAT, 2005Q1(C)]

Given that

$$\log_{10} 2 = 0.3010 \text{ to 4 d.p. and that } 10^{0.2} < 2,$$

it is possible to deduce that

(A) 2^{100} begins in a 1 and is 30 digits long.

(B) 2^{100} begins in a 2 and is 30 digits long.

(C) 2^{100} begins in a 1 and is 31 digits long.

(D) 2^{100} begins in a 2 and is 31 digits long.

[MAT, 2005Q1(D)]

In the range $0 \leq x < 2\pi$ the equation $\cos(\sin x) = \frac{1}{2}$ has

(A) no solution.

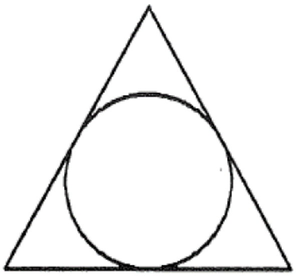
(B) one solution.

(C) two solutions.

(D) three solutions.

[MAT, 2005Q1(E)]

A circle is inscribed in an equilateral triangle as shown in the diagram below. The area of the circle, as a percentage of the triangle's, is approximately



(A) 40%

(B) 50%

(C) 60%

(D) 70%

[MAT, 2005Q1(F)]

The fact that

$$6 \times 7 = 42,$$

is a counter-example to which of the following statements?

(A) the product of any two odd integers is odd.

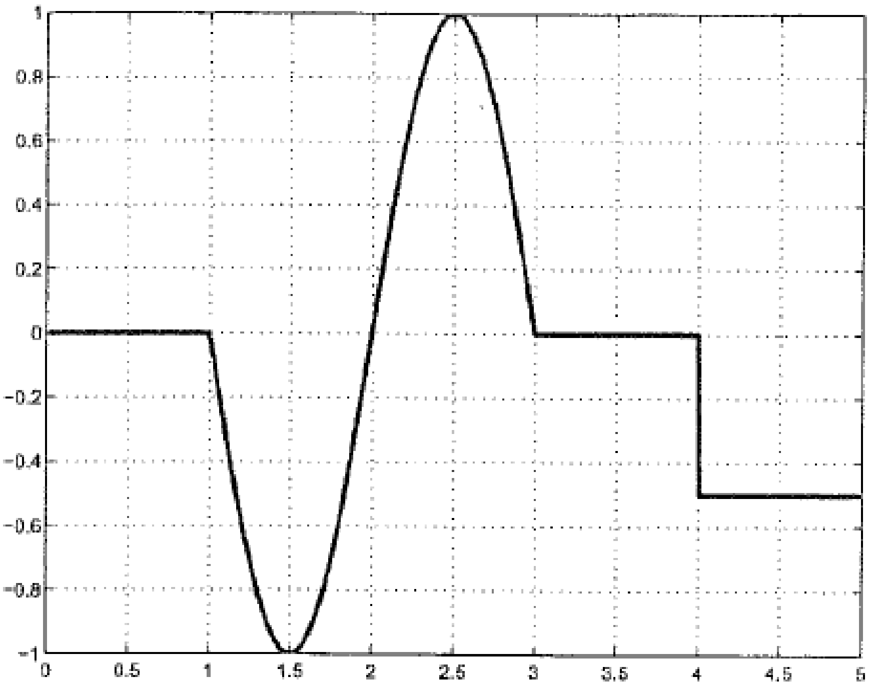
(B) if the product of two integers is not a multiple of 4 then the integers are not consecutive.

(C) if the product of two integers is a multiple of 4 then the integers are not consecutive.

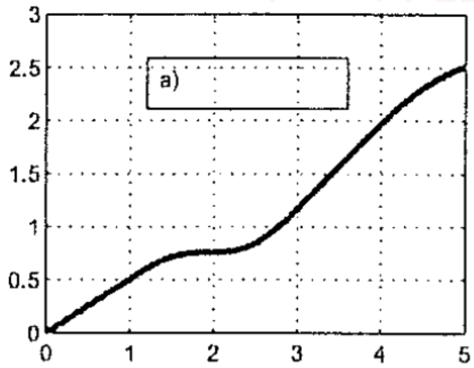
(D) any even integer can be written as the product of two even integers.

[MAT, 2005Q1(G)]

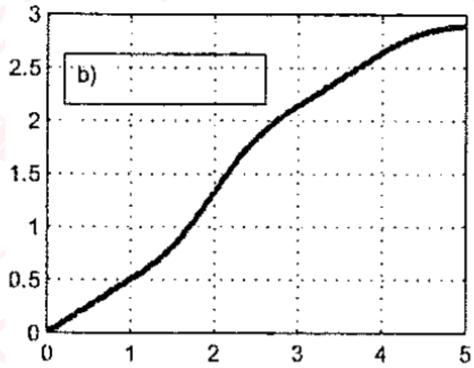
A motorist drives along a curvy road. The acceleration $a(t)$ of her car as a function of time is graphed below.



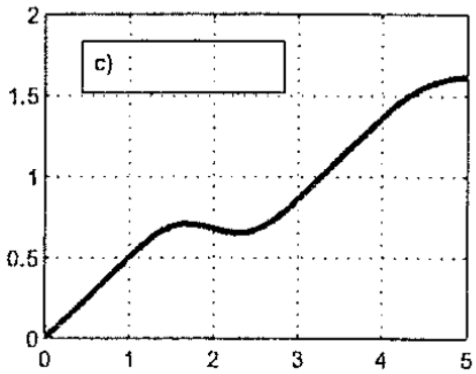
One of the graphs below represents the distance $d(t)$ travelled by the motorist as a function of time. Which is the correct graph?



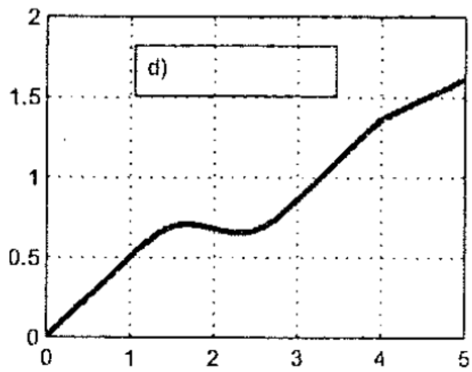
(A)



(B)



(C)



(D)

[MAT, 2005Q1(H)]

The four digit number 2652 is such that any two consecutive digits from it make a multiple of 13. Another number N has this same peoperty, is 100 digits long, and begins in a 9. What is the last digit of N?

(A) 2

(B) 3

(C) 6

(D) 9

[MAT, 2005Q1(I)]

The curve with equation

$$x^{17} + x^3 + y^4 + y^{12} = 2$$

has

(A) neither the x-axis nor y-axis as a line of symmetry.

(B) the x-axis but not the y-axis as a line of symmetry.

(C) the y-axis but not the x-axis as a line of symmetry.

(D) both axes as lines of symmetry.

[MAT, 2005Q1(J)]

The numbers x and y satisfy

$$(x-1)^2 + y^2 \leq 1.$$

The largest that $x + y$ can be is

(A) 2

(B) $1 + \sqrt{2}$

(C) 3

(D) $2 + \sqrt{2}$

[MAT, 2005Q2]

(i) Show, with working, that

$$x^3 - (1 + \cos\theta + \sin\theta)x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)x - \sin\theta\cos\theta, \qquad (1)$$

equals

$$(x-1)(x^2 - (\cos\theta + \sin\theta)x + \cos\theta\sin\theta).$$

Deduce that the cubic in (1) has roots

$$1, \qquad \cos\theta, \qquad \sin\theta$$

(ii) Give the roots when $\theta = \frac{\pi}{3}$.

(iii) Find all values of θ in the range $0 \leq \theta < 2\pi$ such that two of the three roots are equal.

(iv) What is the greatest possible difference between two of the roots, and for what values of θ in the range $0 \leq \theta < 2\pi$ does this greates difference occur?

Show that for each such θ the cubic (1) is the same.

[MAT, 2005Q3]

(i) Find the co-ordinates of the turning points of

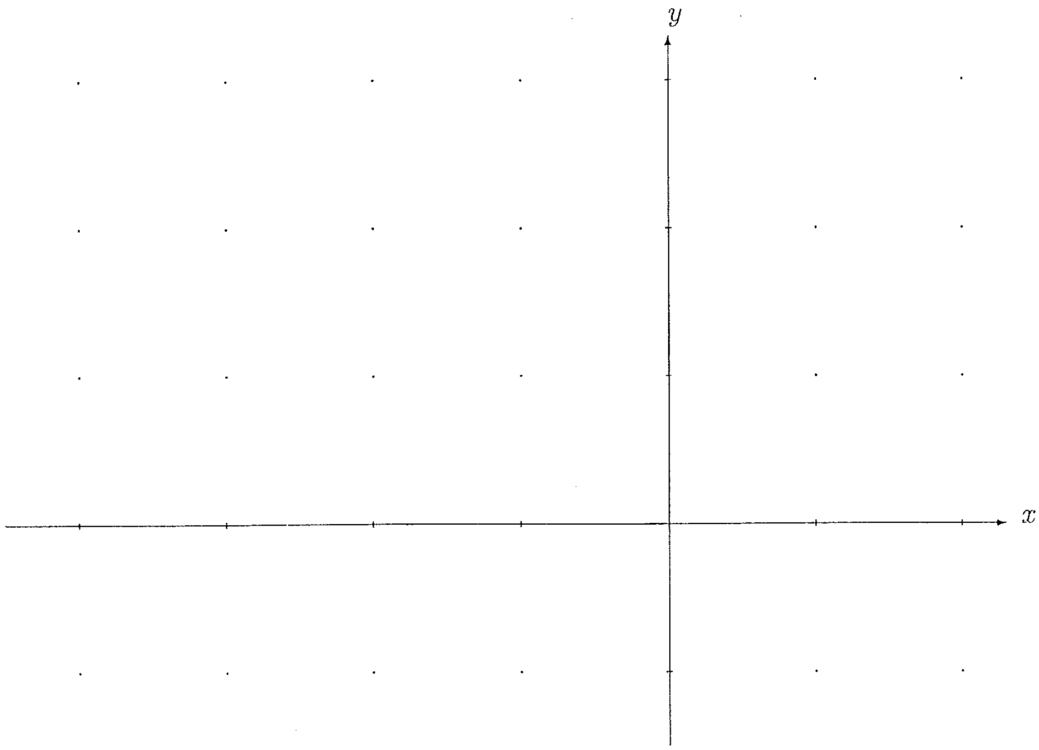
$$f(x) = e^x(2x^2 - x - 1).$$

(ii) Sketch the graph of $y = f(x)$ on the axes below for the range $-4 \le x \le 2$.

(iii) Now consider

$$g(x) = \begin{cases} e^x(2x^2 - x - 1) & \text{if } x < 1; \\ \sin(x - 1) & \text{if } x \ge 1. \end{cases}$$

Determine, with explanations, the maximum and minimum values of $g(x)$ as x varies over the real numbers.



[MAT, 2005Q4]

An $n \times n$ square array contains 0s and 1s. Such a square is given below with $n = 3$.

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ \hline 1 & 1 & 0 \\ \hline \end{array}$$

Two types of operation C and R may be performed on such an array.

- The first operation C takes the first and second columns (on the left) and replaces them with a single column by comparing the two elements in each row as follows: if the two elements are the same then C replaces them with a 1, and if they differ C replaces them with a 0.
- The second operation R takes the first and second rows (from the top) and replaces them with a single row by comparing the two elements in each column as follows: if the two elements are the same then R replaces them with a 1, and if they differ R replaces them with a 0.

By way of example, the effects of performing R then C on the square above are given below.

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ \hline 1 & 1 & 0 \\ \hline \end{array} \xrightarrow{R} \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline \end{array} \xrightarrow{C} \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}$$

(i) If R then C are performed (in that order) on a 2×2 array then only a single number (0 or 1) remains.

(a) Write down the eight 2×2 arrays which, when R then C are performed, produce a 1.

(b) By grouping your answers accordingly, show that if $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}$ is amongst your answers to part (a) then so is $\begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline \end{array}$.

Explain why this means that doing R then C on a 2×2 array produces the same answer as doing C first then R.

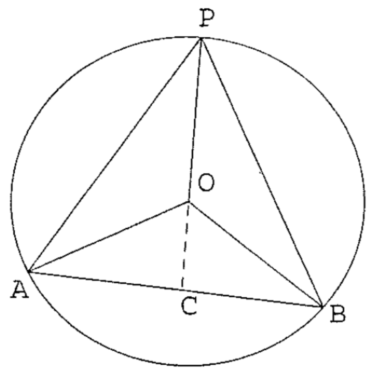
(ii) Consider now an $n \times n$ square array containing 0s and 1s, and the effects of performing R then C or C then R on the square.

(a) Explain why the effect on the right $n - 2$ columns is the same whether the order is R then C or C then R. [This then also applies to the bottom $n - 2$ rows.]

(b) Deduce that performing R then C on an $n \times n$ square produces the same result as performing C then R.

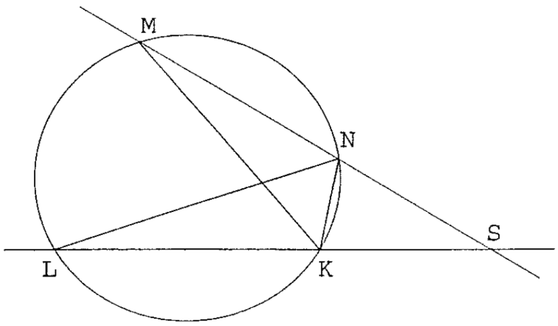
[MAT, 2005Q5]

(i) Three points P, A, B lie on a circle which has centre O. The point C is where PO extends to meet AB as shown in the diagram below.



Show that $\angle AOC = 2\angle APC$ and $\angle BOC = 2\angle BPC$. Why does this mean that $\angle APB$ is independent of the choice of the point P?

(ii) Four points K, L, M, N lie on a circle and the lines LK and MN meet outside the circle at a point S, as shown in the diagram below.

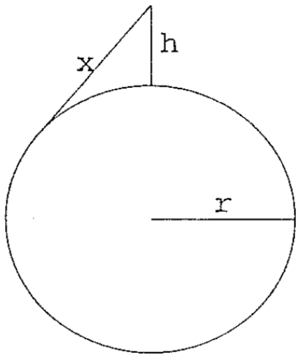


Using part (i) and the Sine Rule show that

$$\frac{KS}{NS} = \frac{SM}{SL}.$$

[You may also assume that part (ii) holds true in the special case when $M = N$ in which case the line SM is the tangent to the circle at M.]

(iii) A tower has height h. Assuming the earth to be a perfect sphere of radius r, determine the greatest distance x from the top of the tower at which an observer can still see it.



MAT 2006

MAT 2006
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

This paper contains 7 questions, of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics applicants should attempt Questions **1,2,3,4,5**.
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Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), or (D) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

[MAT, 2006Q1(A)]

Which of the following number is largest?

(A) $((2^3)^2)^3$

(B) $(2^3)^{(2^3)}$

(C) $2^{\left((3^2)^3\right)}$

(D) $2^{\left(3^{(2^3)}\right)}$

[MAT, 2006Q1(B)]

The equation

$$(2 + x - x^2)^2 = 16$$

has:

(A) no real roots.

(B) one real root.

(C) two real roots.

(D) three real roots.

[MAT, 2006Q1(C)]

The function f is defined for whole positive numbers and satisfies $f(1) = 1$ and also the rules

$$f(2n) = f(n),$$
$$f(2n + 1) = f(n) + 1$$

for all values of n. It follows that $f(9)$ equals

(A) 1

(B) 2

(C) 3

(D) 4

[MAT, 2006Q1(D)]

The function

$$y = x^2 \ln x$$

defined for $x > 0$ satisfies

(A) $x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2y + x^2$, (and only this part).

(B) $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ for all x, (and only this part).

(C) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq 0$ for all x, (and only this part).

(D) all of the above.

[MAT, 2006Q1(E)]

The cubic

$$x^3 + ax + b$$

has both $(x-1)$ and $(x-2)$ as factors. Then

(A) $a = -7$ and $b = 6$.

(B) $a = -3$ and $b = 2$.

(C) $a = 0$ and $b = -2$.

(D) $a = 5$ and $b = 4$.

[MAT, 2006Q1(F)]

The inequality

$$\frac{x^2+1}{x^2-1} < 1$$

is true:

(A) for no values of x.

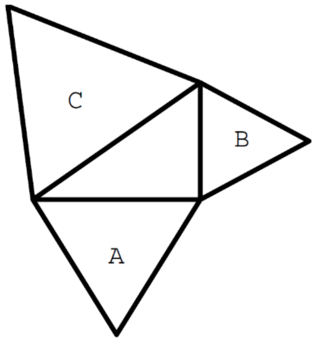
(B) whenever $-1 < x < 1$.

(C) whenever $x > 1$.

(D) for all values of x.

[MAT, 2006Q1(G)]

Three equilateral triangles with areas A, B, C are drawn on the sides of a right-angled triangle as in the diagram below.



These areas are related by the equation:

(A) $A^{1/2} + B^{1/2} = C^{1/2}$

(B) $A + B = C$

(C) $A^{3/2} + B^{3/2} = C^{3/2}$

(D) $A^2 + B^2 = C^2$

[MAT, 2006Q1(H)]

How many solutions does the equation

$$2 = \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots$$

Have in the range $0 \leq x < 2\pi$?

(A) 0

(B) 1

(C) 2

(D) 3

[MAT, 2006Q1(I)]

The equation

$$|x| + |x - 1| = 0$$

has

(A) no solutions.

(B) one solution.

(C) two solutions.

(D) three solutions.

[MAT, 2006Q1(J)]

The two circles with equations

$$x^2 + y^2 = 1, \qquad (x-a)^2 + (y-b)^2 = r^2$$

(where $r > 0$) do *not* intersect if

(A) $\sqrt{a^2+b^2} + r < 1$, (and only this part).

(B) $\sqrt{a^2+b^2} + 1 < r$, (and only this part).

(C) $\sqrt{a^2+b^2} - r > 1$, (and only this part).

(D) all of the above.

[MAT, 2006Q2]

The real numbers x and y satisfy the equation

$$x^2 + xy + y^2 = 1. \qquad (1)$$

(i) If $y = 1$ then find the possible values of x

(ii) For what values of y are there two possible (real) values of x?

(iii) Find the largest possible value of y which satisfies (1) and the corresponding value of x

(iv) Show that, for all values of θ, the numbers

$$x = \frac{1}{\sqrt{3}}\cos\theta + \sin\theta\,,$$
$$y = \frac{1}{\sqrt{3}}\cos\theta - \sin\theta,$$

satisfy the equation (1).

[MAT, 2006Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS}.

Let

$$f(x) = x^3 - 3x^2 + 2x.$$

(i) On the axes below, sketch the curve $y = f(x)$ for the range $-1 < x < 3$, carefully labelling any turning points.

(ii) The equation $f(x) = k$ has exactly one positive solution and exactly one negative solution. Find k.

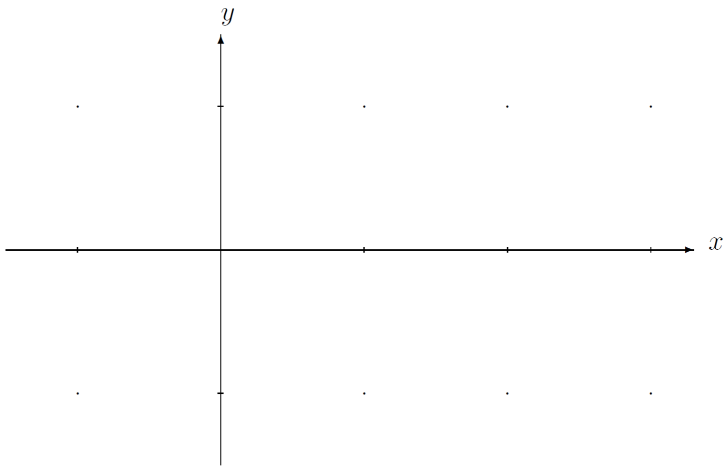
For x in the range $0 \leq x \leq 2$ the functions $g(x)$ and $h(x)$ are defined by

$$g(x) = \int_0^x f(t)\,\mathrm{d}t,$$

$$h(x) = \int_0^x |f(t)|\,\mathrm{d}t.$$

(iii) Find the value X_1 of x in the range $0 \leq x \leq 2$ for which $g(x)$ is greatest. Calculate $g(X_1)$.

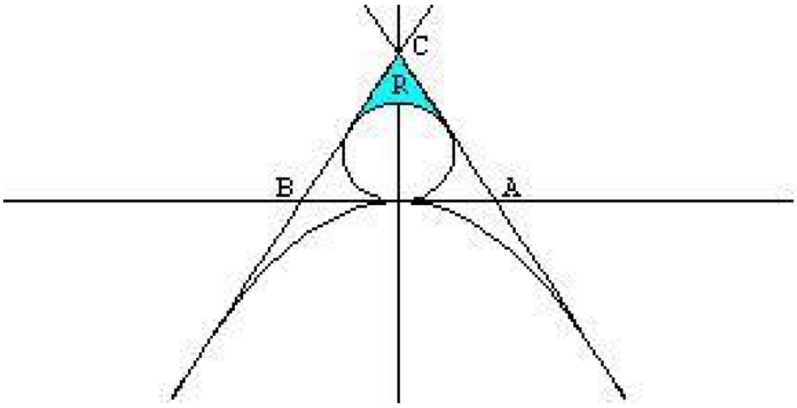
(iv) Find the value X_2 of x in the range $0 \leq x \leq 2$ for which $h(x)$ is greatest. [You are not asked to calculate $h(X_2)$.]



[MAT, 2006Q4]

For applicants in {Math, Math & Statistics, and Math & Philosophy} only. Not for {Math & CS, and CS}

In the diagram below are drawn the circle $x^2 + (y-1)^2 = 1$ and the parabola $y = -\frac{1}{4}x^2$



(i) Find the equation of the tangent to the parabola $y = -\frac{1}{4}x^2$ at the point $\left(t, -\frac{1}{4}t^2\right)$.

(ii) Show that, when $t = 2\sqrt{3}$, this tangent is also a tangent to the circle $x^2 + (y-1)^2 = 1$.

By symmetry the tangent to the parabola at $\left(-2\sqrt{3}, -3\right)$ is also a tangent to the circle. These tangents are also shown on the diagram above.

(iii) The first and second tangents meet the x-axis at A and B respectively; the two tangents intersect at C. Find the angle CAB.

Deduce that the area of the shaded region R, bounded by the circle and the two tangents, is

$$\sqrt{3} - \frac{\pi}{3}.$$

[MAT, 2006Q5]

For any four numbers a, b, c, d the symbol

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array}$$

represents a number that depends on a, b, c, and d and has the following properties:

$$\begin{array}{c|c} s\times a & s\times b \\ \hline c & d \end{array} = s \times \begin{array}{c|c} a & b \\ \hline c & d \end{array} \tag{1}$$

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} = \begin{array}{c|c} d & b \\ \hline c & a \end{array} \tag{2}$$

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} = \begin{array}{c|c} a & c \\ \hline b & d \end{array} \tag{3}$$

$$\begin{array}{c|c} a+x & b+y \\ \hline c & d \end{array} = \begin{array}{c|c} a & b \\ \hline c & d \end{array} + \begin{array}{c|c} x & y \\ \hline c & d \end{array} \tag{4}$$

You may assume nothing else about $\begin{array}{c|c} a & b \\ \hline c & d \end{array}$.

(i) Use property (1) to show that $\begin{array}{c|c} 0 & 0 \\ \hline c & d \end{array} = 0$.

(ii) Hence use properties (2) and (3) to show that each of

$$\begin{array}{c|c} a & 0 \\ \hline c & 0 \end{array}, \quad \begin{array}{c|c} 0 & b \\ \hline 0 & d \end{array}, \quad \begin{array}{c|c} a & b \\ \hline 0 & 0 \end{array}$$

is also zero.

(iii) Show that

$$\begin{array}{c|c} a & b \\ \hline s\times c & s\times d \end{array} = s \times \begin{array}{c|c} a & b \\ \hline c & d \end{array}$$

(iv) Show that

$$\begin{array}{c|c} a & b \\ \hline c+x & d+y \end{array} = \begin{array}{c|c} a & b \\ \hline c & d \end{array} + \begin{array}{c|c} a & b \\ \hline x & y \end{array}$$

[MAT, 2006Q6]

For applicants in {Math & CS, CS} only.

Portia has three boxes made from gold, silver and lead. She has placed a prize in one of these boxes and challenges a friend, Bassanio, to find the prize.

She explains that on each box there is a message which may be true or false. On the basis of these messages Bassanio should be able to choose the box with the prize in.

(i) Initially suppose that there are only two boxes, gold and silver, one of which contains the prize. The messages read:

Gold: The prize is not in here.

Silver: Exactly one of these messages is true.

Which box contains the prize? Explain your answer. [Hint: consider separately the cases where the message on the silver box is true or false.]

(ii) Now suppose that there are all three boxes and that Portia has left the following messages on them:

Gold: The prize is in here.

Silver: The prize is in here.

Lead: At least two of these messages are false.

Which box should Bassanio choose? Explain your answer. [Hint: show that the message on the lead box cannot be false.]

(iii) In this version of the challenge, Portia puts a dagger into one of the boxes. Bassanio must choose a box that does not contain the dagger. The messages on the boxes now read as follows:

Gold: The dagger is in this box.

Silver: The dagger is not in this box.

Lead: At most one of these messages is true.

Which box should Bassanio choose? Explain your answer.

[MAT, 2006Q7]

For applicants in {CS} only.

The game of ABC involves creating "words" formed from the letters **A**, **B**, **C**, according to certain rules. All forms of the game start with the word **AB**.

In the simple form of the game, there is a single rule:

If the current word is of the form $\mathbf{A}x$, for some word x,
then it may be replaced with the word $\mathbf{A}xx$. (1)

For example the word **ABC** could be replaced by **ABCBC**.

(i) Show how to produce the word **ABBBB**.

(ii) Describe precisely all the words that can be produced in the simple form of the game. You may want to write $\mathbf{B}^n$ as a shorthand for the word formed from n copies of **B**; e.g. $\mathbf{AB}^4 =$ **ABBBB**.

In the intermediate form of the game, there is a second rule:

If the current word is of the form $x\mathbf{BBB}$, for some word x,
then it may be replaced by the word $\mathbf{C}x$. (2)

(iii) Show how to produce the word **CCABB**.

(iv) Describe precisely all the words that can be produced in the intermediate form of the game. Explain your answer.

In the advanced form of the game, there is a third rule:

If the current word is of the form $x\mathbf{B}$, for some word x,
then it may be replaced by the word x. (3)

(v) Describe precisely all the words that can be produced in the advanced form of the game. Explain your answer.

MAT 2007

MAT 2007
On-line Exam

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TIME ALLOWED: 150 MINUTES

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	(A)	(B)	(C)	(D)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

[MAT, 2007Q1(A)]

Let r and s be integers. Then

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}}$$

is an integer if

(A) $r + s \leq 0$

(B) $s \leq 0$

(C) $r \leq 0$

(D) $r \geq s$

[MAT, 2007Q1(B)]

The greatest value which the function

$$f(x) = (3\sin^2(10x + 11) - 7)^2$$

takes, as x varies over all real values, equals

(A) -9

(B) 16

(C) 49

(D) 100

[MAT, 2007Q1(C)]

The number of solutions x to the equations

$$7\sin x + 2\cos^2 x = 5,$$

in the range $0 \leq x < 2\pi$, is

(A) 1

(B) 2

(C) 3

(D) 4

[MAT, 2007Q1(D)]

The point on the circle

$$(x-5)^2 + (y-4)^2 = 4$$

which is closest to the circle

$$(x-1)^2 + (y-1)^2 = 1$$

is

(A) (3.4,2.8).

(B) (3,4).

(C) (5,2).

(D) (3.8,2.4).

[MAT, 2007Q1(E)]

If x and n are integers then

$$(1-x)^n(2-x)^{2n}(3-x)^{3n}(4-x)^{4n}(5-x)^{5n}$$

is

(A) negative when $n > 5$ and $x < 5$.

(B) negative when n is odd and $x > 5$.

(C) negative when n is a multiple of 3 and $x > 5$.

(D) negative when n is even and $x < 5$.

[MAT, 2007Q1(F)]

The equation

$$8^x + 4 = 4^x + 2^{x+2}$$

has

(A) no real solutions.

(B) one real solution.

(C) two real solutions.

(D) three real solutions.

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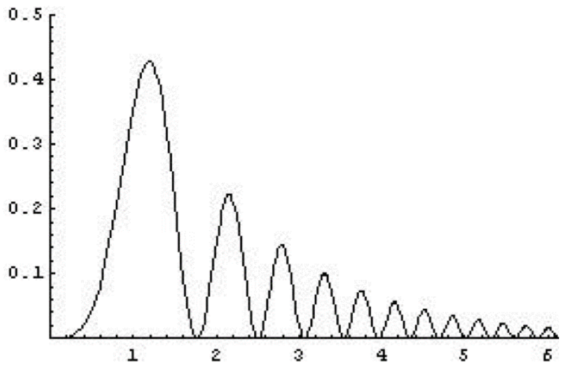
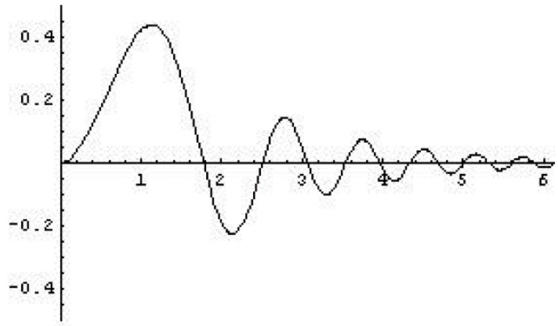
[MAT, 2007Q1(G)]

On which of the axes below is a sketch of the graph

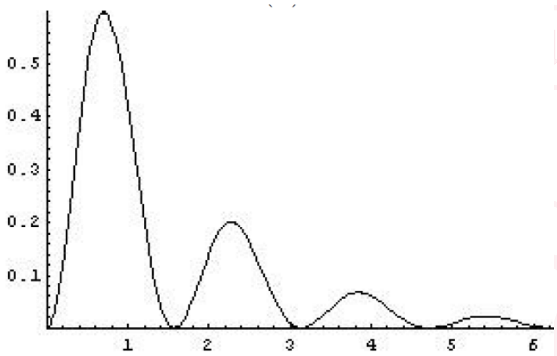
$$y = 2^{-x}\sin^2(x^2)\,?$$



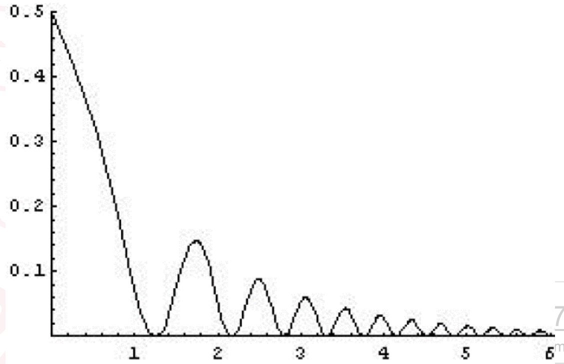
(A)



(B)



(C)



(D)

[MAT, 2007Q1(H)]

Given a function $f(x)$, you are told that

$$\int_0^1 3f(x)\,\mathrm{d}x + \int_1^2 2f(x)\,\mathrm{d}x = 7,$$
$$\int_0^2 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x = 1.$$

It follows that $\int_0^2 f(x)\,\mathrm{d}x$ equals

(A) -1

(B) 0

(C) $\frac{1}{2}$

(D) 2

[MAT, 2007Q1(I)]

Given that a and b are positive and

$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$$

then the greatest possible value of a is

(A) $\frac{1}{10}$

(B) 1

(C) $\sqrt{10}$

(D) $10^{\sqrt{2}}$

[MAT, 2007Q1(J)]

The inequality

$$(n+1) + (n^4+2) + (n^9+3) + (n^{16}+4) + \cdots + (n^{10000}+100) > k$$

is true for all $n \geq 1$. It follows that

(A) $k < 1300$

(B) $k^2 < 101$

(C) $k \geq 101^{10000}$

(D) $k < 5150$

[MAT, 2007Q2]

Let

$$f_n(x) = (2 + (-2)^n)x^2 + (n + 3)x + n^2$$

where n is a positive integer and x is any real number

(i) Write down $f_3(x)$.

Find the maximum value of $f_3(x)$.
For what values of n does $f_n(x)$ have a maximum value (as x varies)?
[Note you are not being asked to calculate the value of this maximum.]

(ii) Write down $f_1(x)$.

Calculate $f_1(f_1(x))$ and $f_1\left(f_1(f_1(x))\right)$.

Find an expression, simplified as much as possible, for

$$f_1\left(f_1\left(f_1(\cdots f_1(x))\right)\right)$$

where f_1 is applied k times. [Here k is a positive integer.]

(iii) Write down $f_2(x)$.

The function

$$f_2\left(f_2\left(f_2(\cdots f_2(x))\right)\right),$$

where f_2 is applied k times, is a polynomial in x. What is the degree of this polynomial?

[MAT, 2007Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS}.

Let

$$I(c) = \int_0^1 ((x-c)^2 + c^2)\,\mathrm{d}x$$

where c is a real number

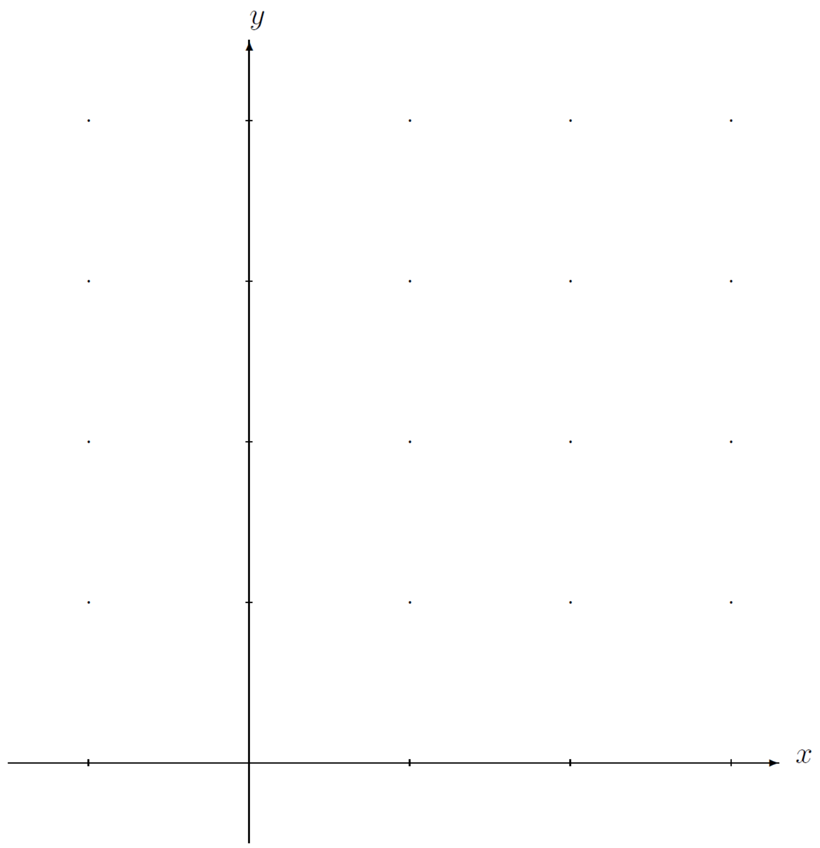
(i) Sketch $y = (x-1)^2 + 1$ for the values $-1 \leq x \leq 3$ on the axes below and show on your graph the area represented by the integral $I(1)$.

(ii) Without explicitly calculating $I(c)$, explain why $I(c) \geq 0$ for any value of c.

(iii) Calculate $I(c)$.

(iv) What is the minimum value of $I(c)$ (as c varies)?

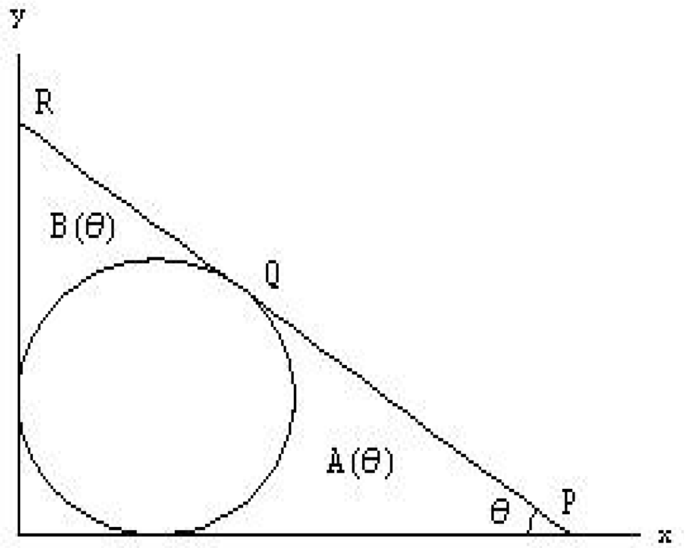
(v) What is the maximum value of $I(\sin\theta)$ as θ varies?



[MAT, 2007Q4]

For applicants in {Math, Math & Statistics, and Math & Philosophy} only. Not for {Math & CS, and CS}

In the diagram below is sketched the circle with centre $(1,1)$ and radius 1 and a line L. The line L is tangential to the circle at Q; further L meets the y-axis at R and the x-axis at P in such a way that the angle OPQ equals θ where $0 < \theta < \frac{\pi}{2}$.



(i) Show that the co-ordinates of Q are

$$(1 + \sin\theta, 1 + \cos\theta),$$

and that the gradient of PQR is $-\tan\theta$.

Write down the equation of the line PQR and so find the co-ordinates of P.

(ii) The region bounded by the circle, the x-axis and PQ has area $A(\theta)$; the region bounded by the circle, the y-axis and QR has area $B(\theta)$. (See diagram.)

Explain why

$$A(\theta) = B\left(\frac{\pi}{2} - \theta\right)$$

for any θ.

Calculate $A\left(\frac{\pi}{4}\right)$.

(iii) Show that

$$A\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3}.$$

[MAT, 2007Q5]

Let $f(n)$ be a function defined, for any integer $n > 0$, as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ \left(f\left(\frac{n}{2}\right)\right)^2 & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 2f(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(i) What is the value of $f(5)$?

The *recursion depth* of $f(n)$ is defined to be the number of other integers m such that the value of $f(m)$ is calculated whilst computing the value of $f(n)$. For example, the recursion depth of $f(4)$ is 3, because the values of $f(2)$, $f(1)$, and $f(0)$ need to be calculated on the way to computing the value of $f(4)$

(ii) What is the recursion depth of $f(5)$?

Now let $g(n)$ be a function, defined for all integers $n \geq 0$, as follows:

$$g(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 + g\left(\frac{n}{2}\right) & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 1 + g(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(iii) What is $g(5)$?

(iv) What is $g(2^k)$, where $k \geq 0$ is an integer? Briefly explain your answer.

(v) What is $g(2^l + 2^k)$ where $l > k \geq 0$ are integers? Briefly explain your answer.

(vi) Explain briefly why the value of $g(n)$ is equal to the recursion depth of $f(n)$.

[MAT, 2007Q6]

For applicants in {CS and Math & CS} only.

Three people called Alf, Beth, and Gemma, sit together in the same room.

One of them always tells the truth.

One of them always tells a lie.

The other one tells truth or lies at random.

In each of the following situations, your task is to determine how each person acts.

(i) Suppose that Alf says "I always tell lies" and Beth says "Yes, that's true, Alf always tells lies".

Who always tells the truth? Who always lies? Briefly explain your answer.

(ii) Suppose instead that Gemma says "Beth always tells the truth" and Beth says "That's wrong."

Who always tells the truth? Who always lies? Briefly explain your answer.

(iii) Suppose instead that Alf says "Beth is the one who behaves randomly" and Gemma says "Alf always lies". Then Beth says "You have heard enough to determine who always tells the truth".

Who always tells the truth? Who always lies? Briefly explain your answer.

[MAT, 2007Q7]

For applicants in {CS} only.

Suppose we have a collection of tiles, each containing two strings of letters, one above the other. A **match** is a list of tiles from the given collection (tiles may be used repeatedly) such that the string of letters along the top is the same as the string of letters along the bottom. For example, given the collection

$$\left\{\begin{bmatrix}\text{AA}\\\hline \text{A}\end{bmatrix}, \begin{bmatrix}\text{B}\\\hline \text{ABA}\end{bmatrix}, \begin{bmatrix}\text{CCA}\\\hline \text{CA}\end{bmatrix}\right\},$$

the list

$$\begin{bmatrix}\text{AA}\\\hline \text{A}\end{bmatrix}\begin{bmatrix}\text{B}\\\hline \text{ABA}\end{bmatrix}\begin{bmatrix}\text{AA}\\\hline \text{A}\end{bmatrix}$$

is a match since the string AABAA occurs both on the top and bottom.

Consider the following set of tiles:

$$\left\{\begin{bmatrix}\text{X}\\\hline \text{U}\end{bmatrix}, \begin{bmatrix}\text{UU}\\\hline \text{U}\end{bmatrix}, \begin{bmatrix}\text{Z}\\\hline \text{X}\end{bmatrix}, \begin{bmatrix}\text{E}\\\hline \text{ZE}\end{bmatrix}, \begin{bmatrix}\text{Y}\\\hline \text{U}\end{bmatrix}, \begin{bmatrix}\text{Z}\\\hline \text{Y}\end{bmatrix}\right\}.$$

(i) What tile must a match begin with?

(ii) Write down all the matches which use four tiles (counting any repetitions).

(iii) Suppose we replace the tile $\begin{bmatrix}\text{E}\\\hline \text{ZE}\end{bmatrix}$ with $\begin{bmatrix}\text{E}\\\hline \text{ZZZE}\end{bmatrix}$.

What is the least number of tiles that can be used in a match?
How many different matches are there using this smallest numbers of tiles?
[Hint: you may find it easiest to construct your matches backwards from right to left.]

Consider a new set of tiles $\left\{\begin{bmatrix}\text{XXXXXXX}\\\hline \text{X}\end{bmatrix}, \begin{bmatrix}\text{X}\\\hline \text{XXXXXXXXXX}\end{bmatrix}\right\}$. (The first tile has seven Xs on top, and the second tile has ten Xs on the bottom.)

(iv) For which numbers n do there exist matches using n tiles? Briefly justify your answer.

MAT 2008

MAT 2008
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

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Further credit cannot be obtained by attempting extra questions.

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1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), or (D) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

[MAT, 2008Q1(A)]

The function

$$y = 2x^3 - 6x^2 + 5x - 7$$

has

(A) no stationary points.

(B) one stationary point.

(C) two stationary points.

(D) three stationary points.

[MAT, 2008Q1(B)]

Which is the smallest of these values?

(A) $\log_{10} \pi$

(B) $\sqrt{\log_{10}(\pi^2)}$

(C) $\left(\frac{1}{\log_{10} \pi}\right)^3$

(D) $\frac{1}{\log_{10} \sqrt{\pi}}$

[MAT, 2008Q1(C)]

The simultaneous equations in x, y

$$(\cos\theta)x - (\sin\theta)y = 2$$

$$(\sin\theta)x + (\cos\theta)y = 1$$

are solvable

(A) for all values of θ in the range $0 \leq \theta < 2\pi$.

(B) except for one value of θ in the range $0 \leq \theta < 2\pi$.

(C) except for two values of θ in the range $0 \leq \theta < 2\pi$.

(D) except for three values of θ in the range $0 \leq \theta < 2\pi$.

[MAT, 2008Q1(D)]

When

$$1 + 3x + 5x^2 + 7x^3 + \cdots + 99x^{49}$$

is divided by $x - 1$ the remainder is

(A) 2000

(B) 2500

(C) 3000

(D) 3500

[MAT, 2008Q1(E)]

The highest power of x in

$$\{[(2x^6 + 7)^3 + (3x^8 - 12)^4]^5 + [(3x^5 - 12x^2)^5 + (x^7 + 6)^4]^6\}^3$$

is

(A) x^{424}

(B) x^{450}

(C) x^{500}

(D) x^{504}

[MAT, 2008Q1(F)]

If the trapezium rule is used to estimate the integral

$$\int_0^1 f(x)\,\mathrm{d}x,$$

by splitting the interval $0 \le x \le 1$ into 10 intervals then an overestimate of the integral is produced. It follows that

(A) the trapezium rule with 10 intervals underestimates $\int_0^1 2f(x)\,\mathrm{d}x$.

(B) the trapezium rule with 10 intervals underestimates $\int_0^1 (f(x) - 1)\,\mathrm{d}x$.

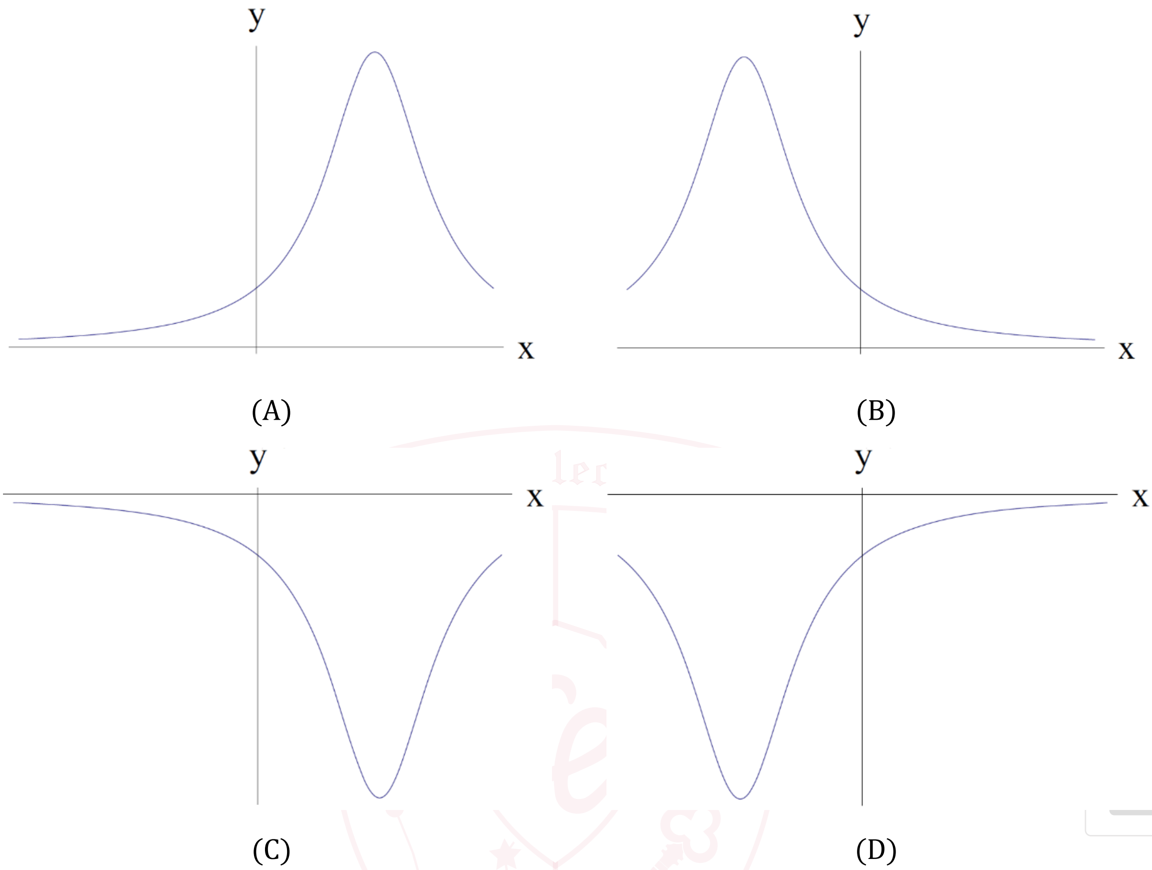
(C) the trapezium rule with 10 intervals underestimates $\int_1^2 f(x-1)\,\mathrm{d}x$.

(D) the trapezium rule with 10 intervals underestimates $\int_0^1 \big(1 - f(x)\big)\,\mathrm{d}x$.

[MAT, 2008Q1(G)]

Which of the graphs below is a sketch of

$$y = \frac{1}{4x - x^2 - 5} \ ?$$



[MAT, 2008Q1(H)]

The equation

$$9^x - 3^{x+1} = k$$

Has one or more real solutions precisely when

(A) $k \geq -\frac{9}{4}$

(B) $k > 0$

(C) $k \leq -1$

(D) $k \geq \frac{5}{8}$

[MAT, 2008Q1(I)]

The function $S(n)$ is defined for positive integers n by

$$S(n) = \text{sum of the digits of } n.$$

For example, $S(723) = 7 + 2 + 3 = 12$. The sum

$$S(1) + S(2) + S(3) + \cdots S(99)$$

equals

(A) 746

(B) 862

(C) 900

(D) 924

[MAT, 2008Q1(J)]

In the range $0 \leq x < 2\pi$ the equation $(3 + \cos x)^2 = 4 - 2\sin^8 x$ has

(A) 0 solutions

(B) 1 solution

(C) 2 solutions

(D) 3 solutions

[MAT, 2008Q2]

(i) Find a pair of positive integers, x_1 and y_1, that solve the equation

$$(x_1)^2 - 2(y_1)^2 = 1.$$

(ii) Given integers a, b, we define two sequences $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ by setting

$$x_{n+1} = 3x_n + 4y_n, \qquad y_{n+1} = ax_n + by_n, \qquad \text{for } n \geq 1.$$

Find *positive* values for a, b such that

$$(x_{n+1})^2 - 2(y_{n+1})^2 = (x_n)^2 - 2(y_n)^2.$$

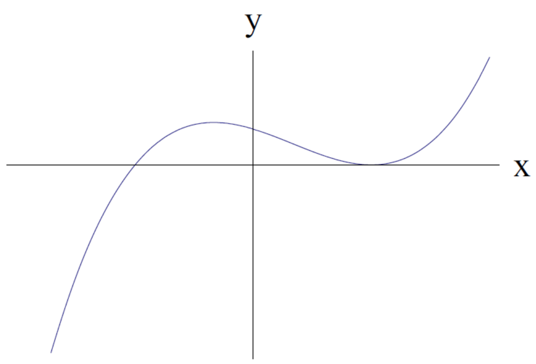
(iii) Find a pair of integers X, Y which satisfy $X^2 - 2Y^2 = 1$ such that $X > Y > 50$.

(iv) (Using the values of a and b found in part (ii)) what is the approximate value of x_n/y_n as n increases?

[MAT, 2008Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy, and Math & CS} only. Not for {CS}

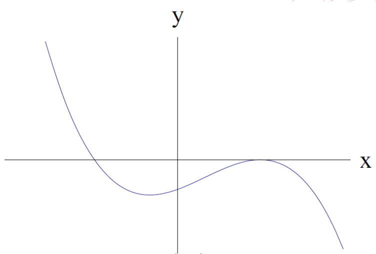
(i) The graph $y = f(x)$ of a certain function has been plotted below



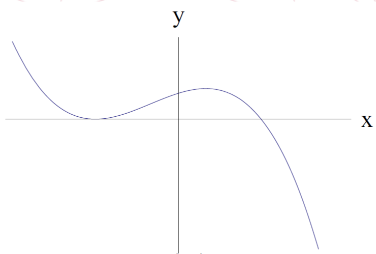
On the next three pairs of axes (A), (B), (C) are graphs of

$$y = f(-x), f(x-1), -f(x)$$

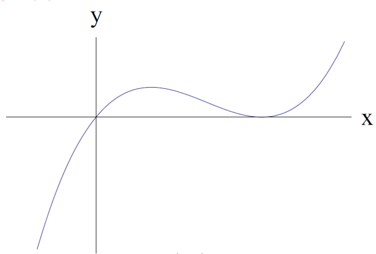
in some order. Say which axes correspond to which graphs



(A)



(B)



(C)

(ii) Sketch, on the axes opposite, graphs of *both* of the following functions

$$y = 2^{-x^2} \text{ and } y = 2^{2x-x^2}$$

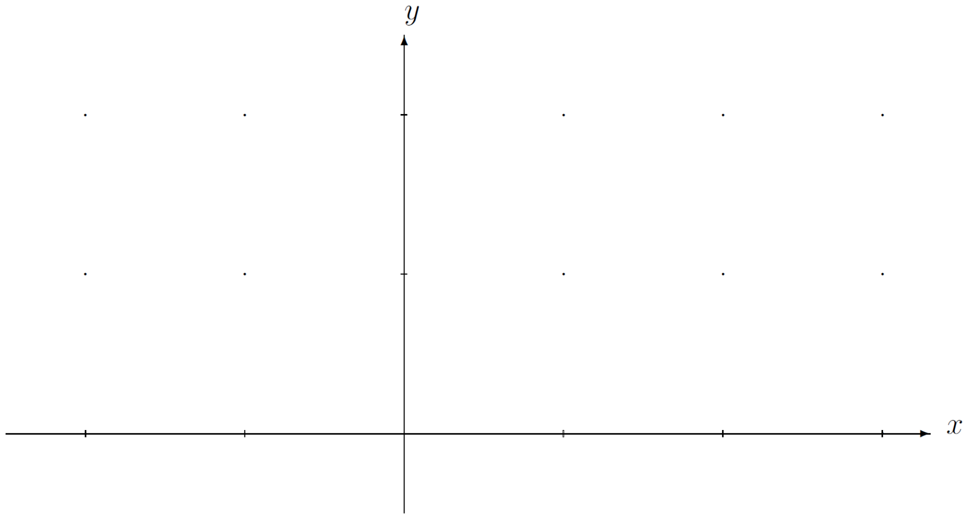
Carefully label any stationary points

(iii) Let c be a real number and define the following integral

$$I(c) = \int_0^1 2^{-(x-c)^2}\,\mathrm{d}x$$

State the value(s) of c for which $I(c)$ is largest. Briefly explain your reasoning

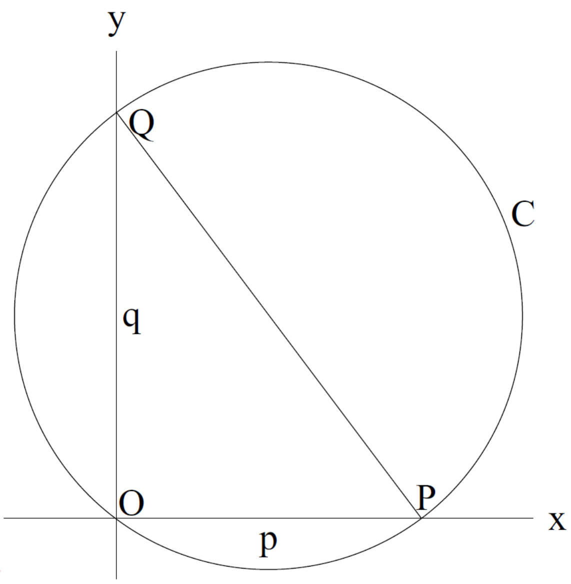
[Note you are not being asked to calculate this maximum value.]



y
x

[MAT, 2008Q4]

For applicants in {Math, Math & Statistics, and Math & Philosophy} only. Not for {Math & CS, and CS}



Let p and q be positive real numbers. Let P denote the point $(p, 0)$ and Q denote the point $(0, q)$.

(i) Show that the equation of the circle C which passes through P, Q and the origin O is

$$x^2 - px + y^2 - qy = 0$$

Find the centre and area of C

(ii) Show that

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} \geq \pi.$$

(iii) Find the angles OPQ and OQP if

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} = 2\pi.$$

[MAT, 2008Q5]

The Millennium school has 1000 students and 1000 student lockers. The lockers are in a line in a long corridor and are numbered from 1 to 1000.

Initially all the lockers are closed (but unlocked).

The first student walks along the corridor and opens every locker.

The second student then walks along the corridor and closes every second locker, i.e. closes lockers 2, 4, 6, etc. At that point there are 500 lockers that are open and 500 that are closed.

The third student then walks along the corridor, changing the state of every third locker. Thus s/he closes locker 3 (which had been left open by the first student), opens locker 6 (closed by the second student), closes locker 9, etc.

All the remaining students now walk by in order, with the kth student changing the state of every kth locker, and this continues until all 1000 students have walked along the corridor.

(i) How many lockers are closed immediately after the third student has walked along the corridor? Explain your reasoning.

(ii) How many lockers are closed immediately after the fourth student has walked along the corridor? Explain your reasoning.

(iii) At the end (after all 1000 students have passed), what is the state of locker 100? Explain your reasoning.

(iv) After the *hundredth* student has walked along the corridor, what is the state of locker 1000? Explain your reasoning.

[MAT, 2008Q6]

For applicants in {CS and Math & CS} only.

(i) A, B and C are three people. One of them is a *liar* who always tells lies, another is a *saint* who always tells the truth, and the third is a *switcher* who sometimes tells the truth and sometimes lies. They make the following statements:

A: I am the liar.

B: A is the liar.

C: I am not the liar.

Who is the liar among A, B and C? Briefly explain your answer.

(ii) P, Q and R are three more people, one a liar, one a saint, and the third a *contrarian* who tells a lie if he is the first to speak or if the preceding speaker told the truth, but otherwise tells the truth. They make the following statements:

P: Q is the liar.

Q: R is the liar.

R: P is the liar.

Who is the liar among P, Q and R? Briefly explain your answer.

(iii) X, Y and Z are three more people, one a liar, one a switcher and one a contrarian. They make the following statements:

X: Y is the liar.

Y: Z is the liar.

Z: X is the liar.

X: Y is the liar.

Y: X is the liar.

Who is the liar among X, Y and Z? Briefly explain your answer.

[MAT, 2008Q7]

For applicants in {CS} only.

Ox-words are sequences of letters a and b that are constructed according to the following rules:

I. The sequence consisting of no letters is an Ox-word.

II. If the sequence W is an Ox-word, then the sequence that begins with a, followed by W and ending in b, written aWb, is an Ox-word.

III. If the sequences U and V are Ox-words, then the sequence U followed by V, written UV, is an Ox-word.

All Ox-words are constructed using these rules. The *length* of an Ox-word is the number of letters that occur in it. For example $aabb$ and $abab$ are Ox-words of length 4.

(i) Show that every Ox-word has an even length.

(ii) List all Ox-words of length 6.

(iii) Let W be an Ox-word. Is the number of occurrences of a in W necessarily equal to the number of occurrences of b in W? Justify your answer.

You may now assume that every Ox-word (of positive length) can be written *uniquely* in the form $aWbW'$ where W and W' are Ox-words.

(iv) For $n \geq 0$, let C_n be the number of Ox-words of length $2n$. Find an expression for C_{n+1} in terms of $C_0, C_1, \cdots, C_n$. Explain your reasoning.

MAT 2009

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On-line Exam

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TIME ALLOWED: 150 MINUTES

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	(A)	(B)	(C)	(D)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

[MAT, 2009Q1(A)]

The smallest value of

$$I(a) = \int_0^1 (x^2 - a)^2 \, dx,$$

as a varies, is

(A) $\frac{3}{20}$

(B) $\frac{4}{45}$

(C) $\frac{7}{13}$

(D) 1

[MAT, 2009Q1(B)]

The point on the circle

$$x^2 + y^2 + 6x + 8y = 75,$$

which is closest to the origin, is at what distance from the origin?

(A) 3

(B) 4

(C) 5

(D) 10

[MAT, 2009Q1(C)]

Given a real constant c, the equation

$$x^4 = (x - c)^2$$

has four real solutions (including possible repeated roots) for

(A) $c \leq \frac{1}{4}$

(B) $-\frac{1}{4} \leq c \leq \frac{1}{4}$

(C) $c \leq -\frac{1}{4}$

(D) all values of c

[MAT, 2009Q1(D)]

The smallest positive integer n such that

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n \geq 100,$$

is

(A) 99

(B) 101

(C) 199

(D) 300

[MAT, 2009Q1(E)]

In the range $0 \leq x < 2\pi$, the equation

$$2^{\sin^2 x} + 2^{\cos^2 x} = 2$$

(A) has 0 solutions.

(B) has 1 solution.

(C) has 2 solutions.

(D) holds for all values of x.

[MAT, 2009Q1(F)]

The equation in x

$$3x^4 - 16x^3 + 18x^2 + k = 0$$

has four real solutions

(A) when $-27 < k < 5$.

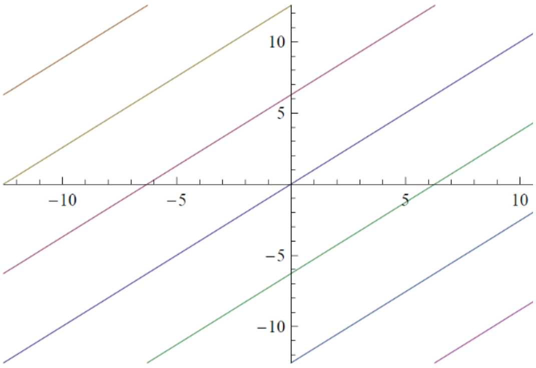
(B) when $5 < k < 27$.

(C) when $-27 < k < -5$.

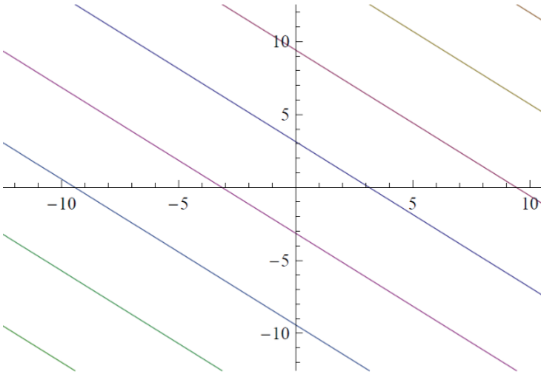
(D) when $-5 < k < 0$.

[MAT, 2009Q1(G)]

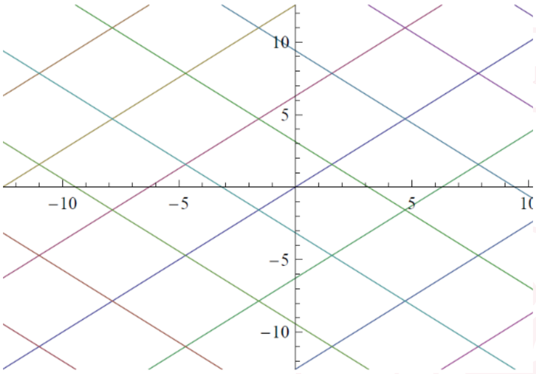
The graph of all those points (x, y) in the xy-plane which satisfy the equation $\sin y = \sin x$ is drawn in



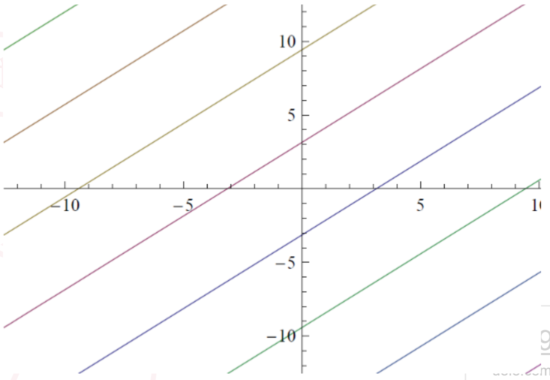
(A)



(B)



(C)



(D)

[MAT, 2009Q1(H)]

When the trapezium rule is used to estimate the integral

$$\int_0^1 2^x \, dx$$

by dividing the interval $0 \le x \le 1$ into N subintervals the answer achieved is

(A) $\frac{1}{2N}\left\{1 + \frac{1}{2^{\frac{1}{N}}+1}\right\}$

(B) $\frac{1}{2N}\left\{1 + \frac{2}{2^{\frac{1}{N}}-1}\right\}$

(C) $\frac{1}{N}\left\{1 - \frac{1}{2^{\frac{1}{N}}-1}\right\}$

(D) $\frac{1}{2N}\left\{\frac{5}{2^{\frac{1}{N}}+1} - 1\right\}$

[MAT, 2009Q1(I)]

The polynomial

$$n^2x^{2n+3} - 25nx^{n+1} + 150x^7$$

has $x^2 - 1$ as a factor

(A) for no values of n.

(B) for $n = 10$ only.

(C) for $n = 15$ only.

(D) for $n = 10$ and $n = 15$ only.

[MAT, 2009Q1(J)]

The number of *pairs of positive integers* x, y which solve the equation

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is

(A) 0

(B) 2^6

(C) $2^9 - 1$

(D) $2^{10} + 2$

[MAT, 2009Q2]

A list of real numbers $x_1, x_2, x_3, \cdots$ is defined by $x_1 = 1, x_2 = 3$ and then for $n \geq 3$ by

$$x_n = 2x_{n-1} - x_{n-2} + 1.$$

So, for example,

$$x_3 = 2x_2 - x_1 + 1 = 2 \times 3 - 1 + 1 = 6.$$

(i) Find the values of x_4 and x_5.

(ii) Find values of real constants A, B, C such that <u>for $n = 1, 2, 3$</u>,

$$x_n = A + Bn + Cn^2. \qquad (*)$$

(iii) Assuming that equation $(*)$ holds true <u>for all $n \geq 1$</u>, find the smallest n such that $x_n \geq 800$.

(iv) A second list of real numbers $y_1, y_2, y_3, \cdots$ is defined by $y_1 = 1$ and

$$y_n = y_{n-1} + 2n.$$

Find, explaining your reasoning, a formula for y_n which holds for $n \geq 2$.

What is the approximate value of x_n / y_n for large values of n?

[MAT, 2009Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS}.

For a positive whole number n, the function $f_n(x)$ is defined by

$$f_n(x) = (x^{2n-1} - 1)^2$$

(i) On the axes provided opposite, sketch the graph of $y = f_2(x)$ labelling where the graph meets the axes.

(ii) On the same axes sketch the graph of $y = f_n(x)$ where n is a large positive integer.

(iii) Determine

$$\int_0^1 f_n(x)\,\mathrm{d}x.$$

(iv) The *positive* constants A and B are such that

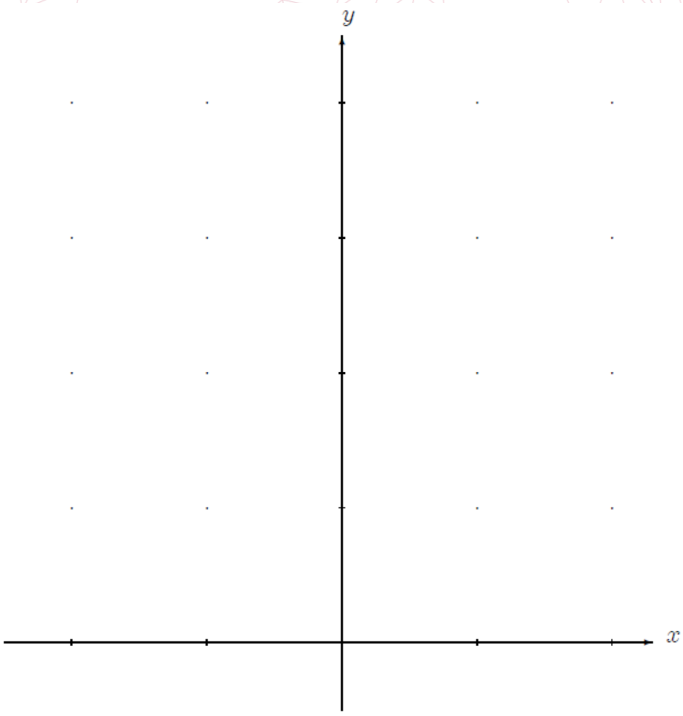
$$\int_0^1 f_n(x)\,\mathrm{d}x \le 1 - \frac{A}{n+B} \quad \text{for all } n \ge 1.$$

Show that

$$(3n-1)(n+B) \ge A(4n-1)n,$$

and explain why $A \le 3/4$.

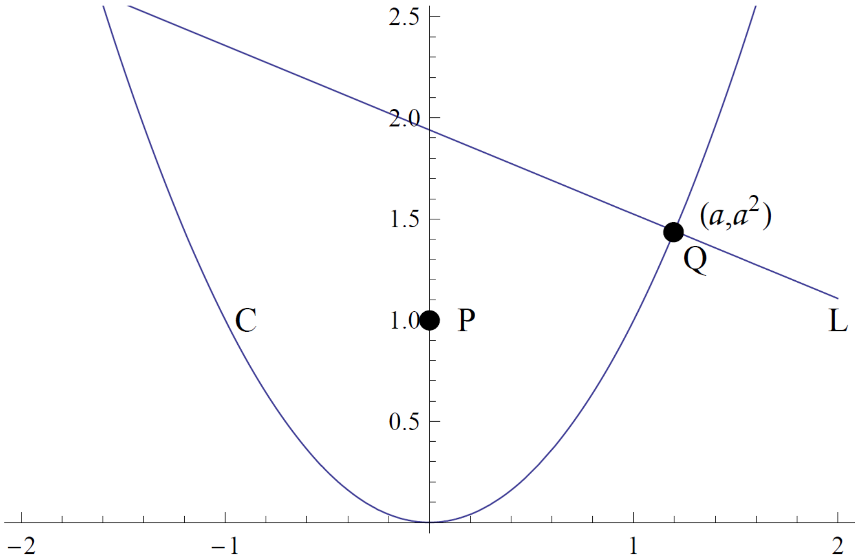
(v) When $A = 3/4$, what is the smallest possible value of B?



[MAT, 2009Q4]

For applicants in {Math, Math & Statistics, Math & Philosophy} only. Not for {Math & CS, and CS}.

As shown in the diagram below: C is the parabola with equation $y = x^2$; P is the point $(0,1)$; Q is the point (a, a^2) on C; L is the normal to C which passes through Q.



(i) Find the equation of L.

(ii) For what values of a does L pass through P?

(iii) Determine $|QP|^2$ as a function of a, where $|QP|$ denotes the distance from P to Q.

(iv) Find the values of a for which $|QP|$ is smallest.

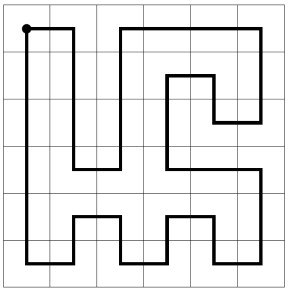
(v) Find a point R, in the xy-plane but not on C, such that $|RQ|$ is smallest for a unique value of a. Briefly justify your answer.

[MAT, 2009Q5]

Given an $n \times n$ grid of squares, where $n > 1$, a *tour* is a path drawn within the grid such that:

- along its way the path moves, horizontally or vertically, from the centre of one square to the centre of an adjacent square;
- the path starts and finishes in the same square;
- the path visits the centre of every other square just once.

For example, below is a tour drawn in a 6×6 grid of squares which starts and finishes in the top-left square.



For parts (i)-(iv) it is assumed that n is *even*.

(i) With the aid of a diagram, show how a tour, which starts and finishes in the top-left square, can be drawn in any $n \times n$ grid.

(ii) Is a tour still possible if the start/finish point is changed to the centre of a different square? Justify your answer.

Suppose now that a robot is programmed to move along a tour of an $n \times n$ grid. The robot understands two commands:

- command R which turns the robot clockwise through a right angle;
- command F which moves the robot forward to the centre of the next square.

The robot has a program, a list of commands, which it performs in the given order to complete a tour; say that, in total, command R appears r times in the program and command F appears f times.

(iii) Initially the robot is in the top-left square pointing to the right. Assuming the first command is an F, what is the value of f? Explain also why $r + 1$ is a multiple of 4.

(iv) Must the results of part (iii) still hold if the robot starts and finishes at the centre of a different square? Explain your reasoning.

(v) Show that a tour of an $n \times n$ grid is not possible when n is odd.

[MAT, 2009Q6]

For applicants in {CS and Math & CS} only.

(i) Alice, Bob, and Charlie make the following statements:

Alice: Bob is lying.

Bob: Charlie is lying.

Charlie: $1+1=2$.

Who is telling the truth? Who is lying? Explain your answer.

(ii) Now Alice, Bob, and Charlie make the following statements:

Alice: Bob is telling the truth.

Bob: Alice is telling the truth.

Charlie: Alice is lying.

What are the possible numbers of people telling the truth? Explain your answer.

(iii) They now make the following statements:

Alice: Bob and Charlie are both lying.

Bob: Alice is telling the truth or Charlie is lying (or both).

Charlie: Alice and Bob are both telling the truth.

Who is telling the truth and who is lying on this occasion? Explain your answer.

[MAT, 2009Q7]

For applicants in {CS} only.

Consider sequences of the letters M, X and W. *Valid* sequences are made up according to the rule that an M and a W can never be adjacent in the sequence. So M, XMXW, and XMMXW are examples of valid sequences, whereas the sequences MW and XWMX are not valid.

(i) Clearly, there are 3 valid sequences of length 1. List all valid sequences of length 2.

(ii) Let $g(n)$ denote the number of valid sequences of length n. Further, let $m(n)$, $x(n)$, $w(n)$ denote the number of valid sequences of length n that start with an M, an X, a W respectively.

Explain why

$$\begin{aligned} m(n) &= w(n), \\ m(n) &= m(n-1) + x(n-1) \quad \text{for } n > 1, \\ x(n) &= 2m(n-1) + x(n-1) \quad \text{for } n > 1, \end{aligned}$$

and write down a formula for $g(n)$ in terms of $m(n)$ and $x(n)$.

Hence compute $g(3)$, and verify that $g(4) = 41$.

(iii) Given a sequence using these letters then we say that it is *reflexive* if the following operation on the sequence does not change it: reverse the letters in the sequence, and then replace each occurrence of M by W and vice versa. So MXW, WXXM and XWXMX are reflexive strings, but MXM and XMXX are not. Let $r(n)$ be the number of valid, reflexive sequences of length n.

If a sequence is reflexive and has odd length, what must the middle letter be? Explain your answer.

Hence, show that

$$r(n) = \begin{cases} x\left(\dfrac{n+1}{2}\right) & \text{if } n \text{ is odd,} \\ x\left(\dfrac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

MAT 2010

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On-line Exam

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TIME ALLOWED: 150 MINUTES

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1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), or (D) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

[MAT, 2010Q1(A)]

The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are precisely

(A) $k \leq 0$

(B) $k \geq 4$

(C) $k \geq 0$ or $k \leq -4$

(D) $-4 \leq k \leq 0$

[MAT, 2010Q1(B)]

The sum of the first $2n$ terms of

$$1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \cdots$$

is

(A) $2^n + 1 - 2^{1-n}$

(B) $2^n + 2^{-n}$

(C) $2^{2n} - 2^{3-2n}$

(D) $\frac{2^n - 2^{-n}}{3}$

[MAT, 2010Q1(C)]

In the range $0 \leq x < 2\pi$, the equation

$$\sin^2 x + 3 \sin x \cos x + 2 \cos^2 x = 0$$

has

(A) 1 solution.

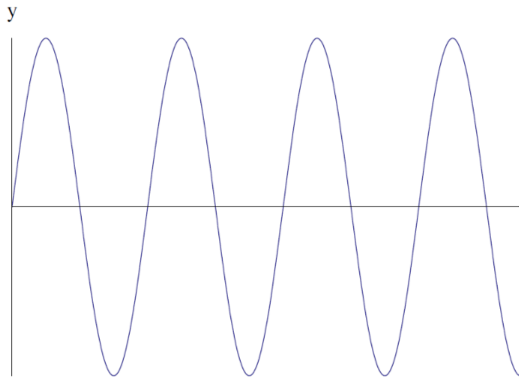
(B) 2 solutions.

(C) 3 solutions.

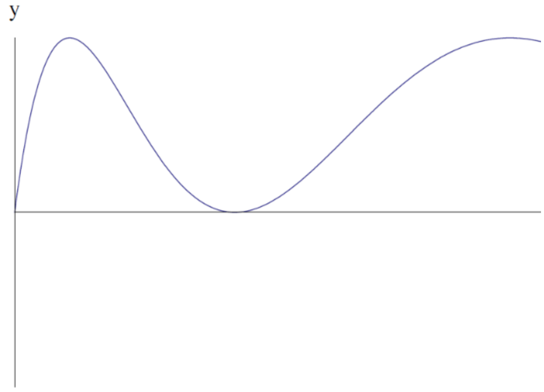
(D) 4 solutions.

[MAT, 2010Q1(D)]

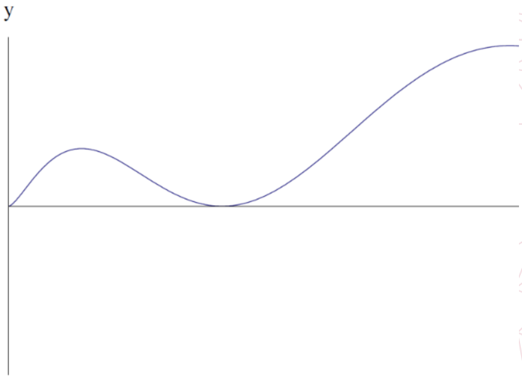
The graph of $y = \sin^2 \sqrt{x}$ is drawn in



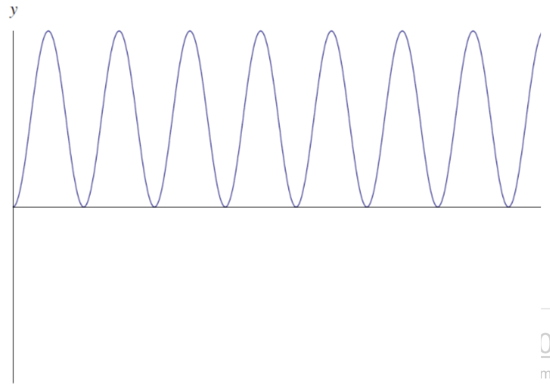
(A)



(B)



(C)



(D)

[MAT, 2010Q1(E)]

Which is the largest of the following four numbers?

(A) $\log_2 3$

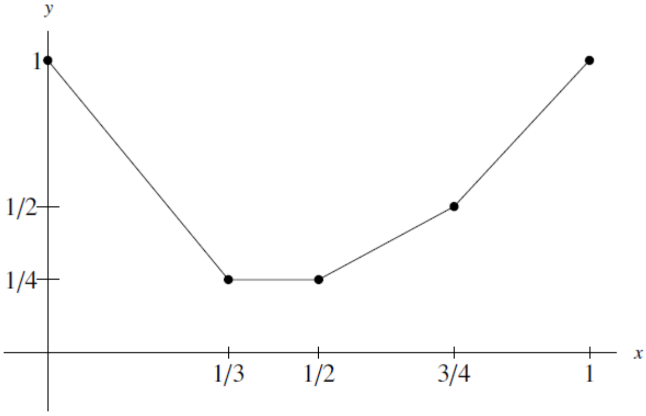
(B) $\log_4 8$

(C) $\log_3 2$

(D) $\log_5 10$

[MAT, 2010Q1(F)]

The graph $y = f(x)$ of a function is drawn below for $0 \le x \le 1$.



The trapezium rule is then used to estimate

$$\int_0^1 f(x)\,dx$$

By dividing $0 \le x \le 1$ into n equal intervals. The estimate calculated will equal the actual integral when

(A) n is a multiple of 4.

(B) n is a multiple of 6.

(C) n is a multiple of 8.

(D) n is a multiple of 12.

[MAT, 2010Q1(G)]

The function f, defined for whole positive numbers, satisfies $f(1) = 1$ and also the rules

$$f(2n) = 2f(n),$$
$$f(2n+1) = 4f(n)$$

for all values of n. How many numbers n satisfy $f(n) = 16$?

(A) 3

(B) 4

(C) 5

(D) 6

[MAT, 2010Q1(H)]

Given a positive integer n and a real number k, consider the following equation in x,

$$(x-1)(x-2)(x-3)\times\cdots\times(x-n)=k.$$

Which of the following statements about this equation is true?

(A) If $n=3$, then the equation has no real solution x for some values of k.

(B) If n is even, then the equation has a real solution x for any given value of k.

(C) If $k\geq 0$ then the equation has (at least) one real solution x.

(D) The equation never has a repeated solution x for any given values of k and n.

[MAT, 2010Q1(I)]

For a positive number a, let

$$I(a)=\int_0^a\left(4-2^{x^2}\right)\mathrm{d}x$$

Then $\mathrm{d}I/\mathrm{d}a=0$ when a equals

(A) $\frac{1+\sqrt{5}}{2}$

(B) $\sqrt{2}$

(C) $\frac{\sqrt{5}-1}{2}$

(D) 1

[MAT, 2010Q1(J)]

Let a,b,c be positive numbers. There are *finitely many positive whole* numbers x,y which satisfy the inequality

$$a^x>cb^y$$

if

(A) $a>1$ or $b<1$

(B) $a<1$ or $b<1$

(C) $a<1$ and $b<1$

(D) $a<1$ and $b>1$

[MAT, 2010Q2]

Suppose that a, b, c are *integers* such that

$$a\sqrt{2} + b = c\sqrt{3}$$

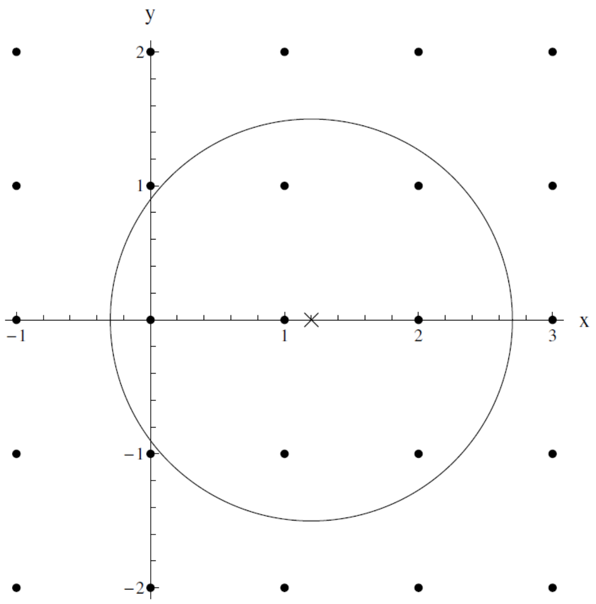
(i) By squaring both sides of the equation, show that $a = b = c = 0$.

[You may assume that $\sqrt{2}$,$\sqrt{3}$ and $\sqrt{2/3}$ are all irrational numbers. An irrational number is one which cannot be written in the form p/q where p and q are integers.]

(ii) Suppose now that m, n, M, N are *integers* such that the distance from the point (m, n) to $(\sqrt{2}, \sqrt{3})$ equals the distance from (M, N) to $(\sqrt{2}, \sqrt{3})$.

Show that $m = M$ and $n = N$.

Given real numbers a, b and a positive number r, let $N(a, b, r)$ be the number of integer pairs x, y such that the distance between the points (x, y) and (a, b) is less than or equal to r. For example, we see that $N(1.2, 0, 1.5) = 7$ in the diagram below.



(iii) Explain why $N(0.5, 0.5, r)$ is a multiple of 4 for any value of r.

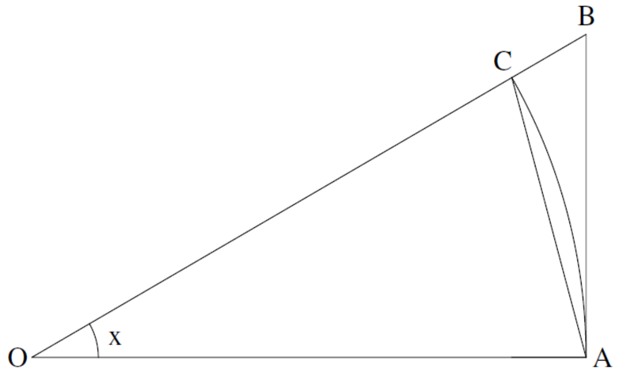
(iv) Let k be any positive integer. Explain why there is a positive number r such that

$$N(\sqrt{2}, \sqrt{3}, r) = k.$$

[MAT, 2010Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy, Math & CS} only. Not for {CS}.

[In this question, you may assume that the derivative of $\sin x$ is $\cos x$.]



(i) In the diagram above OA and OC are of length 1 and subtend an angle x at O. The angle BAO is a right angle and the circular arc from A to C, centred at O, is also drawn.

By consideration of various areas in the above diagram, show, for $0 < x < \frac{\pi}{2}$, that

$$x \cos x < \sin x < x.$$

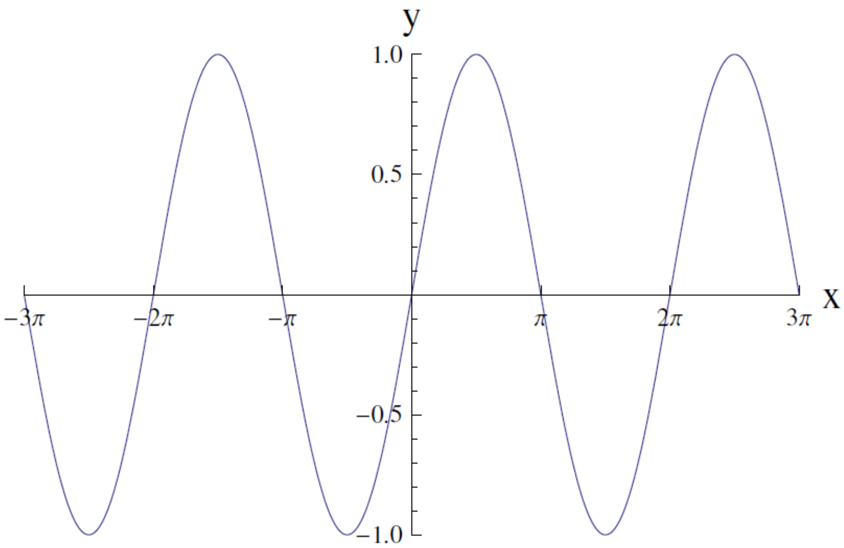
(ii) Sketch, on the axes provided on the opposite page, the graph of

$$y = \frac{\sin x}{x}, \qquad 0 < x < 4\pi.$$

Justify your value that y takes as x becomes small.

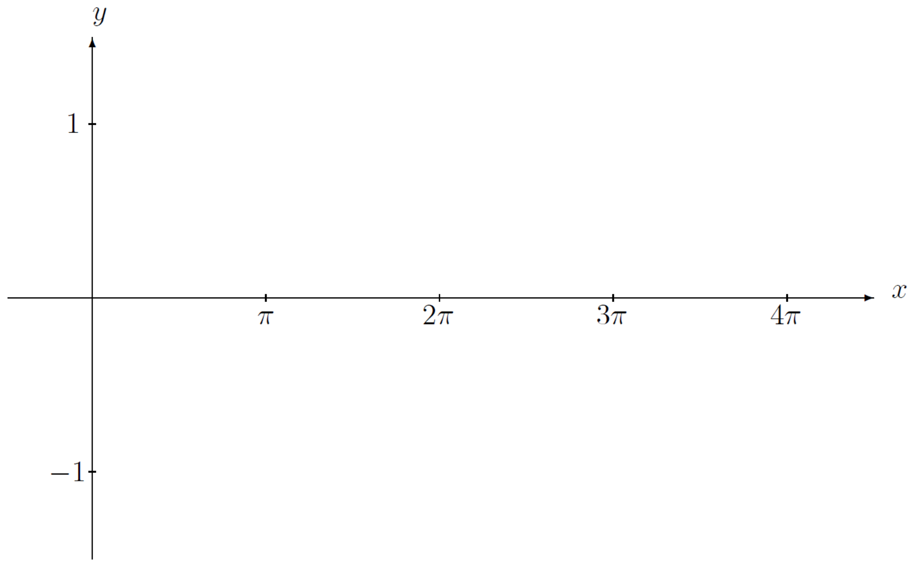
[You do not need to determine the coordinates of the turning points.]

(iii) Drawn below is a graph of $y = \sin x$. Sketch on the same axes the line $y = cx$ where $c > 0$ is such that the equation $\sin x = cx$ has *exactly 5 solutions.*



(iv) Draw the line $y = c$ on the axes on the opposite page.

(v) If X is the largest of the five solutions of the equation $\sin x = cx$, explain why $\tan X = X$.


y
1
-1
x
π
2π
3π
4π

[MAT, 2010Q4]

For applicants in {Math, Math & Statistics, and Math & Philosophy} only. Not for {Math & CS, and CS}

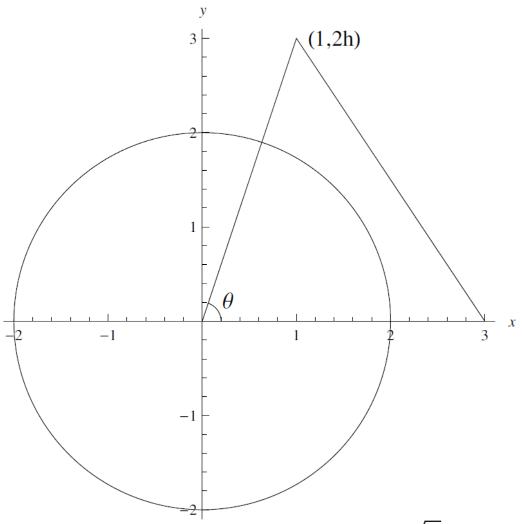


Diagram when $h > 2/\sqrt{5}$

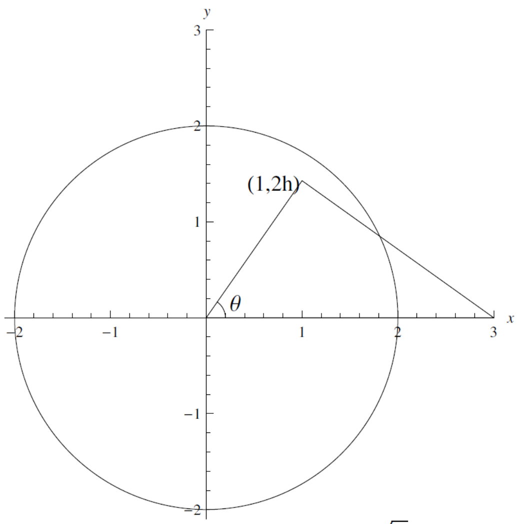


Diagram when $h < \sqrt{3}/2$

The three corners of a triangle T are $(0,0)$, $(3,0)$, $(1,2h)$ where $h > 0$. The circle C has equation $x^2 + y^2 = 4$. The angle of the triangle at the origin is denoted as θ. The circle and triangle are drawn in the diagrams above for different values of h.

(i) Express $\tan\theta$ in terms of h.

(ii) Show that the point $(1,2h)$ lies inside C when $h < \sqrt{3}/2$.

(iii) Find the equation of the line connecting $(3,0)$ and $(1,2h)$. Show that this line is tangential to the circle C when $h = 2/\sqrt{5}$.

(iv) Suppose now that $h > 2/\sqrt{5}$. Find the area of the region inside both C and T in terms of θ.

(v) Now let $h = 6/7$. Show that the point $(8/5,\ 6/5)$ lies on both the line (from part (iii)) and the circle C.

Hence show that the area of the region inside both C and T equals

$$\frac{27}{35} + 2\alpha$$

where α is an angle whose tangent, $\tan\alpha$, you should determine.

[You may use the fact that the area of a triangle with corners $(0,0)$, (a,b), (c,d) equals $\frac{1}{2}|ad - bc|$.]

[MAT, 2010Q5]

This question concerns calendar dates of the form

$$d_1d_2/m_1m_2/y_1y_2y_3y_4$$

in the order day/month/year.

The question specifically concerns those dates which contain no repetitions of a digit. For example, the date 23/05/1967 is one such date but 07/12/1974 is not such a date as both $1 = m_1 = y_1$ and $7 = d_2 = y_3$ are repeated digits.

We will use the Gregorian Calendar throughout (this is the calendar system that is standard throughout most of the world; see below.)

(i) Show that there is no date with no repetition of digits in the years from 2000 to 2099.

(ii) What was the last date before today with no repetition of digits? Explain your answer.

(iii) When will the next such date be? Explain your answer.

(iv) How many such dates were there in years from 1900 to 1999? Explain your answer.

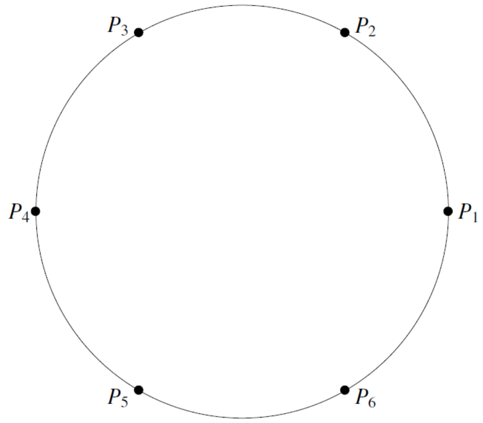
[The Gregorian Calendar uses 12 months, which have, respectively, 31, 28 or 29, 31, 30, 31, 30, 31, 31, 30, 31, 30 and 31 days. The second month (February) has 28 days in years that are not divisible by 4, or that are divisible by 100 but not 400 (such as 1900); it has 29 days in the other years (leap years).]

[MAT, 2010Q6]

For applicants in {CS and Math & CS} only.

In the questions below, the people involved make statements about each other. Each person is either a *saint* (S) who always tells the truth or a *liar* (L) who always lies.

(i) Six people, $P_1, P_2, \dots, P_6$ sit in order around a circular table with P_1 sitting to P_6's right, as shown in the diagram below.



(a) Suppose all six people say "the person directly opposite me is telling the truth". One possibility is that all six are lying. But, in total, how many different possibilities are there? Explain your reasoning.

(b) Suppose now that all six people say "the person to my left is lying". In how many different ways can this happen? Explain your reasoning.

(ii) Now n people $Q_1, Q_2, \cdots, Q_n$ sit in order around a circular table with Q_1 sitting to Q_n's right.

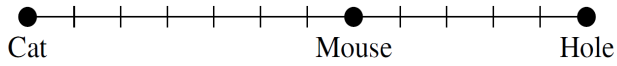
(a) Suppose that all n people make the statement "the person on my left is lying *and* the person on my right is telling the truth". Explain why everyone is lying.

(b) Suppose now that every person makes the statement "either the people to my left and right are both lying *or* both are telling the truth". If at least one person is lying, show that n is a multiple of three.

[MAT, 2010Q7]

For applicants in {CS} only.

In a game of *Cat and Mouse*, a cat starts at position 0, a mouse starts at position m and the mouse's hole is at position h. Here m and h are integers with $0 < m < h$. By way of example, a starting position is shown below where $m = 7$ and $h = 12$.



With each turn of the game, one of the mouse or cat (but not both) advances one position towards the hole *on the condition that the cat is always strictly behind the mouse and never catches it*. The game ends when the mouse reaches the safety of its hole at position h.

This question is about calculating the number, $g(h, m)$, of different sequences of moves that make a game of Cat and Mouse.

Let C denote a move of the cat and M denote a move of the mouse. Then, for example, $g(3,1) = 2$ as MM and MCM are the only possible games. Also $CMCCM$ is *not* a valid game when $h = 4$ and $m = 2$ as the mouse would be caught on the fourth turn.

(i) Write down the five valid games when $h = 4$ and $m = 2$.

(ii) Explain why $g(h, h-1) = h - 1$ for $h \geq 2$.

(iii) Explain why $g(h, 2) = g(h, 1)$ for $h \geq 3$.

(iv) By considering the possible first moves of a game, explain why

$$g(h, m) = g(h, m+1) + g(h-1, m-1) \quad \text{when } 1 < m < h - 1.$$

(v) Below is a table with certain values of $g(h, m)$ filled in. Complete the remainder of the table and verify that $g(6, 1) = 42$.

m \ h	2	3	4	5	6
5					5
4				4	
3			3		
2		2			
1	1				

MAT 2011

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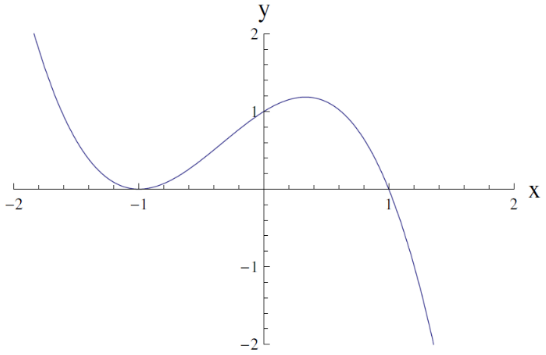
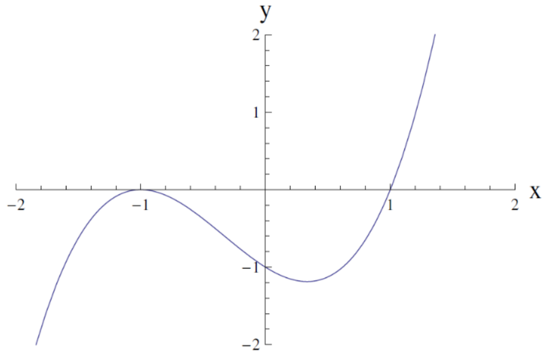
	(A)	(B)	(C)	(D)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

[MAT, 2011Q1(A)]

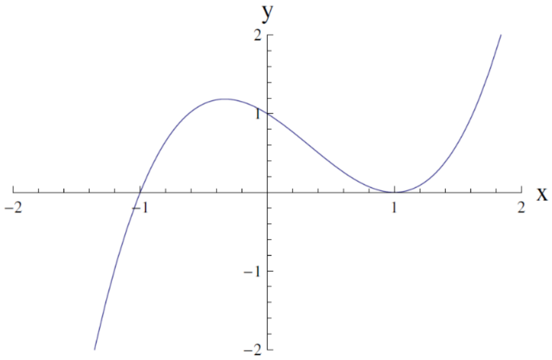
A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axes?



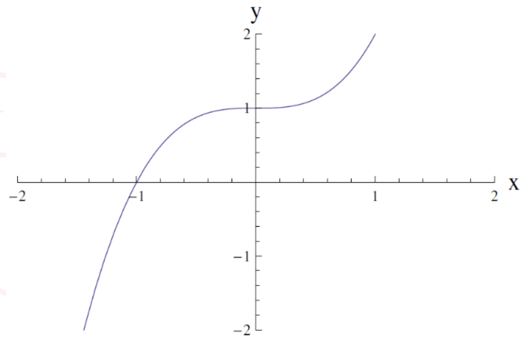
(A)



(B)



(C)



(D)

[MAT, 2011Q1(B)]

A rectangle has perimeter P and area A. The values P and A must satisfy

(A) $P^3 > A$

(B) $A^2 > 2P + 1$

(C) $P^2 \geq 16A$

(D) $PA \geq A + P$

[MAT, 2011Q1(C)]

The sequence x_n is given by the formula

$$x_n = n^3 - 9n^2 + 631$$

The largest value of n for which $x_n > x_{n+1}$ is

(A) 5

(B) 7

(C) 11

(D) 17

[MAT, 2011Q1(D)]

The fraction of the interval $0 \le x \le 2\pi$, for which one (or both) of the inequalities

$$\sin x \ge \frac{1}{2}, \quad \sin 2x \ge \frac{1}{2}$$

is true, equals

(A) $\frac{1}{3}$

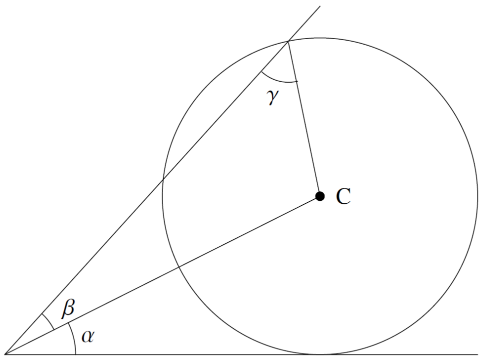
(B) $\frac{13}{24}$

(C) $\frac{7}{12}$

(D) $\frac{5}{8}$

[MAT, 2011Q1(E)]

The circle in the diagram has centre C. Three angles α, β, γ are also indicated



The angles α, β, γ are related by the equation:

(A) $\cos\alpha = \sin(\beta + \gamma)$

(B) $\sin\beta = \sin\alpha \sin\gamma$

(C) $\sin\beta\,(1 - \cos\alpha) = \sin\gamma$

(D) $\sin(\alpha + \beta) = \cos\gamma \sin\alpha$

[MAT, 2011Q1(F)]

Given θ in the range $0 \le \theta < \pi$, the equation

$$x^2 + y^2 + 4x\cos\theta + 8y\sin\theta + 10 = 0$$

represents a circle for

(A) $0 < \theta < \frac{\pi}{3}$

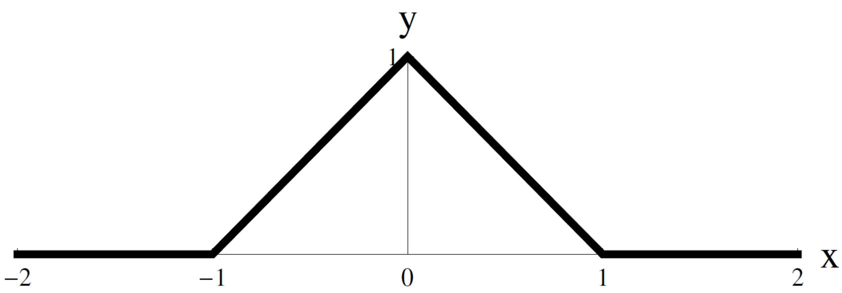
(B) $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$

(C) $0 < \theta < \frac{\pi}{2}$

(D) all values of θ

[MAT, 2011Q1(G)]

A graph of the function $y = f(x)$ is sketched on the axes below:



The value of $\int_{-1}^{1} f(x^2 - 1)\,dx$ equals

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) $\frac{3}{5}$

(D) $\frac{2}{3}$

[MAT, 2011Q1(H)]

The number of *positive* values x which satisfy the equation

$$x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$$

is

(A) 0

(B) 1

(C) 2

(D) 3

[MAT, 2011Q1(I)]

In the range $0 \leq x < 2\pi$ the equation $\sin^8 x + \cos^6 x = 1$ has

(A) 3 solutions

(B) 4 solutions

(C) 6 solutions

(D) 8 solutions

[MAT, 2011Q1(J)]

The function $f(n)$ is defined for positive integers n according to the rules

$$f(1) = 1, \qquad f(2n) = f(n), \qquad f(2n+1) = \left(f(n)\right)^2 - 2.$$

The value of $f(1) + f(2) + f(3) + \cdots + f(100)$ is

(A) -86

(B) -31

(C) 23

(D) 58

[MAT, 2011Q2]

Suppose that x satisfies the equation

$$x^3 = 2x + 1. \qquad (*)$$

(i) Show that

$$x^4 = x + 2x^2 \text{ and } x^5 = 2 + 4x + x^2.$$

(ii) For every integer $k \geq 0$, we can uniquely write

$$x^k = A_k + B_k x + C_k x^2$$

where A_k, B_k, C_k are integers. So, in part (i), it was shown that

$$A_4 = 0, B_4 = 1, C_4 = 2 \text{ and } A_5 = 2, B_5 = 4, C_5 = 1.$$

Show that

$$A_{k+1} = C_k, B_{k+1} = A_k + 2C_k, C_{k+1} = B_k.$$

(iii) Let

$$D_k = A_k + C_k - B_k.$$

Show that $D_{k+1} = -D_k$ and hence that

$$A_k + C_k = B_k + (-1)^k.$$

(iv) Let $F_k = A_{k+1} + C_{k+1}$. Show that

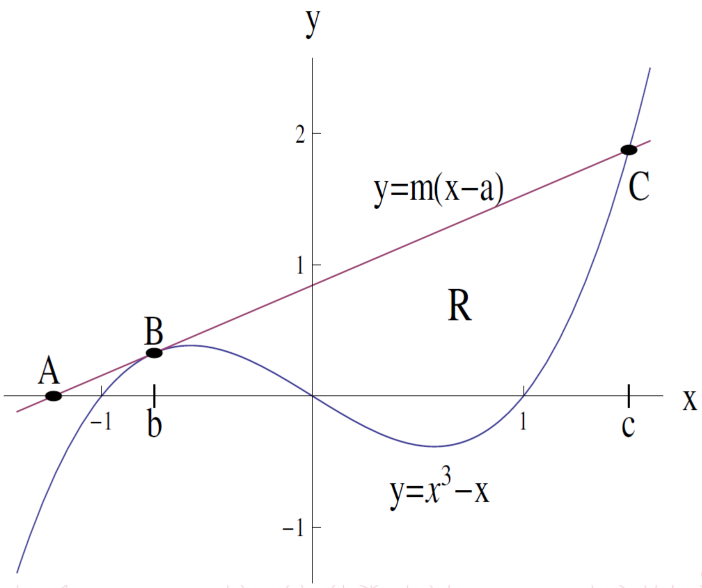
$$F_k + F_{k+1} = F_{k+2}.$$

[MAT, 2011Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS, and CS & Philosophy}

The graphs of $y = x^3 - x$ and $y = m(x-a)$ are drawn on the axes below. Here $m > 0$ and $a \leq -1$.

The line $y = m(x-a)$ meets the x-axis at $A = (a, 0)$, *touches* the cubic $y = x^3 - x$ at B and intersects again with the cubic at C. The x-coordinates of B and C are respectively b and c.



(i) Use the fact that the line and cubic *touch* when $x = b$, to show that $m = 3b^2 - 1$.

(ii) Show further that

$$a = \frac{2b^3}{3b^2 - 1}.$$

(iii) If $a = -10^6$, what is the approximate value of b?

(iv) Using the fact that

$$x^3 - x - m(x-a) = (x-b)^2(x-c)$$

(which you need not prove), show that $c = -2b$.

(v) R is the finite region bounded above by the line $y = m(x-a)$ and bounded below by the cubic $y = x^3 - x$. For what value of a is the area of R largest?

Show that the largest possible area of R is $\frac{27}{4}$.

[MAT, 2011Q4]

For applicants in {Math, Math & Statistics, and Math & Philosophy} only. Not for {Math & CS, CS, and CS & Philosophy}

Let Q denote the quarter-disc of points (x, y) such that $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 1$ as drawn in Figures A and B below.

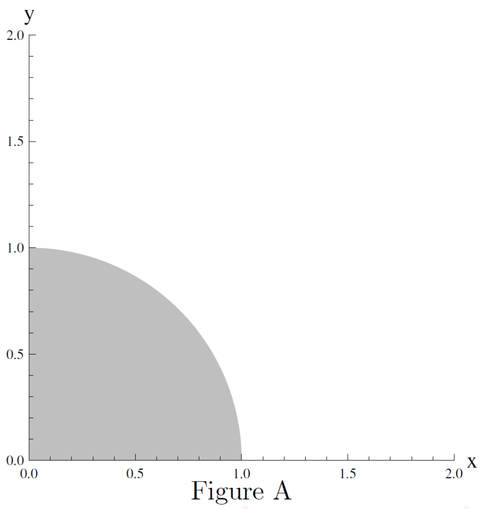


Figure A

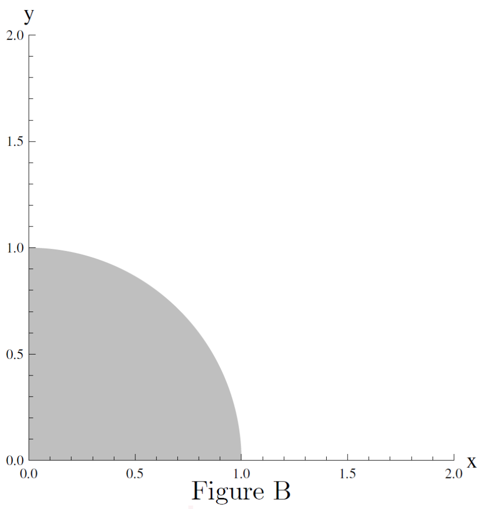


Figure B

(i) On the axes in Figure A, sketch the graphs of

$$x + y = \frac{1}{2}, \qquad x + y = 1, \qquad x + y = \frac{3}{2}.$$

What is the largest value of $x + y$ achieved at points (x, y) in Q? Justify your answer.

(ii) On the axes in Figure B, sketch the graphs of

$$xy = \frac{1}{4}, \qquad xy = 1, \qquad xy = 2.$$

What is the largest value of $x^2 + y^2 + 4xy$ achieved at points (x, y) in Q?

What is the largest value of $x^2 + y^2 - 6xy$ achieved at points (x, y) in Q?

(iii) Describe the curve

$$x^2 + y^2 - 4x - 2y = k$$

where $k > -5$.

What is the *smallest* value of $x^2 + y^2 - 4x - 2y$ achieved at points (x, y) in Q?

[MAT, 2011Q5]

An $n \times n$ *grid* consists of squares arranged in n rows and n columns; for example, a chessboard is an 8×8 grid. Let us call a *semi-grid* of size n the lower left part of an $n \times n$ grid — that is, the squares located on or below the grid's diagonal. For example, Figure C shows an example of a semi-grid of size 4.

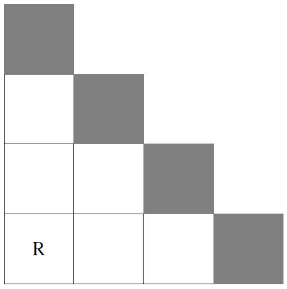


Figure C

Let us suppose that a robot is located in the lower-left corner of the grid. The robot can move only up or right, and its goal is to reach one of the *goal squares*, which are all located on the semi-grid's diagonal. In the example shown in Figure C, the robot is initially located in the square denoted with R, and the goal squares are shown in grey. Let us call a *solution* a sequence of the robot's moves that leads the robot from the initial location to some goal square.

(i) Write down all 8 solutions for a robot on a semi-grid of size 4.

(ii) Devise a concise way of representing the possible journeys of the robot in a semi-grid of size n. In your notation, which of the journeys are solutions?

(iii) Write down a formula for the number of possible solutions in a semi-grid of size n. Explain why your formula is correct.

Now let us change the problem slightly and redefine a goal square as any square that can be described as follows:

- the lower-left square is not a goal square;
- each square that is located immediately above or immediately to the right of a non-goal square is a goal square; and
- each square that is located immediately above or immediately to the right of a goal square is a non-goal square.

Furthermore, let us assume that, upon reaching a goal square, the robot may decide to stop or to continue moving (provided that there are more allowed moves).

(iv) With these modifications in place, write down all the solutions in a semi-grid of size 4, and all the solutions in a semi-grid of size 5.

(v) How many solutions are there now in a semi-grid of size n, where n is a positive integer? You may wish to consider separately the cases where n is even or odd.

[MAT, 2011Q6]

For applicants in {CS and Math & CS} only.

Alice, Bob, Charlie and Diane are playing together when one of them breaks a precious vase. They all know who broke the vase. When questioned they make the following statements:

Alice: It was Bob.

Bob: It was Diane.

Charlie: It was not me.

Diane: What Bob says is wrong.

Each statement is either true or false.

(i) Explain why at least one of the four must be lying.

(ii) Explain why at least one of them must be telling the truth.

(iii) Let us suppose that exactly one of the four is lying, so the other three are telling the truth. Who is lying? Who did break the vase? Explain your answer.

(iv) Let us now suppose that exactly one of the four is telling the truth, so the other three are lying. Who is telling the truth? Who did break the vase? Explain your answer.

(v) Let us now suppose that two of the statements are true and two are false. List the people who might now have broken the vase. Justify your answers.

(vi) Hence show that if we don't know how many of the four statements are true, then any one of the four could have broken the vase.

[MAT, 2011Q7]

For applicants in {CS and CS & Philosophy} only.

Alice and Bob have a large bag of coins which they use to play a game called HT-2. In this game, Alice and Bob take turns placing one coin at a time on the table, each to the right of the previous one; thus they build a row of coins that grows to the right. Alice always places the first coin. Each coin is placed head-up (H) or tail-up (T), and cannot be flipped or moved once it has been placed.

A player loses the game if he or she places a coin that results in two adjacent coins having the same pattern of heads and tails as another adjacent pair somewhere in the row (reading a pattern from left to right). For example, Bob has lost this game by producing a second instance of HT (where a and b denote coins placed by Alice and Bob respectively):

$$\begin{matrix} a & b & a & b & a & b \\ H & H & T & T & H & T \end{matrix}$$

and Alice has lost this game by producing a second instance of TT (overlapping pairs can count as repeats):

$$\begin{matrix} a & b & a & b & a \\ T & H & T & T & T \end{matrix}$$

(i) What is the smallest number of coins that might be placed in a game of HT-2 (including the final coin that causes a player to lose)? What is the largest number? Justify each answer.

(ii) Bob can always win a game of HT-2 by adopting a particular strategy. Describe the strategy.

For any positive integer n, there is a game HT-n with the same rules as HT-2, except that the game is lost by a player who creates an unbroken sequence of n heads and tails that appears elsewhere in the row. For example, Bob has lost this game of HT-3 by producing a second instance of THT:

$$\begin{matrix} a & b & a & b & a & b & a & b \\ H & H & T & T & H & T & H & T \end{matrix}$$

(iii) Suppose n is odd, and Bob chooses to play by always duplicating Alice's previous play (and Alice knows that this is Bob's strategy). Show that Alice can always win.

In these games, a maximum time of one minute is allowed for each turn.

(iv) Can we be certain that a game of HT-6 will be finished within two hours? Justify your answer.

MAT 2012

MAT 2012
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

This paper contains 7 questions, of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics applicants should attempt Questions **1,2,3,4,5**.
- Mathematics & Computer Science applicants should attempt Questions **1,2,3,5,6**.
- Computer Science *or* Computer Science & Philosophy applicants should attempt **1,2,5,6,7**.

Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), or (D) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

[MAT, 2012Q1(A)]

Which of the following lines is a tangent to the circle with equation

$$x^2 + y^2 = 4?$$

(A) $x + y = 2$

(B) $y = x - 2\sqrt{2}$

(C) $x = \sqrt{2}$

(D) $y = \sqrt{2} - x$

[MAT, 2012Q1(B)]

Let $N = 2^k \times 4^m \times 8^n$ where k, m, n are positive whole numbers. Then N will definitely be a square number whenever

(A) k is even

(B) $k + n$ is odd

(C) k is odd but $m + n$ is even

(D) $k + n$ is even

[MAT, 2012Q1(C)]

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3

Which is the smallest of the following numbers?

(A) $\left(\sqrt{3}\right)^3$

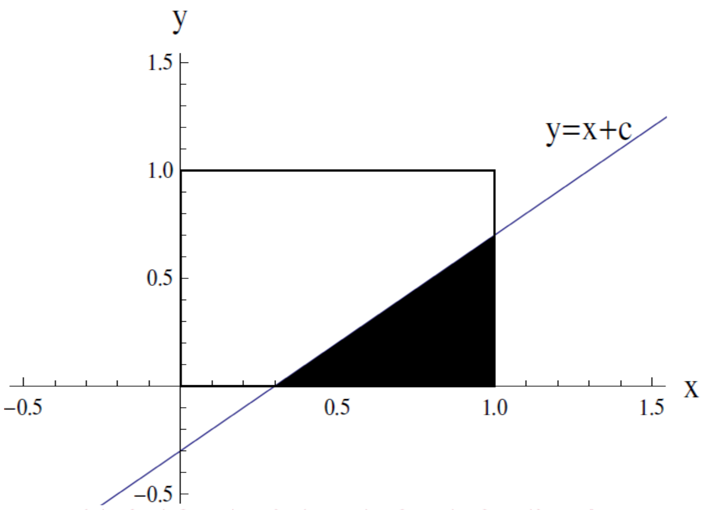
(B) $\log_3(9^2)$

(C) $\left(3\sin\frac{\pi}{3}\right)^2$

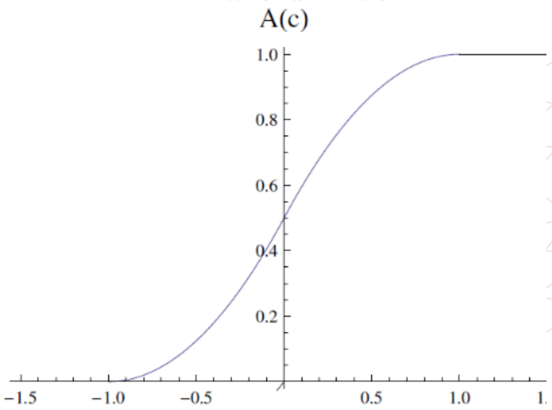
(D) $\log_2(\log_2(8^5))$

[MAT, 2012Q1(D)]

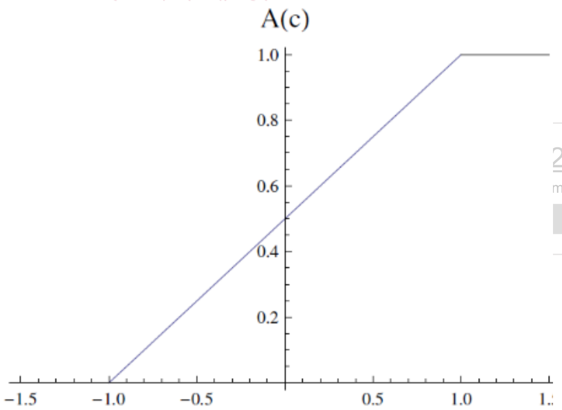
Shown below is a diagram of the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$ and the line $y = x + c$. The shaded region is the region of the square which lies below the line; this shaded region has area $A(c)$.



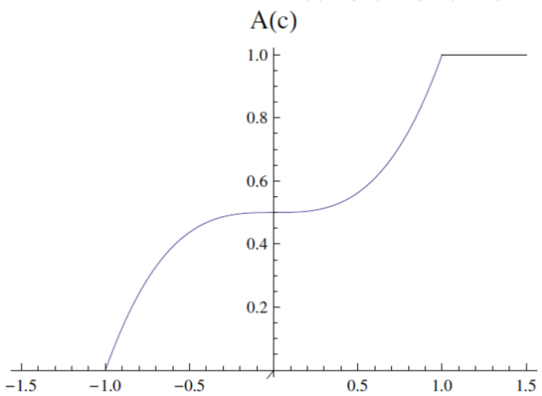
Which of the following graphs shows $A(c)$ as c varies?



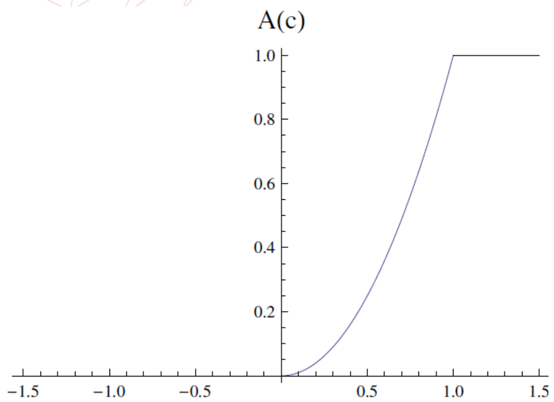
(A)



(B)



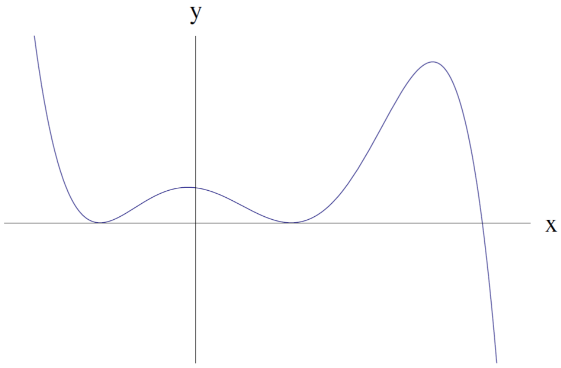
(C)



(D)

[MAT, 2012Q1(E)]

Which one of the following equations could possibly have the graph given below?



(A) $y = (3-x)^2(3+x)^2(1-x)$

(B) $y = -x^2(x-9)(x^2-3)$

(C) $y = (x-6)(x-2)^2(x+2)^2$

(D) $y = (x^2-1)^2(3-x)$

[MAT, 2012Q1(F)]

Let

$$T = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx\right) \times \left(\int_{\pi}^{2\pi} \sin x \, dx\right) \times \left(\int_{0}^{\frac{\pi}{8}} \frac{dx}{\cos 3x}\right).$$

Which of the following is true?

(A) $T = 0$

(B) $T < 0$

(C) $T > 0$

(D) T is not defined

[MAT, 2012Q1(G)]

There are *positive* real numbers x and y which solve the equations

$$2x + ky = 4$$
$$x + y = k$$

for

(A) all values of k

(B) no values of k

(C) $k = 2$ only

(D) only $k > -2$

[MAT, 2012Q1(H)]

In the region $0 < x \leq 2\pi$, the equation

$$\int_0^x \sin(\sin t)\,\mathrm{d}t = 0$$

has

(A) no solution

(B) one solution

(C) two solutions

(D) three solutions

[MAT, 2012Q1(I)]

The vertices of an equilateral triangle are labelled X, Y and Z. The points X, Y and Z lie on a circle of circumference 10 units. Let P and A be the numerical values of the triangle's perimeter and area, respectively. Which of the following is true?

(A) $\frac{A}{P} = \frac{5}{4\pi}$

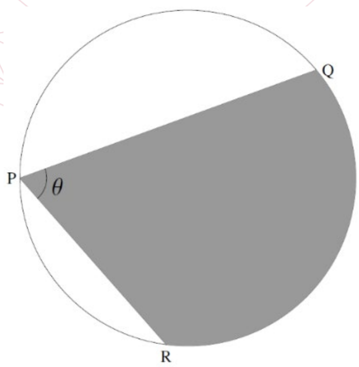
(B) $P < A$

(C) $\frac{P}{A} = \frac{10}{3\pi}$

(D) P^2 is rational

[MAT, 2012Q1(J)]

If two chords QP and RP on a circle of radius 1 meet in an angle θ at P, for example as drawn in the diagram below.



then the largest possible area of the shaded region RPQ is

(A) $\theta\left(1 + \cos\left(\frac{\theta}{2}\right)\right)$

(B) $\theta + \sin\theta$

(C) $\frac{\pi}{2}(1 - \cos\theta)$

(D) θ

[MAT, 2012Q2]

Let

$$f(x) = x + 1 \text{ and } g(x) = 2x.$$

We will, for example, write fg to denote the function "perform g then perform f" so that

$$fg(x) = f\big(g(x)\big) = 2x + 1.$$

If $i \geq 0$ is an integer we will, for example, write f^i to denote the function which performs f i times, so that

$$f^i(x) = \underbrace{fff\cdots f}_{i \text{ times}}(x) = x + i.$$

(i) Show that

$$f^2g(x) = gf(x).$$

(ii) Note that

$$gf^2g(x) = 4x + 4.$$

Find all the other ways of combining f and g that result in the function $4x + 4$.

(iii) Let $i, j, k \geq 0$ be integers. Determine the function

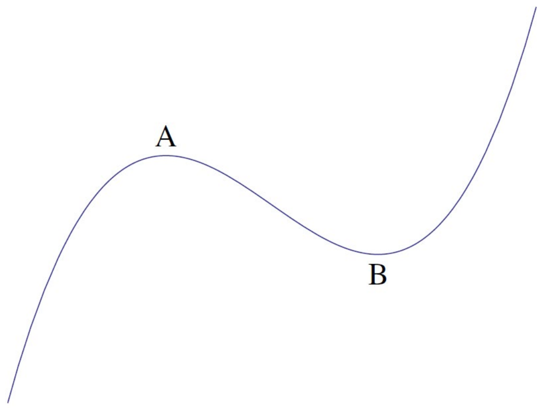
$$f^igf^jgf^k(x)$$

(iv) Let $m \geq 0$ be an integer. How many different ways of combining the functions f and g are there that result in the function $4x + 4m$?

[MAT, 2012Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS, and CS & Philosophy}.

Let $f(x) = x^3 + ax^2 + bx + c$, where the coefficients a, b and c are real numbers. The figure below shows a section of the graph of $y = f(x)$. The curve has two distinct turning points; these are located at A and B, as shown. (Note that the axes have been omitted deliberately.)



(i) Find a condition on the coefficients a, b, c such that the curve has two distinct turning points if, and only if, this condition is satisfied.

It may be assumed from now on that the condition on the coefficients in (i) is satisfied.

(ii) Let x_1 and x_2 denote the x coordinates of A and B, respectively. Show that

$$x_2 - x_1 = \frac{2}{3}\sqrt{a^2 - 3b}.$$

(iii) Suppose now that the graph of $y = f(x)$ is translated so that the turning point at A now lies at the origin. Let $g(x)$ be the cubic function such that $y = g(x)$ has the translated graph. Show that

$$g(x) = x^2\left(x - \sqrt{a^2 - 3b}\right).$$

(iv) Let R be the area of the region enclosed by the x-axis and the graph $y = g(x)$.

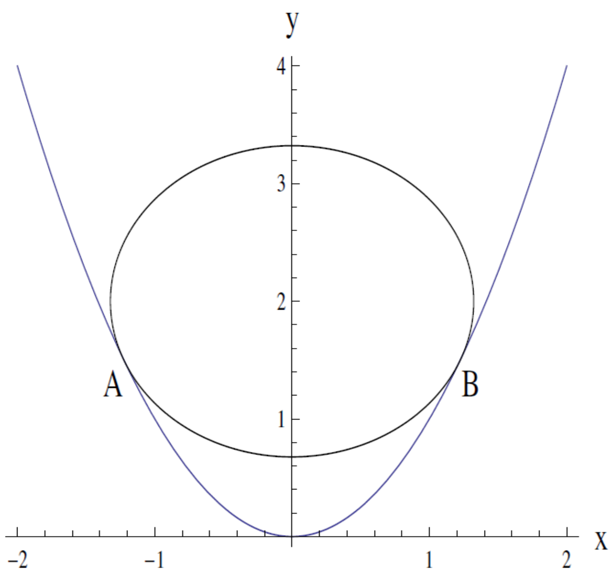
Show that if a and b are rational then R is also rational.

(v) Is it possible for R to be a non-zero rational number when a and b are both irrational? Justify your answer.

[MAT, 2012Q4]

For applicants in {Math, Math & Statistics, and Math & Philosophy} only. Not for {Math & CS, CS, and CS & Philosophy}.

The diagram below shows the parabola $y = x^2$ and a circle with centre $(0,2)$ just 'resting' on the parabola. By 'resting' we mean that the circle and parabola are tangential to each other at the points A and B.



(i) Let (x, y) be a point on the parabola such that $x \neq 0$. Show that the gradient of the line joining this point to the centre of the circle is given by

$$\frac{x^2 - 2}{x}.$$

(ii) With the help of the result from part (i), or otherwise, show that the coordinates of B are given by

$$\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right).$$

(iii) Show that the area of the sector of the circle enclosed by the radius to A, the minor arc AB and the radius to B is equal to

$$\frac{7}{4}\cos^{-1}\left(\frac{1}{\sqrt{7}}\right).$$

(iv) Suppose now that a circle with centre $(0, a)$ is resting on the parabola, where $a > 0$. Find the range of values of a for which the circle and parabola touch at two distinct points.

(v) Let r be the radius of a circle with centre $(0, a)$ that is resting on the parabola. Express a as a function of r, distinguishing between the cases in which the circle is, and is not, in contact with the vertex of the parabola.

[MAT, 2012Q5]

A particular robot has three commands:

F: Move forward a unit distance;

L: Turn left 90°;

R: Turn right 90°.

A *program* is a sequence of commands. We consider particular programs P_n (for $n \geq 0$) in this question. The basic program P_0 just instructs the robot to move forward:

$$P_0 = \mathbf{F}.$$

The program P_{n+1}(for $n \geq 0$) involves performing P_n, turning left, performing P_n again then turning right:

$$P_{n+1} = P_n\mathbf{L}P_n\mathbf{R}.$$

So, for example, $P_1 = \mathbf{FLFR.}$

(i) Write down the program P_2.

(ii) How far does the robot travel during the program P_n? In other words, how many **F** commands does it perform?

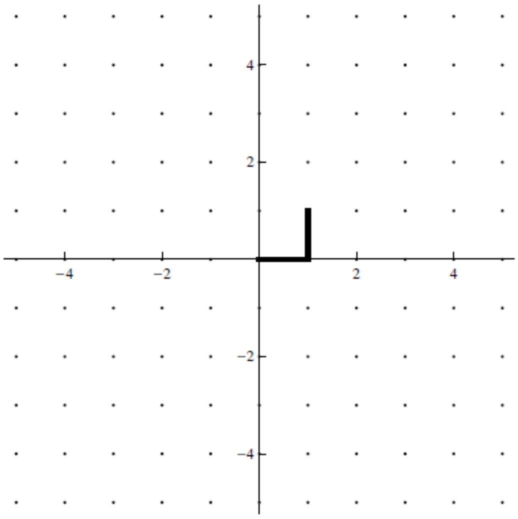
(iii) Let l_n be the total number of commands in P_n; so, for example, $l_0 = 1$ and $l_1 = 4$. Write down an equation relating l_{n+1} to l_n. Hence write down a formula for l_n in terms of n. No proof is required. Hint: consider $l_n + 2$.

(iv) The robot starts at the origin, facing along the positive x-axis. What direction is the robot facing after performing the program P_n?

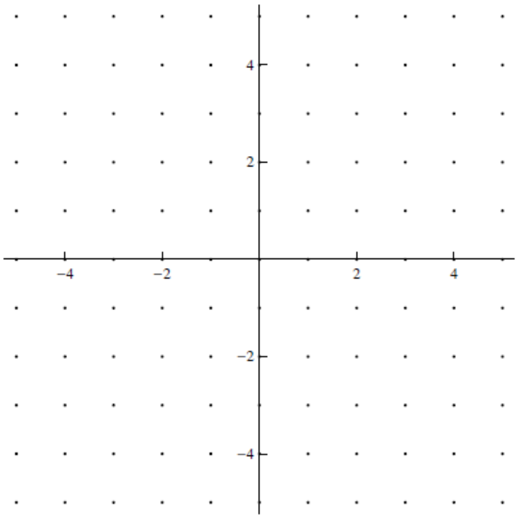
(v) The left-hand diagram on the opposite page shows the path the robot takes when it performs the program P_1. On the right-hand diagram opposite, draw the path it takes when it performs the program P_4.

(vi) Let (x_n, y_n) be the position of the robot after performing the program P_n, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$. Give an equation relating (x_{n+1}, y_{n+1}) to (x_n, y_n).

What is (x_8, y_8)? What is (x_{8k}, y_{8k})?



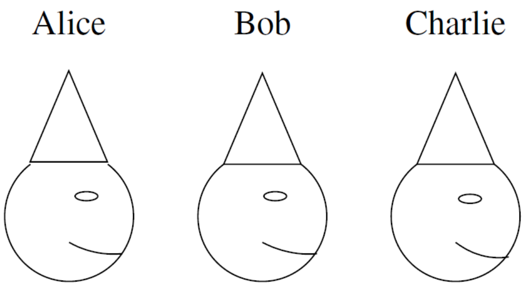
P_1



P_4

[MAT, 2012Q6]

For applicants in {CS, Math & CS, and CS & Philosophy} only.



Alice, Bob and Charlie are well-known expert logicians; they always tell the truth.

They are sat in a row, as illustrated above. In each of the scenarios below, their father puts a red or blue hat on each of their heads. Alice can see Bob's and Charlie's hats, but not her own; Bob can see only Charlie's hat; Charlie can see none of the hats. All three of them are aware of this arrangement.

(i) Their father puts a hat on each of their heads and says: "Each of your hats is either red or blue. At least one of you has a red hat." Alice then says "I know the colour of my hat." What colour is each person's hat? Explain your answer.

(ii) Their father puts a new hat on each of their heads and again says: "Each of your hats is either red or blue. At least one of you has a red hat." Alice then says "I don't know the colour of my hat." Bob then says "I don't know the colour of my hat." What colour is Charlie's hat? Explain your answer.

(iii) Their father puts a new hat on each of their heads and says: "Each of your hats is either red or blue. At least one of you has a red hat, and at least one of you has a blue hat." Alice says "I know the colour of my hat." Bob then says "Mine is red." What colour is each person's hat? Explain your answer.

(iv) Their father puts a new hat on each of their heads and says: "Each of your hats is either red or blue. At least one of you has a red hat, and at least one of you has a blue hat." Alice then says "I don't know the colour of my hat." Bob then says "My hat is red". What colour is Charlie's hat? Explain your answer.

(v) Their father puts a new hat on each of their heads and says: "Each of your hats is either red or blue. Two of you who are seated adjacently both have red hats." Alice then says "I don't know the colour of my hat." What colour is Charlie's hat? Explain your answer.

[MAT, 2012Q7]

For applicants in {CS and CS & Philosophy} only.

Amy and Brian play a game together, as follows. They take it in turns to write down a number from the set $\{0,1,2\}$, with Amy playing first. On each turn (except Amy's first turn), the player must not repeat the number just played by the previous player.

In their first version of the game, Brian wins if, after he plays, the sum of all the numbers played so far is a multiple of 3. For example, Brian will win after the sequence

$$\boxed{2,0}\boxed{1,2}\boxed{1,0}$$

(where we draw a box around each round) because the sum of the numbers is 6. Amy wins if Brian has not won within five rounds; for example, Amy wins after the sequence

$$\boxed{2,0}\boxed{1,2}\boxed{1,2}\boxed{0,2}\boxed{1,2}$$

(i) Show that if Amy starts by playing either 1 or 2, then Brian can immediately win.

(ii) Suppose, instead, Amy starts by playing 0. Show that Brian can always win within two rounds.

They now decide to change the rules so that Brian wins if, after he plays, the sum of all the numbers played so far is *one less than a multiple of 3*. Again, Amy wins if Brian has not won within five rounds. It is still the case that a player must not repeat the number just played previously.

(iii) Show that if Amy starts by playing either 0 or 2, then Brian can immediately win.

(iv) Suppose, instead, Amy starts by playing 1. Explain why it cannot benefit Brian to play 2, assuming Amy plays with the best strategy.

(v) So suppose Amy starts by playing 1, and Brian then plays 0. How should Amy play next?

(vi) Assuming both play with the best strategies, who will win the game? Explain your answer.

MAT 2013

MAT 2013
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

This paper contains 7 questions, of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5**.
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Directions under A take priority over any directions in B which are relevant to you.

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Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), (D), or (E) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2013Q1(A)]

For what values of the real number a does the quadratic equation

$$x^2 + ax + a = 1$$

have distinct real roots?

(A) $a \neq 2$

(B) $a > 2$

(C) $a = 2$

(D) all values of a

[MAT, 2013Q1(B)]

The graph of $y = \sin x$ is reflected first in the line $x = \pi$ and then in the line $y = 2$. The resulting graph has equation

(A) $y = \cos x$

(B) $y = 2 + \sin x$

(C) $y = 4 + \sin x$

(D) $y = 2 - \cos x$

[MAT, 2013Q1(C)]

The functions f, g and h are related by

$$f'(x) = g(x+1), \qquad g'(x) = h(x-1).$$

It follows that $f''(2x)$ equals

(A) $h(2x+1)$

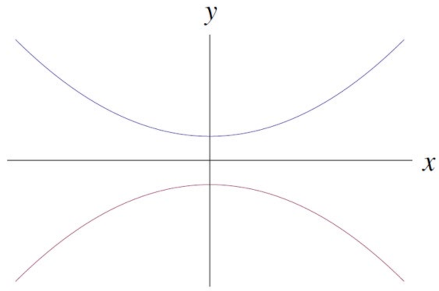
(B) $2h'(2x)$

(C) $h(2x)$

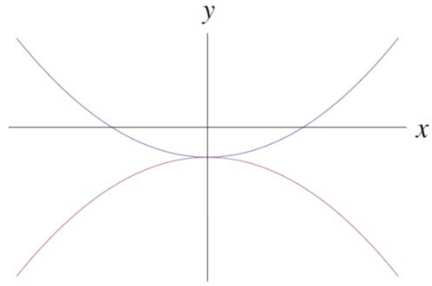
(D) $4h(2x)$

[MAT, 2013Q1(D)]

Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



(A)



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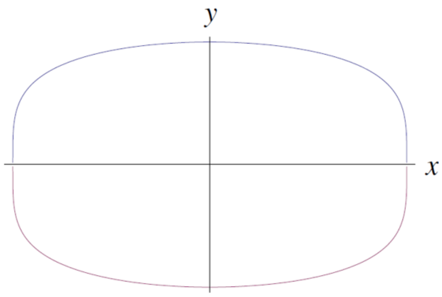
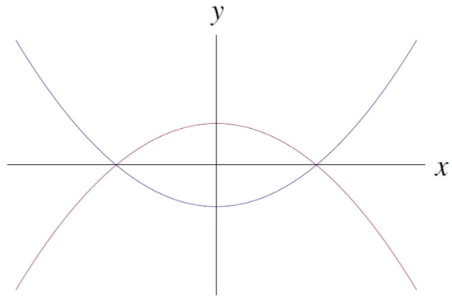


(C)



(D)

[MAT, 2013Q1(E)]

The expression

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}[(2x-1)^4(1-x)^5] + \frac{\mathrm{d}}{\mathrm{d}x}[(2x+1)^4(3x^2-2)^2]$$

is a polynomial of degree

(A) 9

(B) 8

(C) 7

(D) less than 7

[MAT, 2013Q1(F)]

Three *positive* numbers a, b, c satisfy

$$\log_b a = 2, \qquad \log_b(c-3) = 3, \qquad \log_a(c+5) = 2.$$

This information

(A) specifies a uniquely.

(B) is satisfied by two values of a.

(C) is satisfied by infinitely many values of a.

(D) is contradictory.

[MAT, 2013Q1(G)]

Let $n \geq 2$ be an integer and $p_n(x)$ be the polynmial

$$p_n(x) = (x-1) + (x-2) + \cdots + (x-n)$$

What is the remainder when $p_n(x)$ is divided by $p_{n-1}(x)$?

(A) $\frac{n}{2}$

(B) $\frac{n+1}{2}$

(C) $\frac{n^2+n}{2}$

(D) $-\frac{n}{2}$

[MAT, 2013Q1(H)]

The area bounded by the graphs

$$y = \sqrt{2-x^2} \quad \text{and} \quad x + (\sqrt{2}-1)y = \sqrt{2}$$

equals

(A) $\frac{\sin\sqrt{2}}{\sqrt{2}}$

(B) $\frac{\pi}{4} - \frac{1}{\sqrt{2}}$

(C) $\frac{\pi}{2\sqrt{2}}$

(D) $\frac{\pi^2}{6}$

[MAT, 2013Q1(I)]

The function $F(k)$ is defined for positive integers by $F(1) = 1$, $F(2) = 1$, $F(3) = -1$ and by the identities

$$F(2k) = F(k), \qquad F(2k+1) = F(k)$$

for $k \geq 2$. The sum

$$F(1) + F(2) + F(3) + \cdots + F(100)$$

equals

(A) -15

(B) 28

(C) 64

(D) 81

[MAT, 2013Q1(J)]

For a real number x we denote by $[x]$ the largest integer less than or equal to x. Let n be a natural number. The integral

$$\int_0^n [2^x]\,\mathrm{d}x$$

equals

($k! = 1 \times 2 \times 3 \times \cdots \times k$ for a positive integer k.)

(A) $\log_2((2^n - 1)!)$

(B) $n2^n - \log_2((2^n)!)$

(C) $n2^n$

(D) $\log_2((2^n)!)$

[MAT, 2013Q2]

(i) Let $k \neq \pm 1$. The function $f(t)$ satisfies the identity

$$f(t) - kf(1-t) = t$$

for all values of t. By replacing t with $1-t$, determine $f(t)$.

(ii) Consider the new identity

$$f(t) - f(1-t) = g(t). \qquad (*)$$

(a) Show that no function $f(t)$ satisfies $(*)$ when $g(t) = t$.

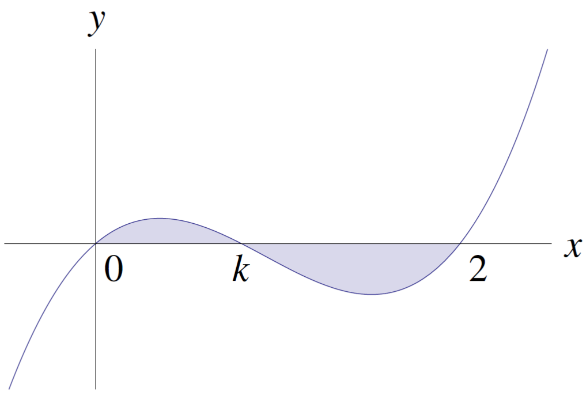
(b) What condition must the function $g(t)$ satisfy for there to be a solution $f(t)$ to $(*)$?

(c) Find a solution $f(t)$ to $(*)$ when $g(t) = (2t-1)^3$.

[MAT, 2013Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS, and CS & Philosophy}.

Let $0 < k < 2$. Below is sketched a graph of $y = f_k(x)$ where $f_k(x) = x(x-k)(x-2)$. Let $A(k)$ denote the area of the shaded region.



(i) Without evaluating them, write down an expression for $A(k)$ in terms of two integrals.

(ii) Explain why $A(k)$ is a polynomial in k of degree 4 or less. [You are not required to calculate $A(k)$ explicitly.]

(iii) Verify that $f_k(1+t) = -f_{2-k}(1-t)$ for any t.

(iv) How can the graph of $y = f_k(x)$ be transformed to the graph of $y = f_{2-k}(x)$?

Deduce that $A(k) = A(2-k)$.

(v) Explain why there are constants a, b, c such that

$$A(k) = a(k-1)^4 + b(k-1)^2 + c.$$

[You are not required to calculate a, b, c explicitly.]

[MAT, 2013Q4]

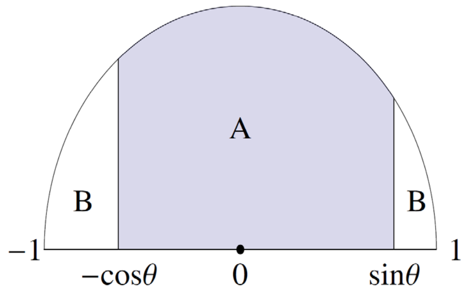
For applicants in {Math, Math & Statistics, Math & Philosophy} only. Not for {Math &CS, CS and CS & Philosophy}.

(i) Let $a > 0$. On the axes opposite, sketch the graph of

$$y = \frac{a+x}{a-x} \quad \text{for} \quad -a < x < a.$$

(ii) Let $0 < \theta < \frac{\pi}{2}$. In the diagram below is the half-disc given by $x^2 + y^2 \leq 1$ and $y \geq 0$. The shaded region A consists of those points with $-\cos\theta \leq x \leq \sin\theta$. The region B is the remainder of the half-disc.

Find the area of A.

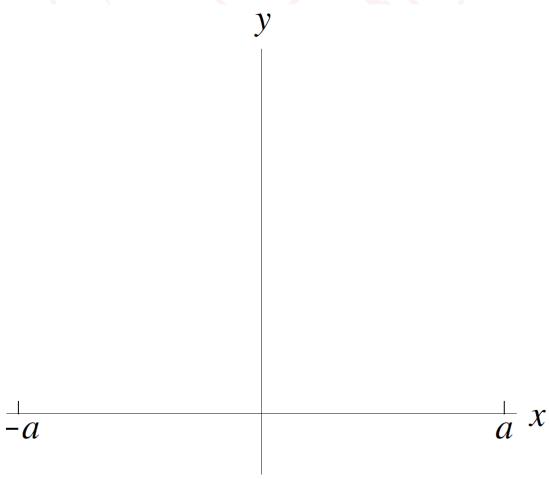


(iii) Assuming *only* that $\sin^2\theta + \cos^2\theta = 1$, show that $\sin\theta\cos\theta \leq 1/2$.

(iv) What is the largest that the ratio

$$\frac{\text{area of } A}{\text{area of } B}$$

can be, as θ varies?



[MAT, 2013Q5]

We define the *digit sum* of a non-negative integer to be the sum of its digits. For example, the digit sum of 123 is $1 + 2 + 3 = 6$.

(i) How many positive integers less than 100 have digit sum equal to 8?

Let n be a positive integer with $n < 10$.

(ii) How many positive integers less than 100 have digit sum equal to n?

(iii) How many positive integers less than 1000 have digit sum equal to n?

(iv) How many positive integers between 500 and 999 have digit sum equal to 8?

(v) How many positive integers less than 1000 have digit sum equal to 8, and one digit at least 5?

(vi) What is the total of the digit sums of the integers from 0 to 999 inclusive?

[MAT, 2013Q6]

For applicants in {CS, Math & CS, CS & Philosophy} only.

Alice, Bob and Charlie are well-known expert logicians; they always tell the truth.

In each of the scenarios below, Charlie writes a whole number on Alice and Bob's foreheads. The difference between the two numbers is one: either Alice's number is one larger than Bob's, or Bob's number is one larger than Alice's. Each of Alice and Bob can see the number on the other's forehead, but can't see their own number.

(i) Charlie writes a number on Alice and Bob's foreheads, and says "Each of your numbers is at least 1. The difference between the numbers is 1."

Alice then says "I know my number."

Explain why Alice's number must be 2. What is Bob's number?

(ii) Charlie now writes new numbers on their foreheads, and says "Each of your numbers is between 1 and 10 inclusive. The difference between the numbers is 1. Alice's number is a prime." (A prime number is a number greater than 1 that is divisible only by 1 and itself.)

Alice then says "I don't know my number."

Bob then says "I don't know my number."

What is Alice's number? Explain your answer.

(iii) Charlie now writes new numbers on their foreheads, and says "Each of your numbers is between 1 and 10 inclusive. The difference between the numbers is 1."

Alice then says "I don't know my number. Is my number a square number?"

Charlie then says "If I told you that, you would know your number."

Bob then says "I don't know my number."

What is Alice's number? Explain your answer.

[MAT, 2013Q7]

For applicants in {CS and CS & Philosophy} only.

AB-words are "words" formed from the letters **A** and **B** according to certain rules. The rules are applied starting with the empty word, containing no letters. The basic rules are:

(1) If the current word is x, then it can be replaced with the word that starts with **A**, followed by x and ending with **B**, written **A**x**B**.

(2) If the current word ends with **B**, the final **B** can be removed.

(i) Show how the word **AAAB** can be produced.

(ii) Describe precisely all the words that can be produced with these two rules. Justify your answer. You might like to write $\mathbf{A}^i$ for the word containing just i consecutive copies of **A**, and similarly for **B**; for example $\mathbf{A}^3\mathbf{B}^2 = \mathbf{AABBB}$.

We now add a third rule:

(3) Reverse the word, and replace every **A** by **B**, and every **B** by **A**.

For example, applying this rule to **AAAB** would give **ABBB**.

(iii) Describe precisely all the words that can be produced with these three rules. Justify your answer.

Finally, we add a fourth rule:

(4) Reverse the word.

(iv) Show that every word consisting of **A**s and **B**s can be formed using these four rules. Hint: show how, if we have produced the word w, we can produce (a) the word **A**w, and (b) the word **B**w; hence deduce the result.

MAT 2014

MAT 2014
On-line Exam

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Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

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	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2014Q1(A)]

The inequality

$$x^4 < 8x^2 + 9$$

is satisfied precisely when

(A) $-3 < x < 3$

(B) $0 < x < 4$

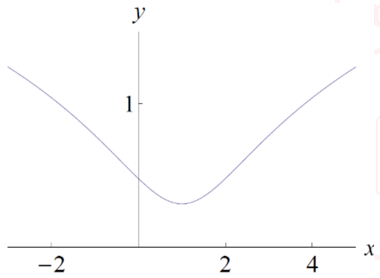
(C) $1 < x < 3$

(D) $-1 < x < 9$

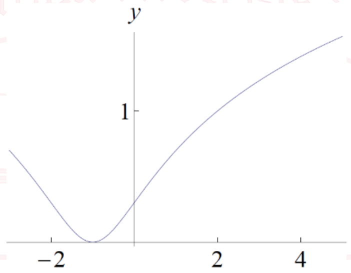
(E) $-3 < x < -1$

[MAT, 2014Q1(B)]

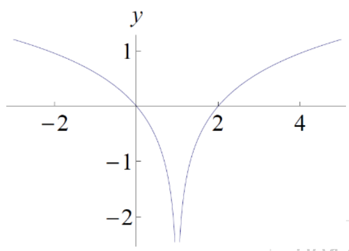
The graph of the function $y = \log_{10}(x^2 - 2x + 2)$ is sketched in



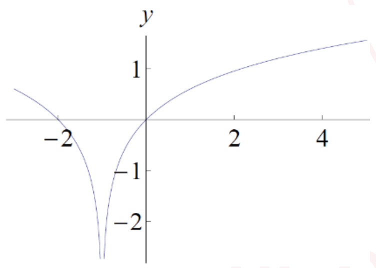
(A)



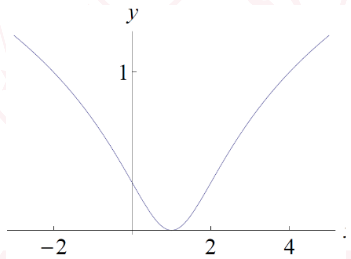
(B)



(C)



(D)



(E)

[MAT, 2014Q1(C)]

The cubic

$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$

has a turning point, that is a minimum, when $x = 1$ precisely for

(A) $k > 0$

(B) $0 < k < 1$

(C) $k > \frac{1}{2}$

(D) $k < 3$

(E) all values of k

[MAT, 2014Q1(D)]

The reflection of the point $(1, 0)$ in the line $y = mx$ has coordinates

(A) $\left(\frac{m^2+1}{m^2-1}, \frac{m}{m^2-1}\right)$

(B) $(1, m)$

(C) (m, m)

(D) $\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$

(E) $(1 - m^2, m)$

[MAT, 2014Q1(E)]

As x varies over the real numbers, the largest value taken by the function

$$(4\sin^2 x + 4\cos x + 1)^2$$

equals

(A) $17 + 12\sqrt{2}$

(B) 36

(C) $48\sqrt{2}$

(D) $64 - 12\sqrt{3}$

(E) 81

[MAT, 2014Q1(F)]

The functions S and T are defined for real numbers by

$$S(x) = x + 1; \text{ and } T(x) = -x.$$

The function S is applied s times and the function T is applied t times, *in some order*, to produce the function

$$F(x) = 8 - x.$$

It is possible to deduce that:

(A) $s = 8$ and $t = 1$

(B) s is odd and t is even

(C) s is even and t is odd

(D) s and t are powers of 2

(E) none of the above

[MAT, 2014Q1(G)]

Let n be a positive integer. The coefficient of x^3y^5 in the expansion of

$$(1 + xy + y^2)^n$$

equals

(A) n

(B) 2^n

(C) $\binom{n}{3}\binom{n}{5}$

(D) $4\binom{n}{4}$

(E) $\binom{n}{8}$

[MAT, 2014Q1(H)]

The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geq 2$,

$$\begin{array}{ll} F(n) = F(n-1) + 2 & \text{if 2 divides } n \text{ but 3 does not divide } n; \\ F(n) = F(n-1) + 3 & \text{if 3 divides } n \text{ but 2 does not divide } n; \\ F(n) = F(n-1) + 4 & \text{if 2 and 3 both divide } n; \\ F(n) = F(n-1) & \text{if neither 2 nor 3 divides } n. \end{array}$$

The value of $F(6000)$ equals

(A) 9827

(B) 10121

(C) 11000

(D) 12300

(E) 12352

[MAT, 2014Q1(I)]

The graph of the function

$$y = 2^{x^2-4x+3}$$

can be obtained from the graph of $y = 2^{x^2}$ by

(A) a stretch parallel to the y-axis followed by a translation parallel to the y-axis.

(B) a translation parallel to the x-axis followed by a stretch parallel to the y-axis.

(C) a translation parallel to the x-axis followed by a stretch parallel to the x-axis.

(D) a translation parallel to the x-axis followed by reflection in the y-axis.

(E) reflection in the y-axis followed by translation parallel to the y-axis.

[MAT, 2014Q1(J)]

For all real numbers x, the function $f(x)$ satisfies

$$6 + f(x) = 2f(-x) + 3x^2\left(\int_{-1}^{1} f(t)\,\mathrm{d}t\right).$$

It follows that $\int_{-1}^{1} f(x)\,\mathrm{d}x$ equals

(A) 4

(B) 6

(C) 11

(D) $\frac{27}{2}$

(E) 23

[MAT, 2014Q2]

Let a and b be real numbers. Consider the cubic equation

$$x^3 + 2bx^2 - a^2x - b^2 = 0. \qquad (*)$$

(i) Show that if $x = 1$ is a solution of $(*)$ then

$$1 - \sqrt{2} \leq b \leq 1 + \sqrt{2}.$$

(ii) Show that there is no value of b for which $x = 1$ is a repeated root of $(*)$.

(iii) Given that $x = 1$ is a solution, find the value of b for which $(*)$ has a repeated root.

(iv) For this value of b, does the cubic

$$y = x^3 + 2bx^2 - a^2x - b^2$$

have a maximum or minimum at its repeated root?

[MAT, 2014Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS, and CS & Philosophy}.

The function $f(x)$ is defined for all real numbers and has the following properties, valid for all x and y:

(A) $f(x+y) = f(x)f(y)$.

(B) $\frac{\mathrm{d}f}{\mathrm{d}x} = f(x)$.

(C) $f(x) > 0$.

Throughout this question, these should be the only properties of f that you use; no marks will be awarded for any use of the exponential function.

Let $a = f(1)$.

(i) Show that $f(0) = 1$.

(ii) Let

$$I = \int_0^1 f(x)\,\mathrm{d}x$$

Show that $I = a - 1$.

(iii) The trapezium rule with n steps is used to produce an estimate I_n for the integral I. Show that

$$I_n = \frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1)$$

Where $b = f\left(\frac{1}{n}\right)$.

(iv) Given that $I_n \geq I$ for all n, show that

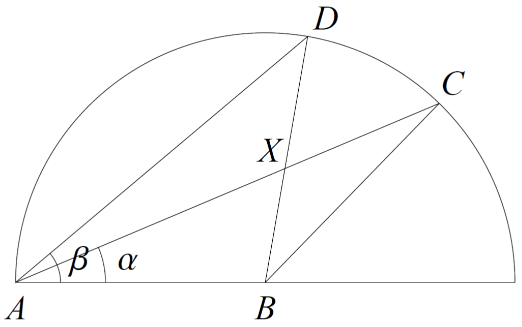
$$a \leq \left(1 + \frac{2}{2n-1}\right)^n.$$

[MAT, 2014Q4]

For applicants in {Math, Math & Statistics, Math & Philosophy} only. Not for {Math & CS, CS and CS & Philosophy}.

In the diagram below is sketched a semicircle with centre B and radius 1. Three points A, C, D lie on the semicircle as shown with α denoting angle CAB and β denoting angle DAB. The triangles ABC and ABD intersect in a triangle ABX.

Throughout the question we shall consider the value of α fixed. Assume for now that $0 < \alpha \leq \beta \leq \frac{\pi}{2}$.



(i) Show that the area of the triangle ABC equals

$$\frac{1}{2}\sin(2\alpha).$$

(ii) Let

$$F = \frac{\text{area of triangle } ABX}{\text{area of triangle } ABC}.$$

Without calculation, explain why, for every k in the range $0 \leq k \leq 1$, there is a unique value of β such that $F = k$.

(iii) Find the value of β such that $F = \frac{1}{2}$.

(iv) Show that

$$F = \frac{\sin(2\beta)\sin\alpha}{\sin(2\beta - \alpha)\sin(2\alpha)}.$$

(v) Suppose now that $0 < \beta < \alpha \leq \frac{\pi}{2}$. Write down, without further calculation, an expression for the area of ABX and hence a formula for F.

[MAT, 2014Q5]

Poets use *rhyme schemes* to describe which lines of a poem rhyme. Each line is denoted by a letter of the alphabet, with the same letter given to two lines that rhyme. To say that a poem has the rhyming scheme ABABCDED, indicates that the first and third lines rhyme, the second and fourth lines rhyme, and the sixth and eighth lines rhyme, but no others.

More precisely, the first line of the poem is given the letter A. If a subsequent line rhymes with an earlier line, it is given the same letter; otherwise, it is given the first unused letter. (For the purposes of this question, you can assume that we have an infinite supply of "letters", not just the 26 letters of the alphabet.)

The purpose of this question is to investigate how many different rhyme schemes there are for poems of n lines. We write r_n for this number.

(i) There are five different rhyming schemes for poems of three lines (so $r_3 = 5$). List them.

Let $c_{n,k}$ denote the number of rhyme schemes for poems with lines n that use exactly k different letters. For example $c_{3,2} = 3$ corresponding to the rhyming schemes AAB, ABA and ABB.

(ii) What is $c_{n,1}$ for $n \geq 1$?

What is $c_{n,n}$?

Explain your answers.

(ii) Suppose that $1 < k < n$. By considering the final letter of a rhyming scheme, explain why

$$c_{n,k} = kc_{n-1,k} + c_{n-1,k-1}.$$

(iii) Write down an equation showing how to calculate r_n in terms of the $c_{n,k}$. Hence calculate r_4.

(iv) Give a formula for $c_{n,2}$ in terms of n (for $n \geq 2$). Justify your answer.

[MAT, 2014Q6]

Alice plays a game 5 times with her friends Sam and Pam. In each game Alice chooses two integers x and y with $1 \leq x \leq y$. She whispers the sum $x + y$ to Sam, and the product $x \times y$ to Pam, so that neither knows what the other was told. Sam and Pam then have to try to work out what the numbers x and y are. Sam and Pam are well known expert logicians.

(i) In the first game, Pam says "I know x and y."

What can we deduce about the values of x and y? Explain your answer.

(ii) In the second game, Pam says "I don't know what x and y are."

Sam then says "I know x and y."

Suppose the sum is 4. What are x and y? Explain your answer.

(iii) In the third game, Pam says "I don't know what x and y are."

Sam then says "I don't know what x and y are."

Pam then says "I now know x and y."

Suppose the product is 4. What are x and y? Explain your answer.

(iv) In the fourth game, Pam says "I don't know what x and y are."

Sam then says "I already knew that."

Pam then says "I now know x and y."

Suppose the product is 8. What are x and y? Explain your answer.

(v) Finally, in the fifth game, Pam says "I don't know what x and y are."

Sam then says "I don't know what x and y are."

Pam then says "I don't know what x and y are."

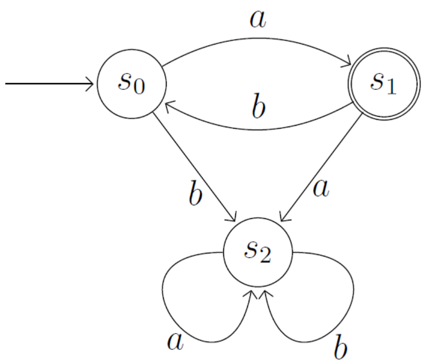
Sam then says "I now know x and y."

Suppose the sum is 5. What are x and y? Explain your answer.

[MAT, 2014Q7]

For applicants in {CS and CS & Philosophy} only.

A *finite automaton* is a mathematical model of a simple computing device. A small finite automaton is illustrated below.



A finite automaton has some *finite* number of states; the above automaton has three states, labelled s_0, s_1 and s_2. The initial state, s_0, is indicated with an incoming arrow. The automaton receives *inputs* (e.g. via button presses), which might cause it to change state. In the example, the inputs are a and b. The state changes are illustrated by arrows; for example, if the automaton is in state s_1 and it receives input b, it changes to state s_0; if it is in state s_2 and receives either input, it remains in state s_2. (For each state, there is precisely one out-going arrow for each input.)

Some of the states are defined to be *accepting states*; in the example, just s_1 is defined to be an accepting state, represented by the double circle. A *word* is a sequence of inputs. The automaton *accepts* a word w if that sequence of inputs leads to an accepting state from the initial state. For example, the above automaton accepts the word aba.

(i) Write down a description of the set of words accepted by the above automaton. A clear but informal description will suffice.

(ii) Suppose we alter the above automaton by swapping accepting and non-accepting states; i.e. we make s_0 and s_2 accepting, and make s_1 non-accepting. Write down a description of the set of words accepted by this new automaton. Again, a clear but informal description will suffice.

(iii) Draw an automaton that accepts all words containing an even number (possibly zero) of a's and any number of b's (and no other words).

(iv) Now draw an automaton that accepts all words containing an even number of a's or an odd number of b's (and no other words).

Let a^n represent n consecutive a's. Let L be the set of all words of the form $a^n b^n$ where $n = 0,1,2,\ldots$; i.e. all words composed of some number of a's followed by the *same* number of b's. We will show that no finite automaton accepts precisely this set of words.

(v) Suppose a particular finite automaton A *does* accept precisely the words in L. Show that if $i \neq j$ then the words a^i and a^j must lead to different states of A.

Hence show that this leads to a contradiction.

MAT 2015

MAT 2015
On-line Exam

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Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

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A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5**.
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6**.
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7**.

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial or Warwick Applicants: if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses: Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year Abroad, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, you should attempt Questions **1,2,3,4,5**.

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Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), (D), or (E) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2015Q1(A)]

Pick a whole number.

Add one.

Square the answer.

Multiply the answer by four.

Subtract three.

Which of the following statements are true regardless of which starting number is chosen?

I The final answer is odd.

II The final answer is one more than a multiple of three.

III The final answer is one more than a multiple of eight.

IV The final answer is not prime.

V The final answer is not one less than a multiple of three.

(A) I, II, and V

(B) I and IV

(C) II and V

(D) I, III, and V

(E) I and V

[MAT, 2015Q1(B)]

Let

$$f(x) = (x + a)^n$$

where a is a real number and n is a positive whole number, and $n > 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will

(A) always be odd

(B) always be even

(C) depend on a but not n

(D) depend on n but not a

(E) depend on both a and n

[MAT, 2015Q1(C)]

Which of the following are true for all real values of x? All arguments are in radians.

$$\begin{array}{ll} \text{I} & \sin\left(\frac{\pi}{2}+x\right)=\cos\left(\frac{\pi}{2}-x\right) \\ \text{II} & 2+2\sin(x)-\cos^2(x)\geq 0 \\ \text{III} & \sin\left(x+\frac{3\pi}{2}\right)=\cos(\pi-x) \\ \text{IV} & \sin(x)\cos(x)\leq\frac{1}{4} \end{array}$$

(A) I and II

(B) I and III

(C) II and III

(D) III and IV

(E) II and IV

[MAT, 2015Q1(D)]

Let

$$f(x)=\int_0^1 (xt)^2\,\mathrm{d}t, \text{ and } g(x)=\int_0^x t^2\,\mathrm{d}t.$$

Let $A>0$. Which of the following statements is true?

(A) $g\big(f(A)\big)$ is always bigger than $f\big(g(A)\big)$.

(B) $f\big(g(A)\big)$ is always bigger than $g\big(f(A)\big)$.

(C) They are always equal.

(D) $f\big(g(A)\big)$ is bigger if $A<1$, and $g\big(f(A)\big)$ is bigger if $A>1$.

(E) $g\big(f(A)\big)$ is bigger if $A<1$ and $f\big(g(A)\big)$ is bigger if $A>1$.

[MAT, 2015Q1(E)]

In the interval $0\leq x\leq 2\pi$, the equation

$$\sin(2\cos(2x)+2)=0$$

has exactly

(A) 2 solutions.

(B) 3 solutions.

(C) 4 solutions.

(D) 6 solutions.

(E) 8 solutions.

[MAT, 2015Q1(F)]

For a real number x we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x. Let

$$f(x) = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor.$$

The smallest number of equal width strips for which the trapezium rule produces an **overestimate** for the integral

$$\int_0^5 f(x)\,\mathrm{d}x$$

is

(A) 2

(B) 3

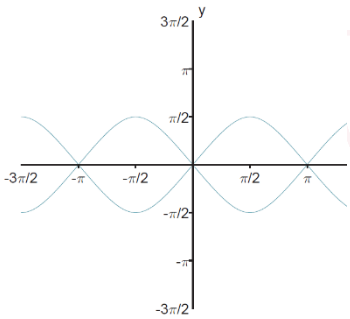
(C) 4

(D) 5

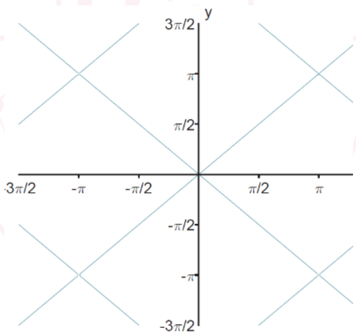
(E) it never produces an overestimate

[MAT, 2015Q1(G)]

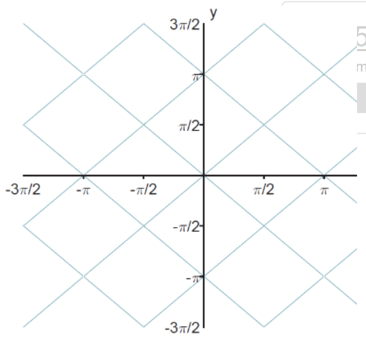
The graph of $\cos^2(x) = \cos^2(y)$ is sketched in



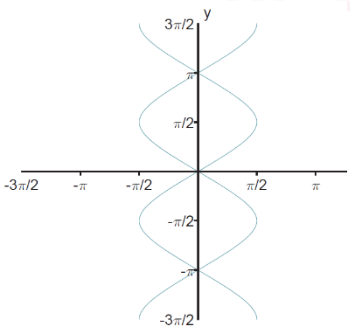
(A)



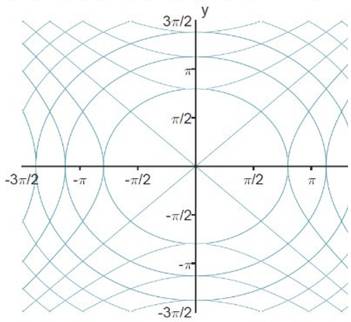
(B)



(C)



(D)



(E)

[MAT, 2015Q1(H)]

How many distinct solutions does the following equation have?

$$\log_{x^2+2}(4-5x^2-6x^3)=2$$

(A) None

(B) 1

(C) 2

(D) 4

(E) Infinitely many

[MAT, 2015Q1(I)]

Into how many regions is the plane divided when the following equations are graphed, not considering the axes?

$$y=x^3$$
$$y=x^4$$
$$y=x^5$$

(A) 6

(B) 7

(C) 8

(D) 9

(E) 10

[MAT, 2015Q1(J)]

Which is the largest of the following numbers?

(A) $\frac{\sqrt{7}}{2}$

(B) $\frac{5}{4}$

(C) $\frac{\sqrt{10!}}{3(6!)}$

(D) $\frac{\log_2(30)}{\log_3(85)}$

(E) $\frac{1+\sqrt{6}}{3}$

[MAT, 2015Q2]

(i) Expand and simplify

$$(a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n)$$

(ii) The prime number 3 has the property that it is one less than a square number. Are there any other prime numbers with this property? Justify your answer.

(iii) Find all the prime numbers that are one more than a cube number. Justify your answer.

(iv) Is $3^{2015} - 2^{2015}$ a prime number? Explain your reasoning carefully.

(v) Is there a positive integer k for which $k^3 + 2k^2 + 2k + 1$ is a cube number? Explain your reasoning carefully.

[MAT, 2015Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only. Not for {CS, and CS & Philosophy}.

In this question we shall investigate when functions are close approximations to each other. We define $|x|$to be equal to x if $x \geq 0$ and to $-x$ if $x \leq 0$. With this notation we say that a function f is an excellent approximation to a function g if

$$|f(x) - g(x)| \leq \frac{1}{320} \quad \text{whenever} \quad 0 \leq x \leq \frac{1}{2};$$

we say that f is a good approximation to a function g if

$$|f(x) - g(x)| \leq \frac{1}{100} \quad \text{whenever} \quad 0 \leq x \leq \frac{1}{2}.$$

For example, any function f is an excellent approximation to itself. If f is an excellent approximation to g then f is certainly a good approximation to g, but the converse need not hold.

(i) Give an example of two functions f and g such that f is a good approximation to g but f is not an excellent approximation to g.

(ii) Show that if

$$f(x) = x \quad \text{and} \quad g(x) = x + \frac{\sin(4x^2)}{400}$$

then f is an excellent approximation to g.

For the remainder of the question we are going to a try to find a good approximation to the exponential function. This function, which we shall call h, satisfies the following equation

$$h(x) = 1 + \int_0^x h(t)\,\mathrm{d}t \quad \text{whenever} \quad x \geq 0.$$

You may not use any other properties of the exponential function during this question, and any attempt to do so will receive no marks.

Let

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

(iii) Show that if

$$g(x) = 1 + \int_0^x f(t)\,\mathrm{d}t$$

then f is an excellent approximation to g.

(iv) Show that for $x \geq 0$

$$h(x) - f(x) = g(x) - f(x) + \int_0^x \big(h(t) - f(t)\big)\,\mathrm{d}t.$$

(v) You are given that $h(x) - f(x)$ has a maximum value on the interval $0 \leq x \leq 1/2$ at $x = x_0$. Explain why

$$\int_0^x \big(h(t) - f(t)\big)\,\mathrm{d}t \le \frac{1}{2}\big(h(x_0) - f(x_0)\big) \quad \text{whenever} \quad 0 \le x \le \frac{1}{2}.$$

(vi) You are also given that $f(x) \le h(x)$ for all $0 \le x \le \frac{1}{2}$. Show that f is a good approximation to h when $0 \le x \le \frac{1}{2}$.

[MAT, 2015Q4]

For applicants in {Math, Math & Statistics, and Math & Philosophy} only. Not for {Math & CS, CS, and CS & Philosophy}.

A circle A passes through the points $(-1,0)$ and $(1,0)$. Circle A has centre (m,h), and radius r.

(i) Determine m and write r in terms of h.

(ii) Given a third point (x_0,y_0) and $y_0 \neq 0$ show that there is a unique circle passing through the three points $(-1,0)$, $(1,0)$, (x_0,y_0).

For the remainder of the question we consider three circles A, B, and C, each passing through the points $(-1,0)$, $(1,0)$. Each circle is cut into regions by the other two circles. For a group of three such circles, we will say the **lopsidedness** of a circle is the fraction of the full area of that circle taken by its largest region.

(iii) Let circle A additionally pass through the point $(1,2)$, circle B pass through $(0,1)$, and let circle C pass through the point $(0,-4)$. What is the lopsidedness of circle A?

(iv) Let $p>0$. Now let A pass through $(1,2p)$, B pass through $(0,1)$, and C pass through $(-1,-2p)$. Show that the value of p minimising the lopsidedness of circle B satisfies the equation

$$(p^2+1)\tan^{-1}\left(\frac{1}{p}\right)-p=\frac{\pi}{6}$$

Note that $\tan^{-1}(x)$ is sometimes written as $\arctan(x)$ and is the value of θ in the range $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ such that $\tan\theta = x$.

[MAT, 2015Q5]

The following functions are defined for all integers a, b and c:

$$p(x) = x + 1$$
$$m(x) = x - 1$$
$$s(x, y, z) = \begin{cases} y & \text{if } x \leq 0 \\ z & \text{if } x > 0. \end{cases}$$

(i) Show that the value of

$$s\Big(s\big(p(0), m(0), m(m(0))\big), s\big(p(0), m(0), p(p(0))\big), s\big(m(0), p(0), m(p(0))\big)\Big)$$

is 2.

Let f be a function defined, for all integers a and b, as follows:

$$f(a, b) = s\Big(b, p(a), p\big(f(a, m(b))\big)\Big).$$

(ii) What is the value of $f(5, 2)$?

(iii) Give a simple formula for the value of $f(a, b)$ for all integers a and all *positive* integers b, and explain why this formula holds.

(iv) Define a function $g(a, b)$ in a similar way to f, using only the functions p, m and s, so that the value of $g(a, b)$ is equal to the sum of a and b for all integers a and all integers $b \leq 0$.

Explain briefly why your function gives the correct value for all such values of a and b.

[MAT, 2015Q6]

For applicants in {CS, Math & CS, CS & Philosophy} only.

The world is divided into two species, vampires and werewolves. Vampires always tell the truth when talking about a vampire, but always lie when talking about a werewolf. Werewolves always tell the truth when talking about a werewolf, but always lie when talking about a vampire. (Note that this does not imply that creatures necessarily lie when speaking to creatures of the other species. Note also that "Zaccaria is a vampire" is a statement about Zaccaria, rather than necessarily about a vampire.)

These facts are well known to both sides, and creatures can tell instinctively which species an individual belongs to.

In your answers to the questions below, you may abbreviate "vampire" and "werewolf" to "V" and "W", respectively.

(i) Azrael says, "Beela is a werewolf." Explain why Azrael must be a werewolf, but that we cannot tell anything about Beela.

(ii) Cesare says, "Dita says 'Elith is a vampire.'" What can we infer about any of the three from this statement? Explain your answer.

(iii) Suppose N creatures (where $N \geq 2$) are sitting around a circular table. Each tells their right-hand neighbour, "You lie about your right-hand neighbour." What can we infer about N? What can we infer about the arrangement of creatures around the table? Explain your answer.

(iv) Consider a similar situation to that in part (iii) (possibly for a different value of N), except that now each tells their right-hand neighbour, "Your right-hand neighbor lies about their right-hand neighbour." Again, what can we infer about N and the arrangement of creatures around the table? Explain your answer.

[MAT, 2015Q7]

For applicants in {CS, CS & Philosophy} only.

In this question we will study a mechanism for producing a set of **words**. We will only consider words containing the letters a and/or b, and that have length at least 1. We will make use of **variables**, which we shall write as capital letters, including a special start variable called S. We will also use rules, which show how a variable can be replaced by a sequence of variables and/or letters. Starting with the start variable S, we repeatedly replace one of the variables according to one of the rules (in any order) until no variables remain

For example suppose the rules are

$$S \to AB, \qquad A \to AA, \qquad A \to a, \qquad B \to bb.$$

We can produce the word $aabb$ as follows; at each point, the variable that is replaced is underlined:

$$\underline{S} \to \underline{A}B \to A\underline{A}B \to \underline{A}aB \to aa\underline{B} \to aabb.$$

(i) Show that the above rules can be used to produce all words of the form $a^n bb$ with $n \geq 1$, where a^n represents n consecutive a's.

Also briefly explain why the rules can be used to produce no other words.

(ii) Give a precise description of the words produced by the following rules.

$$S \to ab, \qquad S \to aSb.$$

(iii) A **palindrome** is a word that reads the same forwards as backwards, for example $bbaabb$. Give rules that produce all palindromes (and no other words).

(iv) Consider the words with the same number of a's as b's; for example, $aababb$. Write down rules that produce these words (and no others).

(v) Suppose you are given a collection of rules that produces the words in L_1, and another collection of rules that produces the words in L_2. Show how to produce a single set of rules that produce all words in L_1 or L_2, or both (and no other words). Hint: you may introduce new variables if you want.

MAT 2016

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On-line Exam

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Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2016Q1(A)]

A sequence (a_n) has first term $a_1 = 1$, and subsequent terms defined by $a_{n+1} = la_n$ for $n \geq 1$. What is the product of the first 15 terms of the sequence?

(A) l^{14}

(B) $15 + l^{14}$

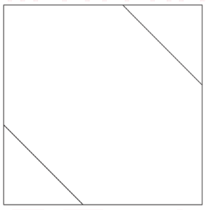
(C) $\frac{1-l^{15}}{1-l}$

(D) l^{105}

(E) $15 + l^{105}$

[MAT, 2016Q1(B)]

An irregular hexagon with all sides of equal length is placed inside a square of side length 1, as shown below (not to scale). What is the length of one of the hexagon sides?



(A) $\sqrt{2} - 1$

(B) $2 - \sqrt{2}$

(C) 1

(D) $\frac{\sqrt{2}}{2}$

(E) $2 + \sqrt{2}$

[MAT, 2016Q1(C)]

The origin lies inside the circle with equation

$$x^2 + ax + y^2 + by = c$$

precisely when

(A) $c > 0$

(B) $a^2 + b^2 > c$

(C) $a^2 + b^2 < c$

(D) $a^2 + b^2 > 4c$

(E) $a^2 + b^2 < 4c$

[MAT, 2016Q1(D)]

How many solutions does $\cos^n(x) + \cos^{2n}(x) = 0$ have in the range $0 \leq x \leq 2\pi$ for an integer $n \geq 1$?

(A) 1 for all n

(B) 2 for all n

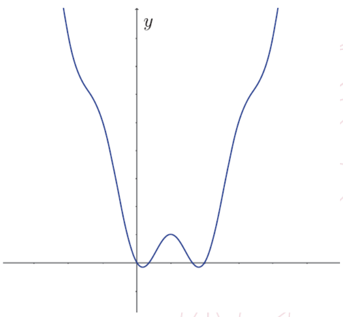
(C) 3 for all n

(D) 2 for even n and 3 for odd n

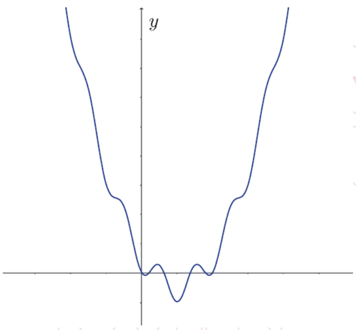
(E) 3 for even n and 2 for odd n

[MAT, 2016Q1(E)]

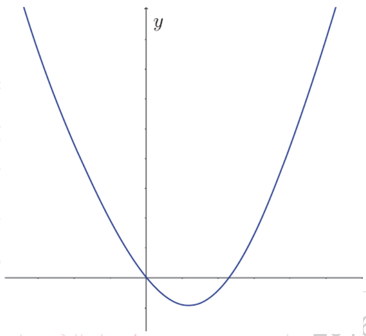
The graph of $y = (x-1)^2 - \cos(\pi x)$ is drawn in



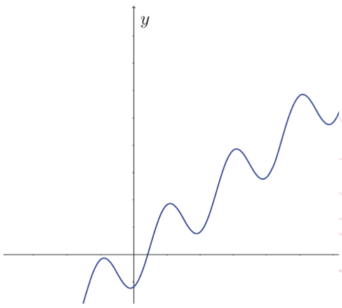
(A)



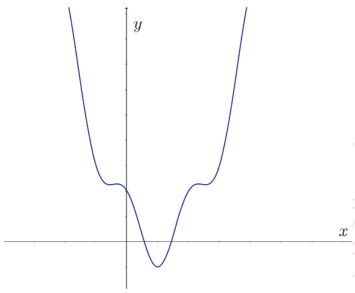
(B)



(C)



(D)



(E)

[MAT, 2016Q1(F)]

Let n be a positive integer. Then $x^2 + 1$ is a factor of

$$(3 + x^4)^n - (x^2 + 3)^n(x^2 - 1)^n$$

for

(A) all n.

(B) even n.

(C) odd n.

(D) $n \geq 3$.

(E) no values of n.

[MAT, 2016Q1(G)]

The sequence (x_n), where $n \geq 0$, is defined by $x_0 = 1$ and

$$x_n = \sum_{k=0}^{n-1} x_k \qquad \text{for } n \geq 1.$$

The sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

equals

(A) 1

(B) $\frac{6}{5}$

(C) $\frac{8}{5}$

(D) 3

(E) $\frac{27}{5}$

[MAT, 2016Q1(H)]

Consider two functions

$$f(x) = a - x^2$$
$$g(x) = x^4 - a$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?

(A) all values of a

(B) $a > 1$

(C) $a > \frac{6}{5}$

(D) $a > \left(\frac{4}{3}\right)^{\frac{3}{2}}$

(E) $a > \left(\frac{6}{5}\right)^{4}$

[MAT, 2016Q1(I)]

Let a and b be positive real numbers. If $x^2 + y^2 \leq 1$ then the largest that $ax + by$ can equal is

(A) $\frac{1}{a} + \frac{1}{b}$

(B) $\max(a, b)$

(C) $\sqrt{a^2 + b^2}$

(D) $a + b$

(E) $a^2 + ab + b^2$

[MAT, 2016Q1(J)]

Let $n > 1$ be an integer. Let $\Pi(n)$ denote the number of distinct prime factors of n and let $x(n)$ denote the final digit of n. For example, $\Pi(8) = 1$ and $\Pi(6) = 2$. Which of the following statements is false?

(A) If $\Pi(n) = 1$, there are some values of $x(n)$ that mean n cannot be prime.

(B) If $\Pi(n) = 1$, there are some values of $x(n)$ that mean n must be prime.

(C) If $\Pi(n) = 1$, there are values of $x(n)$ which are impossible.

(D) If $\Pi(n) + x(n) = 2$, we cannot tell if n is prime.

(E) If $\Pi(n) = 2$, all values of $x(n)$ are possible.

[MAT, 2016Q2]

Let

$$A(x) = 2x + 1, \qquad B(x) = 3x + 2.$$

(i) Show that $A\big(B(x)\big) = B\big(A(x)\big)$.

(ii) Let n be a positive integer. Determine $A^n(x)$ where

$$A^n(x) = \underbrace{A(A(A \cdots A}_{n \text{ times}}(x) \cdots).$$

Put your answer in the simplest form possible.

A function $F(x) = 108x + c$ (where c is a positive integer) is produced by repeatedly applying the functions $A(x)$ and $B(x)$ in some order.

(iii) In how many different orders can $A(x)$ and $B(x)$ be applied to produce $F(x)$?

Justify your answer.

(iv) What are the possible values of c? Justify your answer.

(v) Are there positive integers $m_1, \ldots, m_k, n_1, \ldots, n_k$ such that

$$A^{m_1}B^{n_1}(x) + A^{m_2}B^{n_2}(x) + \cdots + A^{m_k}B^{n_k}(x) = 214x + 92 \qquad \text{for all } x?$$

Justify your answer.

[MAT, 2016Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy, Math & CS} only.

In this question we fix a real number α which will be the same throughout. We say that a function f is **bilateral** if

$$f(x) = f(2\alpha - x)$$

for all x.

(i) Show that if $f(x) = (x - \alpha)^2$ for all x then the function f is bilateral.

(ii) On the other hand show that if $f(x) = x - \alpha$ for all x then the function f is *not* bilateral.

(iii) Show that if n is a non-negative integer and a and b are any real numbers then

$$\int_a^b x^n \, dx = -\int_b^a x^n \, dx.$$

(iv) Hence show that if f is a polynomial (and a and b are any reals) then

$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx.$$

(v) Suppose that f is any bilateral function. By considering the area under the graph of $y = f(x)$ explain why for any $t \geq \alpha$ we have

$$\int_\alpha^t f(x) \, dx = \int_{2\alpha - t}^\alpha f(x) \, dx.$$

If f is a function then we write G for the function defined by

$$G(t) = \int_\alpha^t f(x) \, dx$$

for all t.

(vi) Suppose now that f is any bilateral polynomial. Show that

$$G(t) = -G(2\alpha - t)$$

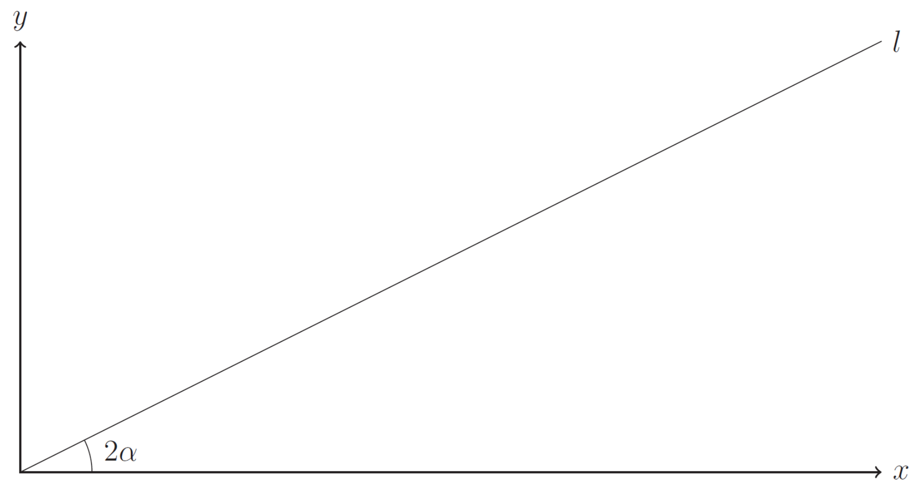
for all t.

(vii) Suppose f is a bilateral polynomial such that G is also bilateral. Show that $G(x) = 0$ for all x.

[MAT, 2016Q4]

For applicants in {Math, Math & Statistics, Math & Philosophy} only.

The line l passes through the origin at angle 2α above the x-axis, where $2\alpha < \frac{\pi}{2}$.



Circles C_1 of radius 1 and C_2 of radius 3 are drawn between l and the x-axis, just touching both lines.

(i) What is the centre of circle C_1?

(ii) What is the equation of circle C_1?

(iii) For what value of α do circles C_1 and C_2 touch?

(iv) For this value of α (for which the circles C_1 and C_2 touch) a third circle, C_3, larger than C_2, is to be drawn between l and the x-axis. C_3 just touches both lines and also touches C_2. What is the radius of this circle C_3?

(v) For the same value of α, what is the area of the region bounded by the x-axis and the circles C_1 and C_2?

[MAT, 2016Q5]

This question concerns the sum s_n defined by

$$s_n = 2 + 8 + 24 + \cdots + n2^n.$$

(i) Let $f(n) = (An + B)2^n + C$ for constants A, B and C yet to be determined, and suppose $s_n = f(n)$ for all $n \geq 1$. By setting $n = 1, 2, 3$, find three equations that must be satisfied by A, B and C.

(ii) Solve the equations from part (i) to obtain values for A, B and C.

(iii) Using these values, show that if $s_k = f(k)$ for some $k \geq 1$ then $s_{k+1} = f(k+1)$.

You may now assume that $f(n) = s_n$ for all $n \geq 1$.

(iv) Find simplified expressions for the following sums:

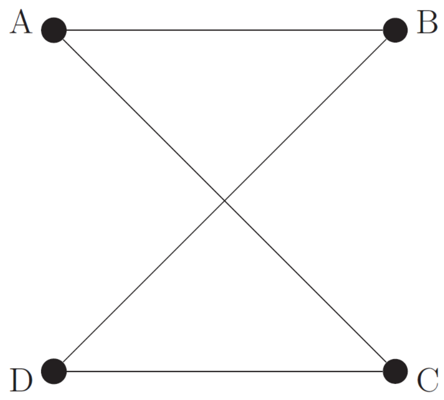
$$t_n = n + 2(n-1) + 4(n-2) + 8(n-3) + \cdots + 2^{n-1}1,$$
$$u_n = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n}.$$

(v) Find the sum

$$\sum_{k=1}^{n} s_k.$$

[MAT, 2016Q6]

For applicants in {CS, Math & CS, CS & Philosophy} only.



Four people A, B, C, D are performing a dance, holding hands in the arrangement shown above. Each dancer is assigned a 1 or a 0 to determine their steps, and there must always be at least a 1 and a 0 in the group of dancers (dancers cannot all dance the same kind of steps). A dancer is **off-beat** if their assigned number plus the numbers assigned to the people holding hands with them is odd. The entire dance is an **off-beat dance** if every dancer is off-beat.

(i) In how many ways can the four dancers perform an off-beat dance? Explain your answer.

A new dance starts and two more people, E and F, join the dance such that each dancer holds hands with their neighbours to form a ring.

(ii) In how many ways can the ring of six dancers perform an off-beat dance? Explain your answer.

(iii) In a ring of n dancers explain why an off-beat dance can only occur if n is a multiple of 3.

(iv) For a new dance a ring of $n > 4$ dancers, each holds hands with dancers one person away from them round the ring (so C holds hands with A and E and D holds hands with B and F and so on). For which values of n can the dance be off-beat?

On another planet the alien inhabitants have three (extendible) arms and still like to dance according to the rules above.

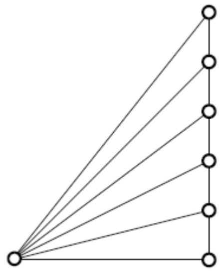
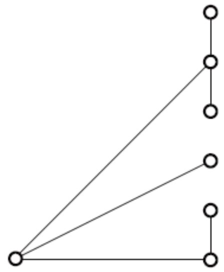
(v) If four aliens dance, each holding hands with each other, how many ways can they perform an off-beat dance?

(vi) Six aliens standing in a ring perform a new dance where each alien holds hands with their direct neighbours and the alien opposite them in the ring. In how many ways can they perform an off-beat dance?

[MAT, 2016Q7]

For applicants in {CS, CS & Philosophy} only.

An n-**fan** consists of a row of n points, the **tips**, in a straight line, together with another point, the **hub**, that is not on the line. The n tips are joined to each other and to the hub with line segments. For example, the left-hand picture here shows a 6-fan.

For a given n-fan, an n-**span** is a subset containing all $n+1$ points and exactly n of the line segments, chosen so that all the points are connected together, with a unique path between any two points. The right-hand picture shows one of many 6-spans obtained from the given 6-fan; in this 6-span, the tips are in "groups" of 3, 1 and 2, with the to "group" containing 3 tips.

(i) Draw all three 2-spans.

(ii) Draw all 3-spans.

(iii) By considering the possible sizes of the top group of tips and how the group is connected to the hub, calculate the number of 4-spans.

(iv) For $n \geq 1$ let z_n denote the number of n-spans. Give an expression for z_n in terms of z_k, where $1 \leq k < n$. Use this expression to show that $z_5 = 55$.

(v) Use this relationship to calculate z_6.

MAT 2017

MAT 2017
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

This paper contains 7 questions, of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5**.
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6**.
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7**.

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial or Warwick Applicants: if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses: Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year Abroad, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, you should attempt Questions **1,2,3,4,5**.

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted**.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), (D), or (E) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2017Q1(A)]

Let

$$f(x) = 2x^3 - kx^2 + 2x - k.$$

For what values of the real number k does the graph $y = f(x)$ have two distinct real stationary points?

(A) $-2\sqrt{3} < k < 2\sqrt{3}$

(B) $k < -2\sqrt{3}$ or $2\sqrt{3} < k$

(C) $k < -\sqrt{21} - 3$ or $\sqrt{21} - 3 < k$

(D) $-\sqrt{21} - 3 < k < \sqrt{21} - 3$

(E) all values of k

[MAT, 2017Q1(B)]

The minimum value achieved by the function

$$f(x) = 9\cos^4 x - 12\cos^2 x + 7$$

equals

(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

[MAT, 2017Q1(C)]

A sequence (a_n) has the property that

$$a_{n+1} = \frac{a_n}{a_{n-1}}$$

for every $n \geq 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is a_{2017}?

(A) $\frac{1}{6}$

(B) $\frac{1}{3}$

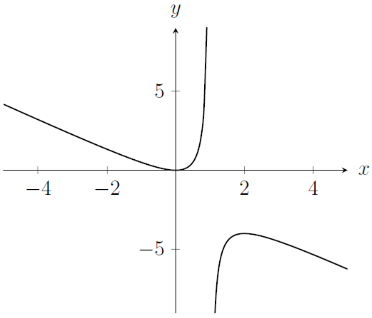
(C) $\frac{1}{2}$

(D) 2

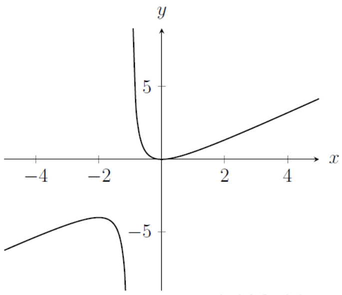
(E) 3

[MAT, 2017Q1(D)]

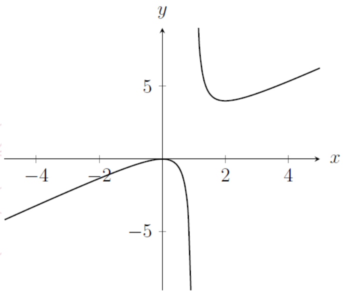
The diagram below shows the graph of $y = f(x)$.



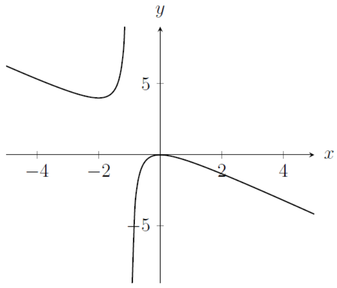
The graph of the function $y = -f(-x)$ is drawn in which of the following diagrams?



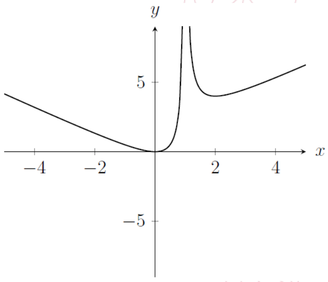
(A)



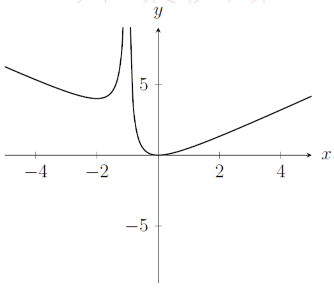
(B)



(C)



(D)



(E)

[MAT, 2017Q1(E)]

Let a and b be *positive integers* such that $a + b = 20$. What is the maximum value that a^2b can take?

(A) 1000

(B) 1152

(C) 1176

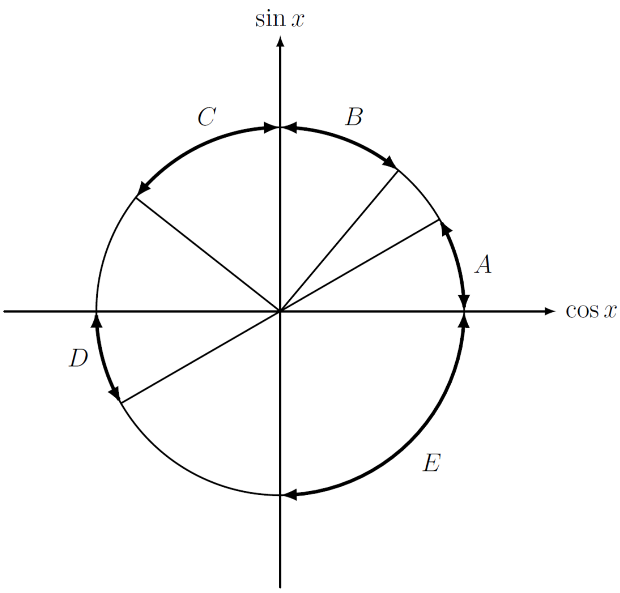
(D) 1183

(E) 1196

[MAT, 2017Q1(F)]

The picture below shows the unit circle, where each point has coordinates $(\cos x, \sin x)$ for some x. Which of the marked arcs corresponds to

$$\tan x < \cos x < \sin x\,?$$



(A) A

(B) B

(C) C

(D) D

(E) E

[MAT, 2017Q1(G)]

For all θ in the range $0 \le \theta < 2\pi$ the line

$$(y-1)\cos\theta = (x+1)\sin\theta$$

divides the disc $x^2 + y^2 \le 4$ into two regions. Let $A(\theta)$ denote the area of the larger region.

Then $A(\theta)$ achieves its maximum value at

(A) one value of θ.

(B) two values of θ.

(C) three values of θ.

(D) four values of θ.

(E) all values of θ.

[MAT, 2017Q1(H)]

In this question a and b are real numbers, and a is non-zero.

When the polynomial $x^2 - 2ax + a^4$ is divided by $x + b$ the remainder is 1.

The polynomial $bx^2 + x + 1$ has $ax - 1$ as a factor.

It follows that b equals

(A) 1 only

(B) 0 or -2

(C) 1 or 2

(D) 1 or 3

(E) -1 or 2

[MAT, 2017Q1(I)]

Let $a, b, c > 0$ and $a \neq 1$. The equation

$$\log_b((b^x)^x) + \log_a\left(\frac{c^x}{b^x}\right) + \log_a\left(\frac{1}{b}\right)\log_a(c) = 0$$

has a repeated root when

(A) $b^2 = 4ac$

(B) $b = \frac{1}{a}$

(C) $c = \frac{b}{a}$

(D) $c = \frac{1}{b}$

(E) $a = b = c$

[MAT, 2017Q1(J)]

Which of these integrals has the largest value? You are not expected to calculate the exact value of any of these.

(A) $\int_0^2 (x^2 - 4)\sin^8(\pi x)\,dx$

(B) $\int_0^{2\pi} (2 + \cos x)^3\,dx$

(C) $\int_0^{\pi} \sin^{100} x\,dx$

(D) $\int_0^{\pi} (3 - \sin x)^6\,dx$

(E) $\int_0^{8\pi} 108(\sin^3 x - 1)\,dx$

[MAT, 2017Q2]

There is a unique real number α that satisfies the equation

$$\alpha^3 + \alpha^2 = 1.$$

[You are not asked to prove this.]

(i) Show that $0 < \alpha < 1$.

(ii) Show that

$$\alpha^4 = -1 + \alpha + \alpha^2$$

(iii) Four functions of α are given in (a) to (d) below. In a similar manner to part (ii), each is equal to a quadratic expression

$$A + B\alpha + C\alpha^2$$

in α; where A, B, C are integers. (So in (ii) we found $A = -1, B = 1, C = 1$) You may assume in each case that the quadratic expression is unique.

In each case below find the quadratic expression in α.

(a) α^{-1}.

(b) The infinite sum

$$1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 - \alpha^5 + \cdots.$$

(c) $(1 - \alpha)^{-1}$.

(d) The infinite product

$$(1 + \alpha)(1 + \alpha^2)(1 + \alpha^4)(1 + \alpha^8)(1 + \alpha^{16}) \cdots.$$

[MAT, 2017Q3]

For applicants in {Math, Math & Statistics, Math &Philosophy and Math & CS} only.

For each positive integer k, let $f_k(x) = x^{1/k}$ for $x \geq 0$.

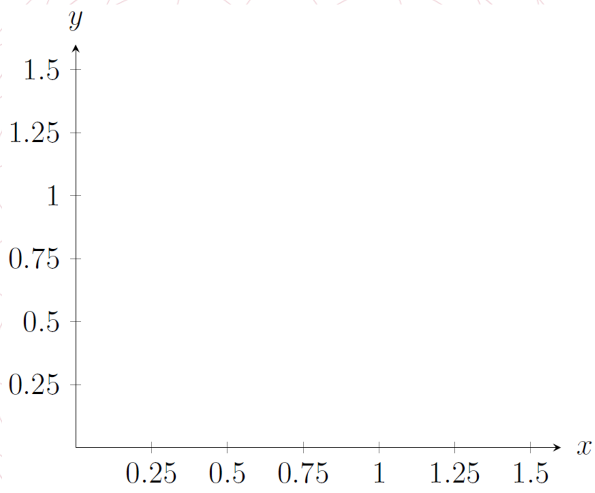
(i) On the same axes (provided below), labelling each curve clearly, sketch $y = f_k(x)$ for $k = 1, 2, 3$, indicating the intersection points.

(ii) Between the two points of intersection in (i), the curves $y = f_k(x)$ enclose several regions. What is the area of the region between $y = f_k(x)$ and $y = f_{k+1}(x)$? Verify that the area of the region between $y = f_1(x)$ and $y = f_2(x)$ is $\frac{1}{6}$.

Let c be a constant where $0 < c < 1$.

(iii) Find the x-coordinates of the points of intersection of the line $y = c$ with $y = f_1(x)$ and of $y = c$ with $y = f_2(x)$.

(iv) The constant c is chosen so that the line $y = c$ divides the region between $y = f_1(x)$ and $y = f_2(x)$ into two regions of equal area. Show that c satisfies the cubic equation $4c^3 - 6c^2 + 1 = 0$. Hence find c.



[MAT, 2017Q4]

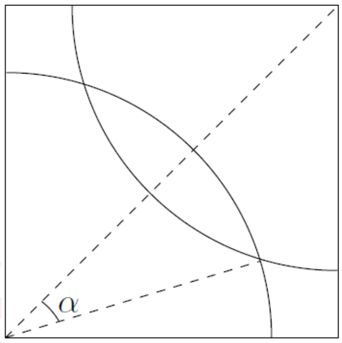
For applicants in {Math, Math & Statistics and Math &Philosophy} only.

A horse is attached by a rope to the corner of a square field of side length 1.

(i) What length of rope allows the horse to reach precisely half the area of the field?

Another horse is placed in the field, attached to the corner diagonally opposite from the first horse. Each horse has a length of rope such that each can reach half the field.

(ii) Explain why the area that both can reach is the same as the area neither can reach.



(iii) The angle α is marked in the diagram above. Show that $\alpha = \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$ and hence show that the area neither can reach is $\frac{4}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right) - \sqrt{\frac{4-\pi}{\pi}}$. Note that $\cos^{-1}$ can also be written as arccos.

A third horse is placed in the field, and the three horses are rearranged. One horse is now attached to the midpoint of the bottom side of the field, and another horse is now attached to the midpoint of the left side of the field. The third horse is attached to the upper right corner.

(iv) Given each horse can access an equal area of the field and that none of the areas overlap, what length of rope must each horse have to minimise the area that no horse can reach?

The horses on the bottom and left midpoints of the field are each replaced by a goat; each goat is attached by a rope of length g to the same midpoint as in part (iii). The remaining horse is attached to the upper right corner with rope length h.

(v) Given that $0 \le h \le 1$, and that none of the animals' areas can overlap, show that $\frac{\sqrt{5}-2}{2} \le g \le \frac{1}{2\sqrt{2}}$ holds if the area that the animals can reach is maximized.

[MAT, 2017Q5]

Ten children, $c_0, c_1, c_2, \cdots, c_9$, are seated clockwise in a circle. The teacher walks clock- wise behind the children with a large bag of sweets. She gives a sweet to child c_1. She then skips a child and gives a sweet to the next child, c_3. Next she skips two children and gives a sweet to the next child, c_6. She continues in this way, at each stage skipping one more child than at the preceding stage before giving a sweet to the next child.

(i) The kth sweet is given to child c_i. Explain why i is the last digit of the number

$$\frac{k(k+1)}{2}.$$

(ii) Let $1 \leq k \leq 18$. Explain why the kth and $(20-k-1)$th sweets are given to the same child.

(iii) Explain why the kth sweet is given to the same child as the $(k+20)$th sweet.

(iv) Which children can never receive any sweets?

When the teacher has given out all the sweets, she has walked exactly 183 times round the circle, and given the last sweet to c_0.

(v) How many sweets were there initially?

(vi) Which children received the most sweets and how many did they receive?

[MAT, 2017Q6]

For applicants in {CS, Math & CS, and CS & Philosophy} only.

You need to pack several items into your shopping bag, without squashing any item. Suppose each item i has a (positive) weight w_i, and a strength s_i which is the maximum weight that can be placed above it without it being squashed. For the purposes of this question, suppose that the items will be arranged one on top of the other within your bag. We will say that a particular packing order is *safe* if no item is squashed, that is, for each item i, s_i is at least the sum of the w_j corresponding to items j placed above item i. For example, suppose we have the following items, packed in the order given

Ordering	Item	w_i	s_i
Top	Apples	5	6
Middle	Bread	4	4
Bottom	Carrots	12	9

This packing is not safe: the bread is squashed, since the weight above it (5) is greater than its strength (4). However, swapping the apples and the bread gives a safe packing.

(i) Which of the other four orderings of apples, bread, and carrots are safe or unsafe?

(ii) Consider the tactic of packing the items in weight order, with the heaviest at the bottom. Show by giving an example that this might not produce a safe packing order, even if a safe packing order exists.

(iii) Now consider the tactic of packing the items in strength order, with the strongest at the bottom. Again show by giving an example that this might not produce a safe packing order, even if one exists.

(iv) Suppose we have a safe packing order, with item j *directly* on top of item i. Suppose further that

$$w_j - s_i \geq w_i - s_j$$

Show that if we swap items i and j, we still have a safe packing order.

(v) Hence suggest a practical method of producing a safe packing order if one exists. Explain why your method works. (Listing all possible orderings is not practical.)

[MAT, 2017Q7]

For applicants in {CS and CS & Philosophy} only.

A simple computer can operate on lists of numbers in several ways.

- Given two lists a and b, it can make the *join* $a + b$, by placing list b after list a. For example if

$$a = (1, 2, 3, 4) \text{ and } b = (5, 6, 7) \text{ then } a + b = (1, 2, 3, 4, 5, 6, 7).$$

- Given a list a it can form the *reverse* sequence $R(a)$ by listing a in reverse order. For example if

$$a = (1, 2, 3, 4) \text{ then } R(a) = (4, 3, 2, 1).$$

(i) Given sequences a and b, express $R(a + b)$ as the join of two sequences. What is $R\big(R(a)\big)$?

- Given a sequence a of length n and $0 \leq k \leq n$, then the kth shuffle S_k of a moves the first k elements of a to the end of the sequence in reverse order. For example

$$S_2(1, 2, 3, 4, 5) = (3, 4, 5, 2, 1) \text{ and } S_3(1, 2, 3, 4, 5) = (4, 5, 3, 2, 1).$$

(ii) Given two sequences a and b, both of length k, express $S_k(a + b)$ as the join of two sequences. What is $S_k\big(S_k(a + b)\big)$?

(iii) Now let $a = (1, 2, 3, 4, 5, 6, 7, 8)$. Write down

$$S_5\big(S_5(a)\big)$$

as the join of three sequences that are either in order or in reverse order. Show that the sequence a is back in its original order after four S_5 shuffles.

(iv) Now let a be a sequence of length n with $k \geq \frac{n}{2}$. Prove, after S_k is performed four times, that the sequence returns to its original order.

(v) Give an example to show that when $k < \frac{n}{2}$, the sequence need not be in its original order after S_k is performed four times. For your example how many times must S_k be performed to first return the sequence to its original order?

MAT 2018

MAT 2018
On-line Exam

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Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2018Q1(A)]

The area of the region bounded by the curve $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis equals

(A) 2

(B) $\frac{5}{2}$

(C) 3

(D) $\frac{10}{3}$

(E) $\frac{16}{3}$

[MAT, 2018Q1(B)]

The function $y = e^{kx}$ satisfies the equation

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = y\frac{\mathrm{d}y}{\mathrm{d}x}$$

for

(A) no values of k.

(B) exactly one value of k.

(C) exactly two distinct values of k.

(D) exactly three distinct values of k.

(E) infinitely many distinct values of k.

[MAT, 2018Q1(C)]

Let a, b, c and d be real numbers. The two curves $y = ax^2 + c$ and $y = bx^2 + d$ have exactly two points of intersection precisely when

(A) $\frac{a}{b} < 1$

(B) $\frac{a}{b} < \frac{c}{d}$

(C) $a < b$

(D) $c < d$

(E) $(d - c)(a - b) > 0$

[MAT, 2018Q1(D)]

If $f(x) = x^2 - 5x + 7$, what are the coordinates of the minimum of $y = f(x-2)$?

(A) $\left(\frac{5}{2}, \frac{3}{4}\right)$

(B) $\left(\frac{9}{2}, \frac{3}{4}\right)$

(C) $\left(\frac{1}{2}, \frac{3}{4}\right)$

(D) $\left(\frac{9}{2}, \frac{-5}{4}\right)$

(E) $\left(\frac{5}{2}, \frac{-5}{4}\right)$

[MAT, 2018Q1(E)]

A circle of radius 2, centred on the origin, is drawn on a grid of points with integer coordinates. Let n be the number of grid points that lie within or on the circle. What is the smallest amount the radius needs to increase by for there to be $2n - 5$ grid points within or on the circle?

(A) $\sqrt{5} - 2$

(B) $\sqrt{6} - 2$

(C) $\sqrt{8} - 2$

(D) 1

(E) $\sqrt{8}$

[MAT, 2018Q1(F)]

A particle moves in the xy-plane, starting at the origin $(0, 0)$. At each turn, the particle may move in one of two ways:

- it may move two to the right and one up, that is, it may be translated by the vector $(2, 1)$, or
- it may move one to the right and two up, that is, it may be translated by the vector $(1, 2)$.

What is the closest the particle may come to the point $(25, 75)$?

(A) 0

(B) $5\sqrt{5}$

(C) $2\sqrt{53}$

(D) 25

(E) 35

[MAT, 2018Q1(G)]

The parabolas with equations $y = x^2 + c$ and $y^2 = x$ touch (that is, meet tangentially) at a single point. It follows that c equals

(A) $\frac{1}{2\sqrt{3}}$

(B) $\frac{3}{4\sqrt[3]{4}}$

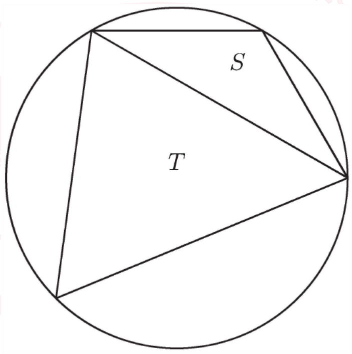
(C) $-\frac{1}{2}$

(D) $\sqrt{5} - \sqrt{3}$

(E) $\sqrt{\frac{2}{3}}$

[MAT, 2018Q1(H)]

Two triangles S and T are inscribed in a circle, as shown in the diagram below.



The triangles have respective areas s and t and S is the smaller triangle so that $s < t$.

The smallest value that

$$\frac{4s^2 + t^2}{5st}$$

can equal is

(A) $\frac{2}{5}$

(B) $\frac{3}{5}$

(C) $\frac{4}{5}$

(D) 1

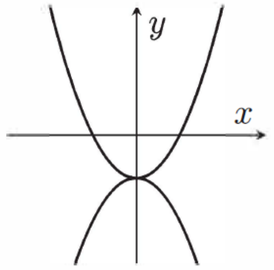
(E) $\frac{3}{2}$

[MAT, 2018Q1(I)]

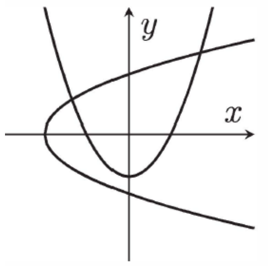
A sketch of the curve

$$(x^8 + 4yx^6 + 6y^2x^4 + 4y^3x^2 + y^4)^2 = 1$$

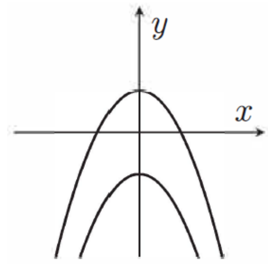
is given below in



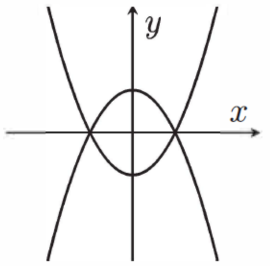
(A)



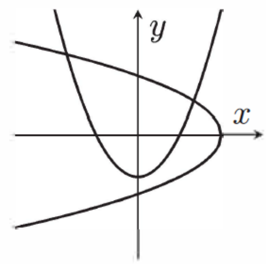
(B)



(C)



(D)



(E)

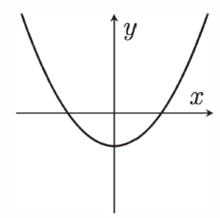
[MAT, 2018Q1(J)]

Which of the following could be the sketch of a curve

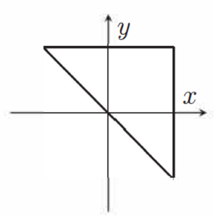
$$p(x) + p(y) = 0$$

for some polynomial p?

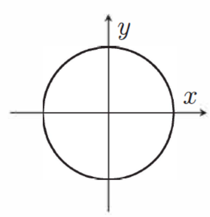
A



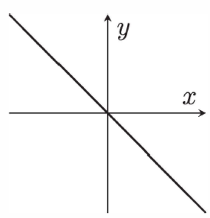
B



C



D



(A) A and D, but not B or C

(B) A and B, but not C or D

(C) C and D, but not A or B

(D) A, C and D, but not B

(E) A, B and C, but not D

[MAT, 2018Q2]

Let S and T denote transformations of the xy-plane

$$S(x,y) = (x+1, y), \qquad T(x,y) = (-y, x),$$

We will write, for example, TS to denote the composition of applying S then T, that is

$$TS(x,y) = T\big(S(x,y)\big),$$

and write T^n to denote n applications of T where n is a positive integer.

(i) Show that $TS(x,y) \neq ST(x,y)$

(ii) For what values of n is it the case that $T^n(x,y) = (x,y)$ for all x, y?

(iii) Show that applications of S and T in some order can produce the transformation

$$U(x,y) = (x-1, y).$$

What is the least number of applications (of S and T in total) that can produce U? Justify your answer.

(iv) Show that for any integers a and b there is some sequence of applications of S and T that maps $(0,0)$ to (a,b).

(v) The parabola C has equation $y = x^2 + 2x + 2$.

What is the equation of the curve obtained by applying S to C?

What is the equation of the curve obtained by applying T to C?

[MAT, 2018Q3]

For applicants in {Math, Math & Statistics, Math &Philosophy and Math & CS} only.

Let $g(x)$ be the function defined by

$$g(x) = \begin{cases} (x-1)^2 + 1 & \text{if } x \geq 0 \\ 3 - (x+1)^2 & \text{if } x \leq 0, \end{cases}$$

and for $x \neq 0$ write $m(x)$ for the gradient of the chord between $(0, g(0))$ and $(x, g(x))$.

(i) Sketch the graph $y = g(x)$ for $-3 \leq x \leq 3$.

(ii) Write down expressions for $m(x)$ in the two cases $x \geq 0$ and $x < 0$.

(iii) Show that $m(x) + 2 = x$ for $x > 0$. What is the value of $m(x) + 2$ when $x < 0$?

(iv) Explain why g has derivative -2 at 0.

(v) Suppose that $p < q$ and that $h(x)$ is a cubic with a maximum at $x = p$ and a minimum at $x = q$. Show that $h'(x) < 0$ whenever $p < x < q$.

Suppose that c and d are real numbers and that there is a cubic $h(x)$ with a maximum at $x = -1$ and a minimum at $x = 1$ such that $h'(0) = -3c$ and $h(0) = d$.

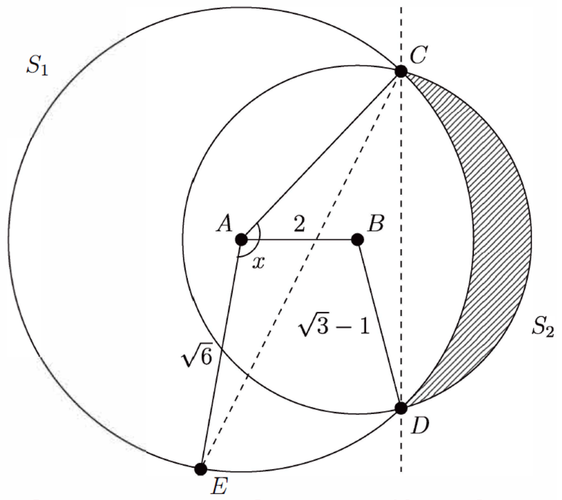
(vi) Show that $c > 0$ and find a formula for $h(x)$ in terms of c and d (and x).

(vii) Show that there are no values of c and d such that the graphs of $y = g(x)$ and $y = h(x)$ are the same for $-3 \leq x \leq 3$.

[MAT, 2018Q4]

For applicants in {Math, Math & Statistics and Math &Philosophy} only.

Consider two circles S_1 and S_2 centred at A and B and with radii $\sqrt{6}$ and $\sqrt{3}-1$, respectively. Suppose that the two circles intersect at two distinct points C and D. Suppose further that the two centres A and B are of distance 2 apart. The sketch below is not to scale.



(i) Find the angle $\angle CBA$, and deduce that A and B lie on the same side of the line CD.

(ii) Show that CD has length $3-\sqrt{3}$ and hence calculate the angle $\angle CAD$.

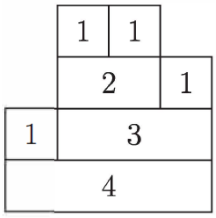
(iii) Show that the area of the region lying inside the circle S_2 and outside of the circle S_1 (that is the shaded region in the picture) is equal to

$$\frac{\pi}{6}(5-4\sqrt{3})+3-\sqrt{3}.$$

(iv) Suppose that a line through C is drawn such that the total area covered by S_1 and S_2 is split into two equal areas. Let E be the intersection of this line with S_1 and x denote the angle $\angle CAE$. You may assume that E lies on the larger arc CD of S_1. Write down an equation which x satisfies and explain why there is a unique solution x.

[MAT, 2018Q5]

Let n be a positive integer. An n-*brick* is a rectangle of height 1 and width n. A 1-*tower* is defined as a 1-brick. An n-*tower*, for $n \geq 2$, is defined as an n-brick on top of which exactly two other towers are stacked: a k_1-tower and a k_2-tower such that $1 \leq k_1 \leq n-1$ and $k_1 + k_2 = n$. The k_1-tower is placed to the left of the k_2-tower so that side-by-side they fit exactly on top of the n-brick. For example, here is a 4-tower:



(i) Draw the four other 4-towers.

(ii) What is the maximum height of an n-tower? Justify your answer.

(iii) The *area* of a tower is defined as the sum of the widths of its bricks. For example, the 4-tower drawn above has area $4 + 4 + 3 + 2 = 13$. Give an expression for the area of an n-tower of maximum height.

(iv) Show that there are infinitely many n such that there is an n-tower of height exactly $1 + \log_2 n$.

(v) Write t_n for the number of n-towers. We have $t_1 = 1$. For $n \geq 2$ give a formula for t_n in terms of t_k for $k < n$. Use your formula to compute t_6.

(vi) Show that t_n is odd if and only if t_{2n} is odd.

[MAT, 2018Q6]

For applicants in {CS, Math & CS, and CS & Philosophy} only.

A positive rational number q is expressed in *friendly form* if it is written as a finite sum of reciprocals of distinct positive integers. For example, $\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$.

(i) Express the following numbers in friendly form: $\frac{2}{3}, \frac{2}{5}, \frac{23}{40}$.

(ii) Let q be a rational number with $0 < q < 1$, and m be the smallest natural number such than $\frac{1}{m} \leq q$. Suppose $q = \frac{a}{b}$ and $q - \frac{1}{m} = \frac{c}{d}$ in their lowest terms. Show that $c < a$.

(iii) Suggest a procedure by which any rational q with $0 < q < 1$ can be expressed in friendly form. Use the result in part (ii) to show that the procedure always works, generating distinct reciprocals and finishing within a finite time.

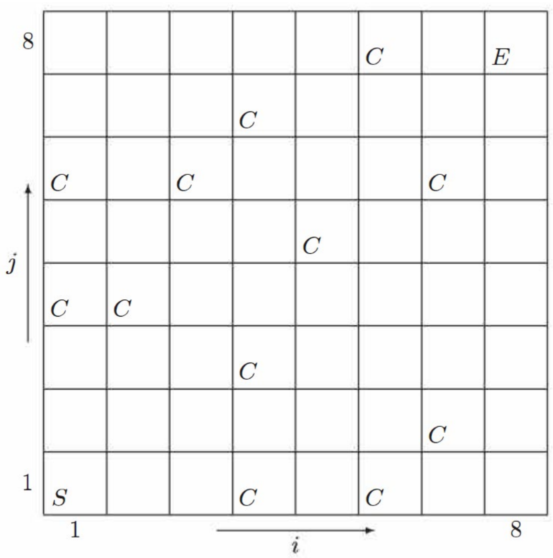
(iv) Demonstrate your procedure by finding a friendly form for $\frac{4}{13}$.

(v) Assuming that $\sum_{n=1}^{N} \frac{1}{n}$ increases without bound as N becomes large, show that every positive rational number can be expressed in friendly form.

[MAT, 2018Q7]

For applicants in {CS and CS & Philosophy} only.

A character in a video game is collecting the magic coins that are attached to a cliff face. The character starts at the bottom left corner S and must reach the top right corner E. One point is scored for each coin C that is collected on the way, and the aim is to reach the top right corner with the highest possible score. At each step the character may move either one cell to the right or one cell up, but never down or to the left.

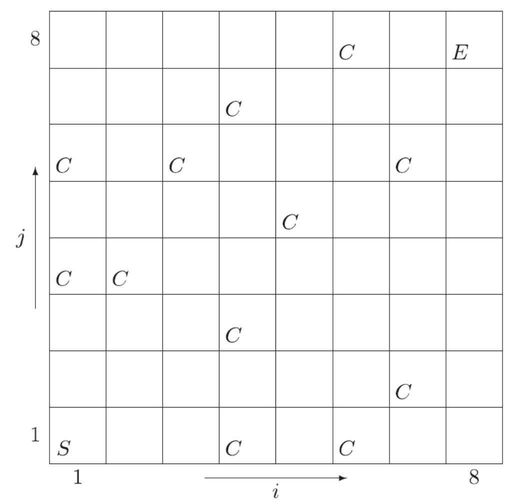


(i) Let $c(i, j) = 1$ if there is a coin at position (i, j), and $c(i, j) = 0$ otherwise. Describe how the maximum score $m(i, j)$ achievable on reaching position (i, j), where $i \geq 2$ and $j \geq 2$, can be determined in terms of the maximum scores $m(i, j-1)$ and $m(i-1, j)$ achievable at the positions immediate below and to the left. Briefly justify your answer.

(ii) Use the result from part (i) to fill in each cell in the diagram above to show the maximum score achievable on reaching that cell. What is the maximum score achievable in the game? *A spare copy of the diagram appears at the end of the question.*

(iii) Given the array of scores $m(i, j)$, describe a method for tracing backwards from E to S a path that, if followed in a forward direction by the character, would achieve the maximum score. Draw one such path across the cliff.

(iv) With the pattern of coins shown, how many different paths from S to E achieve the maximum score? Describe a method for computing the number of such paths.



MAT 2019

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	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2019Q1(A)]

The equation

$$x^3 - 300x = 3000$$

has

(A) no real solutions.

(B) exactly one real solution.

(C) exactly two real solutions.

(D) exactly three real solutions.

(E) infinitely many real solutions.

[MAT, 2019Q1(B)]

The product of a square number and a cube number is

(A) always a square number, and never a cube number.

(B) always a cube number, and never a square number.

(C) sometimes a square number, and sometimes a cube number.

(D) never a square number, and never a cube number.

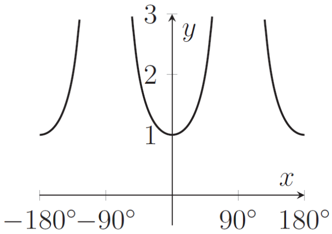
(E) always a cube number, and always a square number.

[MAT, 2019Q1(C)]

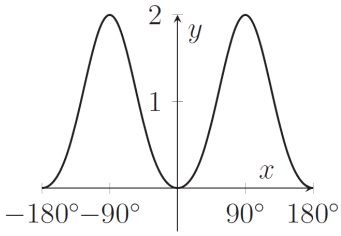
The graph of

$$y = \sin^2 x + \sin^4 x + \sin^6 x + \sin^8 x + \cdots$$

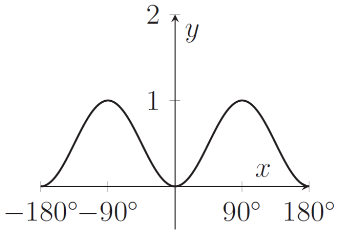
is sketched in



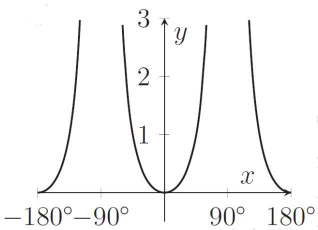
(A)



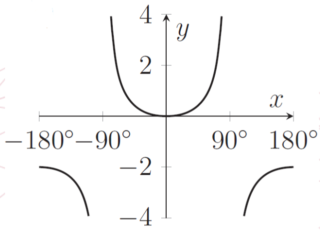
(B)



(C)



(D)



(E)

[MAT, 2019Q1(D)]

The area between the parabolas with equations $y = x^2 + 2ax + a$ and $y = a - x^2$ equals 9. The possible values of a are

(A) $a = 1$

(B) $a = -3$ or $a = 3$

(C) $a = -3$

(D) $a = -1$ or $a = 1$

(E) $a = 1$ or $a = 3$

[MAT, 2019Q1(E)]

The graph of

$$\sin y - \sin x = \cos^2 x - \cos^2 y$$

(A) is empty.

(B) is non-empty but includes no straight lines.

(C) includes precisely one straight line.

(D) includes precisely two straight lines.

(E) includes infinitely many straight lines.

[MAT, 2019Q1(F)]

In the interval $0 \leq x < 360°$, the number of solutions of the equation

$$\sin^3 x + \cos^2 x = 0$$

is

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

[MAT, 2019Q1(G)]

Let $a, b, c > 0$. The equations

$$\log_a b = c, \qquad \log_b a = c + \frac{3}{2}, \qquad \log_c a = b,$$

(A) specify a, b and c uniquely.

(B) specify c uniquely but have infinitely many solutions for a and b.

(C) specify c and a uniquely but have infinitely many solutions for b.

(D) specify a and b uniquely but have infinitely many solutions for c.

(E) have no solutions for a, b and c.

[MAT, 2019Q1(H)]

The triangle ABC is right-angled at B and the side lengths are positive numbers in geometric progression. It follows that $\tan \angle BAC$ is either

(A) $\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{1-\sqrt{5}}{2}}$

(B) $\sqrt{\frac{1+\sqrt{3}}{2}}$ or $\sqrt{\frac{\sqrt{3}-1}{2}}$

(C) $\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{\sqrt{5}-1}{2}}$

(D) $-\sqrt{\frac{1+\sqrt{5}}{2}}$ or $\sqrt{\frac{1+\sqrt{5}}{2}}$

(E) $\sqrt{\frac{1+\sqrt{3}}{2}}$ or $\sqrt{\frac{1-\sqrt{3}}{2}}$

[MAT, 2019Q1(I)]

The positive real numbers x and y satisfy $0 < x < y$ and

$$x2^x = y2^y$$

for

(A) no pairs x and y.

(B) exactly one pair x and y.

(C) exactly two pairs x and y.

(D) exactly four pairs x and y.

(E) infinitely many pairs x and y.

[MAT, 2019Q1(J)]

An equilateral triangle has centre O and side length 1. A straight line through O intersects the triangle at two distinct points P and Q. The minimum possible length of PQ is

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) $\frac{\sqrt{3}}{3}$

(D) $\frac{2}{3}$

(E) $\frac{\sqrt{3}}{2}$

[MAT, 2019Q2]

For k a positive integer, we define the polynomial $p_k(x)$ as

$$p_k(x) = (1+x)(1+x^2)(1+x^3) \times \cdots \times \left(1+x^k\right) = a_0 + a_1 x + \cdots + a_N x^N,$$

denoting the coefficients of $p_k(x)$ as $a_0, \ldots, a_N$.

(i) Write down the degree N of $p_k(x)$ in terms of k.

(ii) By setting $x = 1$, or otherwise, explain why

$$a_{\max} \geq \frac{2^k}{N+1}$$

where $a_{\max}$ denotes the largest of the coefficients $a_0, \ldots, a_N$.

(iii) Fix $i \geq 0$. Explain why the value of a_i eventually becomes constant as k increases.

A student correctly calculates for $k = 6$ that $p_6(x)$ equals

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 4x^7 + 4x^8 + 5x^9 + 5x^{10} + 5x^{11} + 5x^{12} + 4x^{13} \\ + 4x^{14} + 4x^{15} + 3x^{16} + 2x^{17} + 2x^{18} + x^{19} + x^{20} + x^{21}.$$

(iv) On the basis of this calculation, the student guesses that

$$a_i = a_{N-i} \text{ for } 0 \leq i \leq N.$$

By substituting x^{-1} for x, or otherwise, show that the student's guess is correct for all positive integers k.

(v) On the basis of the same calculation, the student guesses that all whole numbers in the range $1, 2, \ldots, a_{\max}$ appear amongst the coefficients $a_0, \ldots, a_N$, for all positive integers k.

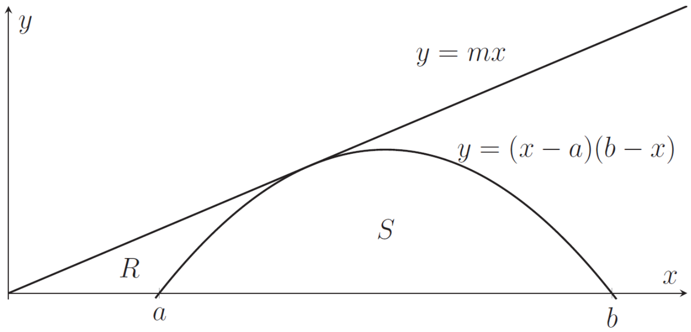
Use part (ii) to show that in this case the student's guess is wrong. Justify your answer.

[MAT, 2019Q3]

For applicants in {Math, Math & Statistics, Math &Philosophy and Math & CS} only.

Let a, b, m be positive numbers with $0 < a < b$. In the diagram below are sketched the parabola with equation $y = (x-a)(b-x)$ and the line $y = mx$. The line is tangential to the parabola.

R is the region bounded by the x-axis, the line and the parabola. S is the region bounded by the parabola and the x-axis.



(i) For $c > 0$, evaluate

$$\int_0^c x(c-x)\,\mathrm{d}x.$$

Without further calculation, explain why the area of region S equals $\frac{(b-a)^3}{6}$.

(ii) The line $y = mx$ meets the parabola tangentially as drawn in the diagram. Show that $m = \left(\sqrt{b} - \sqrt{a}\right)^2$.

(iii) Assume now that $a = 1$ and write $b = \beta^2$ where $\beta > 1$. Given that the area of R equals $(2\beta + 1)(\beta - 1)^2/6$, show that the areas of regions R and S are equal precisely when

$$(\beta - 1)^2(\beta^4 + 2\beta^3 - 4\beta - 2) = 0. \qquad (*)$$

Explain why there is a solution β to $(*)$ in the range $\beta > 1$.

Without further calculation, deduce that for any $a > 0$ there exists $b > a$ such that the area of region S equals the area of region R.

[MAT, 2019Q4]

For applicants in {Math, Math & Statistics and Math &Philosophy} only.

In this question we will consider subsets S of the xy-plane and points (a, b) which may or may not be in S. We will be interested in those points of S which are nearest to the point (a, b). There may be many such points, a unique such point, or no such point.

(i) Let S be the disc $x^2 + y^2 \leq 1$. For a given point (a, b), find the unique point of S which is closest to (a, b).

[You will need to consider separately the cases when $a^2 + b^2 > 1$ and when $a^2 + b^2 \leq 1$.]

(ii) Describe (without further justification) an example of a subset S and a point (a, b) such that there is no point of S nearest to (a, b).

(iii) Describe (without further justification) an example of a subset S and a point (a, b) such that there is more than one point of S nearest to (a, b).

(iv) Let S denote the line with equation $y = mx + c$. Obtain an expression for the distance of (a, b) from a general point $(x, mx + c)$ of S.

Show that there is a unique point of S nearest to (a, b).

(v) For some subset S, and for any point (a, b), the nearest point of S to (a, b) is

$$\left(\frac{a + 2b - 2}{5}, \frac{2a + 4b + 1}{5}\right).$$

Describe the subset S.

(vi) Say now that S has the property that

for any two points P and Q in S the line segment PQ is also in S.

Show that, for a given point (a, b), there cannot be two distinct points of S which are nearest to (a, b).

[MAT, 2019Q5]

This question is about counting the number of ways of partitioning a set of n elements into subsets, each with at least two and at most n elements. If n and k are integers with $1 \leq k \leq n$, let $f(n,k)$ be the number of ways of partitioning a set of n elements into k such subsets. For example, $f(5,2) = 10$ because the allowable partitions of $\{1,2,3,4,5\}$ are

$$\begin{aligned}
&\{1,5\},\{2,3,4\},\{1,2,5\},\{3,4\},\\
&\{2,5\},\{1,3,4\},\{3,4,5\},\{1,2\},\\
&\{3,5\},\{1,2,4\},\{1,3,5\},\{2,4\},\\
&\{4,5\},\{1,2,3\},\{2,4,5\},\{1,3\},\\
&\{1,4,5\},\{2,3\},\\
&\{2,3,5\},\{1,4\}.
\end{aligned}$$

(i) Explain why $f(n,k) = 0$ if $k > n/2$.

(ii) What is the value of $f(n,1)$ and why?

(iii) In forming an allowable partition of $\{1,2,\ldots,n+1\}$ into subsets of at least two elements, we can either

- pair $n+1$ with one other element, leaving $n-1$ elements to deal with, or
- take an allowable partition of $\{1,2,\ldots,n\}$ and add $n+1$ to one of the existing subsets, making a subset of size three or more.

Use this observation to find an equation for $f(n+1,k)$ in terms of $f(n-1,k-1)$ and $f(n,k)$ that holds when $2 \leq k < n$.

(iv) Use this equation to compute the value of $f(7,3)$.

(v) Give a formula for $f(2n,n)$ in terms of n when $n \geq 1$ and show that it is correct.

[MAT, 2019Q6]

For applicants in {CS, Math & CS, and CS & Philosophy} only.

A flexadecimal number consists of a sequence of digits, with the rule that the rightmost digit must be 0 or 1, the digit to the left of it is 0, 1, or 2, the third digit (counting from the right) must be at most 3, and so on. As usual, we may omit leading digits if they are zero. We write flexadecimal numbers in angle brackets to distinguish them from ordinary, decimal numbers. Thus $\langle 34101\rangle$ is a flexadecimal number, but $\langle 231\rangle$ is not, because the digit 3 is too big for its place. (If flexadecimal numbers get very long, we will need 'digits' with a value more than 9.)

The number 1 is represented by $\langle 1\rangle$ in flexadecimal. To add 1 to a flexadecimal number, work from right to left. If the rightmost digit d_1 is 0, replace it by 1 and finish. Otherwise, replace d_1 by 0 and examine the digit d_2 to its left, appending a zero at the left if needed at any stage. If $d_2 < 2$, then increase it by 1 and finish, but if $d_2 = 2$, then replace it by 0, and again move to the left. The process stops when it reaches a digit that can be increased without becoming too large. Thus, the numbers 1 to 4 are represented as $\langle 1\rangle$, $\langle 10\rangle$, $\langle 11\rangle$, $\langle 20\rangle$.

(i) Write the numbers from 5 to 13 in flexadecimal.

(ii) Describe a workable procedure for converting flexadecimal numbers to decimal, and explain why it works. Demonstrate your procedure by converting $\langle 1221\rangle$ to decimal.

(iii) Describe a workable procedure for converting decimal numbers to flexadecimal, and demonstrate it by converting 255 to flexadecimal.

(iv) We could add flexadecimal numbers by converting them to decimal, adding the decimal numbers and converting the result back again. Describe instead a procedure for addition that works directly on the digits of two flexadecimal numbers, and demonstrate it by performing the addition $\langle 1221\rangle + \langle 201\rangle$.

(v) Given a flexadecimal number, how could you test whether it is a multiple of 3 without converting it to decimal?

(vi) If the $\langle 100000\rangle$ arrangements of the letters $abcdef$ are listed in alphabetical order and numbered $\langle 0\rangle : abcdef$, $\langle 1\rangle : abcdfe$, $\langle 10\rangle : abcedf$, etc., what arrangement appears in position $\langle 34101\rangle$ in the list?

[MAT, 2019Q7]

For applicants in {CS and CS & Philosophy} only.

You are given two identical black boxes, each with an N-digit display and each with two buttons marked A and B. Button A resets the display to 0, and button B updates the display using a complex, unknown but fixed, function f, so that pressing button A then repeatedly pressing button B displays a fixed sequence

$$x_0 = 0, \quad x_1 = f(x_0), \quad x_2 = f(x_1), \ldots,$$

the same for both boxes. In general $x_i = f^i(0)$ where f^i denotes applying function f repeatedly i times.

You have no pencil and paper, and the display has too many digits for you to remember more than a few displayed values, but you can compare the displays on the two boxes to see if they are equal, and you can count the number of times you press each button.

(i) Explain briefly why there must exist integers i, j with $0 \le i < j$ such that $x_i = x_j$.

(ii) Show that if $x_i = x_j$ then $x_{i+s} = x_{j+s}$ for any $s \ge 0$.

(iii) Let m be the smallest number such that x_m appears more than once in the sequence, and let $p > 0$ be the smallest number such that $x_m = x_{m+p}$. Show that if $i \ge m$ and $k \ge 0$ then $x_{i+kp} = x_i$.

(iv) Given integers i, j with $0 \le i < j$, show that $x_i = x_j$ if and only if $i \ge m$ and $j - i$ is a multiple of p. [Hint: let r be the remainder on dividing $j - i$ by p, and argue that $r = 0$.]

(v) You conduct an experiment where (after resetting both boxes) you repeatedly press button B once on one box and button B twice on the other box and compare the displays, thus determining the smallest number $u > 0$ such that $x_u = x_{2u}$. What relates the value of u to the (unknown) values of m and p?

(vi) Once u is known, what experiment would you perform to determine the value of m?

(vii) Once u and m are known, what experiment would tell you the value of p?

MAT 2020

MAT 2020
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

This paper contains 7 questions, of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5**.
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6**.
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7**.

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial or Warwick Applicants: if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses: Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year Abroad, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, you should attempt Questions **1,2,3,4,5**.

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted**.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), (D), or (E) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2020Q1(A)]

A square has centre $(3, 4)$ and one corner at $(1, 5)$. Another corner is at

(A) $(1, 3)$

(B) $(5, 5)$

(C) $(4, 2)$

(D) $(2, 2)$

(E) $(5, 2)$

[MAT, 2020Q1(B)]

What is the value of $\int_0^1 (e^x - x)(e^x + x)\,dx$?

(A) $\frac{3e^2-2}{6}$

(B) $\frac{3e^2+2}{6}$

(C) $\frac{2e^2-3}{6}$

(D) $\frac{3e^2-5}{6}$

(E) $\frac{e^2+3}{6}$

[MAT, 2020Q1(C)]

The sum $1 - 4 + 9 - 16 + \cdots + 99^2 - 100^2$ equals

(A) -101

(B) -1000

(C) -1111

(D) -4545

(E) -5050

[MAT, 2020Q1(D)]

The largest value achieved by $3\cos^2 x + 2\sin x + 1$ equals

(A) $\frac{11}{5}$

(B) $\frac{13}{3}$

(C) $\frac{12}{5}$

(D) $\frac{14}{9}$

(E) $\frac{12}{7}$

[MAT, 2020Q1(E)]

A line is tangent to the parabola $y = x^2$ at the point (a, a^2) where $a > 0$. The area of the region bounded by the parabola, the tangent line, and the x-axis equals

(A) $\frac{a^2}{3}$

(B) $\frac{2a^2}{3}$

(C) $\frac{a^3}{12}$

(D) $\frac{5a^3}{6}$

(E) $\frac{a^4}{10}$

[MAT, 2020Q1(F)]

Which of the following expressions is equal to $\log_{10}(10 \times 9 \times 8 \times \cdots \times 2 \times 1)$?

(A) $1 + 5\log_{10} 2 + 4\log_{10} 6$

(B) $1 + 4\log_{10} 2 + 2\log_{10} 6 + \log_{10} 7$

(C) $2 + 2\log_{10} 2 + 4\log_{10} 6 + \log_{10} 7$

(D) $2 + 6\log_{10} 2 + 4\log_{10} 6 + \log_{10} 7$

(E) $2 + 6\log_{10} 2 + 4\log_{10} 6$

[MAT, 2020Q1(G)]

A cubic has equation $y = x^3 + ax^2 + bx + c$ and has turning points at $(1, 2)$ and $(3, d)$ for some d. What is the value of d?

(A) -4

(B) -2

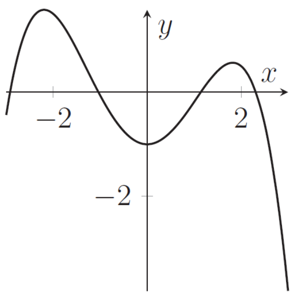
(C) 0

(D) 2

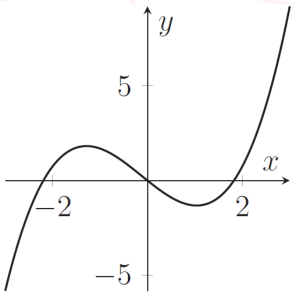
(E) 4

[MAT, 2020Q1(H)]

The following five graphs are, in some order, plots of $y = f(x)$, $y = g(x)$, $y = h(x)$, $y = \frac{\mathrm{d}f}{\mathrm{d}x}$ and $y = \frac{\mathrm{d}g}{\mathrm{d}x}$; that is, three unknown functions and the derivatives of the first two of those functions. Which graph is a plot of $h(x)$?



(A)



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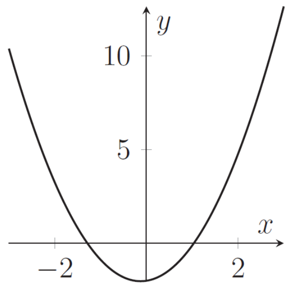
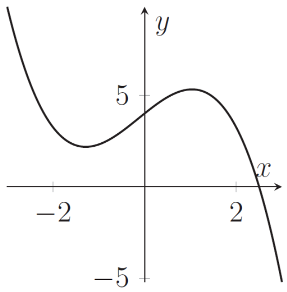


(C)



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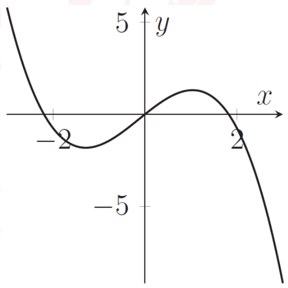


(E)

[MAT, 2020Q1(I)]

In the range $-90° < x < 90°$, how many values of x are there for which the sum to infinity

$$\frac{1}{\tan x} + \frac{1}{\tan^2 x} + \frac{1}{\tan^3 x} + \cdots$$

equals $\tan x$?

(A) 0

(B) 1

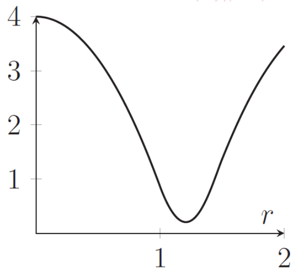
(C) 2

(D) 3

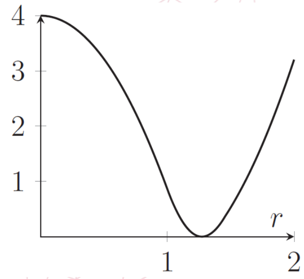
(E) 4

[MAT, 2020Q1(J)]

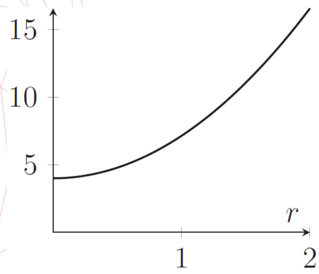
Consider a square with side length 2 and centre $(0,0)$, and a circle with radius r and centre $(0,0)$. Let $A(r)$ be the area of the region that is inside the circle but outside the square, and let $B(r)$ be the area of the region that is inside the square but outside the circle. Which of the following is a sketch of $A(r) + B(r)$?



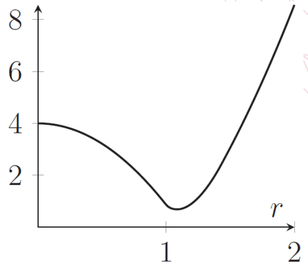
(A)



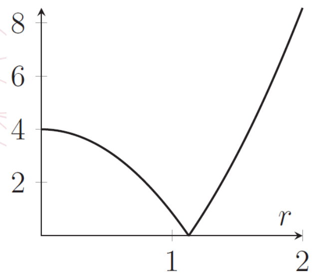
(B)



(C)



(D)



(E)

[MAT, 2020Q2]

The functions $f(n)$ and $g(n)$ are defined for positive integers n as follows:

$$f(n) = 2n + 1, \qquad g(n) = 4n.$$

This question is about the set S of positive integers that can be achieved by applying, in some order, a combination of fs and gs to the number 1. For example as

$$gfg(1) = gf(4) = g(9) = 36,$$

and

$$ffgg(1) = ffg(4) = ff(16) = f(33) = 67,$$

then both 36 and 67 are in S.

(i) Write out the binary expansion of 100 (one hundred).

[Recall that binary is base 2. Every positive integer n can be uniquely written as a sum of powers of 2, where a given power of 2 can appear no more than once. So, for example, $13 = 2^3 + 2^2 + 2^0$ and the binary expansion of 13 is 1101.]

(ii) Show that 100 is in S by describing explicitly a combination of fs and gs that achieves 100.

(iii) Show that 200 is not in S.

(iv) Show that, if n is in S, then there is only one combination of applying fs and gs in order to achieve n. (So, for example, 67 can only be achieved by applying g then g then f then f in that order.)

(v) Let u_k be the number of elements n of S that lie in the range $2^k \leq n < 2^{k+1}$.

Show that

$$u_{k+2} = u_{k+1} + u_k$$

for $k \geq 0$.

(vi) Let s_k be the number of elements n of S that lie in the range $1 \leq n < 2^{k+1}$.

Show that

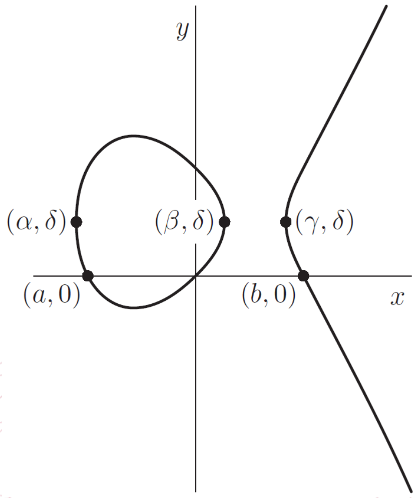
$$s_{k+2} = s_{k+1} + s_k + 1$$

for $k \geq 0$.

[MAT, 2020Q3]

For applicants in {Math, Math & Statistics, Math &Philosophy and Math & CS} only.

Below is a sketch of the curve S with equation $y^2 - y = x^3 - x$. The curve crosses the x-axis at the origin and at $(a, 0)$ and at $(b, 0)$ for some real numbers $a < 0$ and $b > 0$. The curve only exists for $\alpha \leq x \leq \beta$ and for $x \geq \gamma$. The three points with coordinates (α, δ), (β, δ), and (γ, δ) are all on the curve



(i) What are the values of a and b?

(ii) By completing the square, or otherwise, find the value of δ.

(iii) Explain why the curve is symmetric about the line $y = \delta$.

(iv) Find a cubic equation in x which has roots α, β, γ. (Your expression for the cubic should not involve α, β, or γ). Justify your answer.

(v) By considering the factorization of this cubic, find the value of $\alpha + \beta + \gamma$.

(vi) Let C denote the circle which has the points (α, δ) and (β, δ) as ends of a diameter. Write down the equation of C. Show that C intersects S at two other points and find their common x-co-ordinate in terms of γ.

[MAT, 2020Q4]

For applicants in {Math, Math & Statistics and Math &Philosophy} only.

(i) A function $f(x)$ is said to be *even* if $f(-x) = f(x)$ for all x. A function is said to be *odd* if $f(-x) = -f(x)$ for all x.

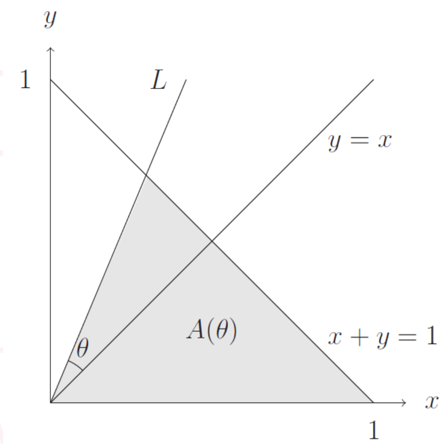
(a) What symmetry does the graph $y = f(x)$ of an even function have?

What symmetry does the graph $y = f(x)$ of an odd function have?

(b) Use these symmetries to show that the derivative of an even function is an odd function, and that the derivative of an odd function is an even function.

[You should not use the chain rule.]

(ii) For $-45° < \theta < 45°$, the line L makes an angle θ with the line $y = x$ as drawn in the figure below. Let $A(\theta)$ denote the area of the triangle which is bounded by the x-axis, the line $x + y = 1$ and the line L.



(a) Let $0 < \theta < 45°$. Arguing geometrically, explain why

$$A(\theta) + A(-\theta) = \frac{1}{2}.$$

(b) For $0 < \theta < 45°$, determine a formula for $A(\theta)$.

(c) Sketch the graph of $A(\theta)$ against θ for $-45° < \theta < 45°$.

(d) In light of the identity in part (ii)(a), what symmetry does the graph of $A(\theta)$ have?

(e) Without explicitly differentiating, explain why $\frac{d^2A}{d\theta^2} = 0$ when $\theta = 0$.

[MAT, 2020Q5]

Miriam and Adam agree to relieve the boredom of the school holidays by eating sweets, but their mother insists they limit their consumption by obeying the following rules.

- Miriam eats as many sweets on any day as there have been sunny days during the holiday so far, including the day in question.
- Adam eats sweets only on rainy days. If day k of the holiday is rainy, then he eats k sweets on that day.

For example, if the holiday is eight days long, and begins Rainy, Sunny, Sunny, ... ,then the tally of sweet consumption might look like this:

Day	1	2	3	4	5	6	7	8	Total
Weather	R	S	S	R	R	S	S	R	
Miriam	0	1	2	2	2	3	4	4	18
Adam	1	0	0	4	5	0	0	8	18

In this case, Miriam and Adam eat the same number of sweets in total

(i) If the holiday has 30 days, 15 of which are sunny and 15 rainy, what arrangement of sunny and rainy days would lead Miriam to eat the greatest number of sweets in total, and what arrangement would lead to the least number? Give the number of sweets that Miriam eats in each case.

(ii) Show that, in the two cases mentioned in part (i), Adam eats the same number of sweets as Miriam.

(iii) Suppose, in a sequence of sunny and rainy days, we arrange to swap a rainy day with a sunny day that immediately follows it. How does the total number of sweets eaten by Miriam change when we make the swap? What about the total number of sweets eaten by Adam?

(iv) If the holiday has 15 sunny days and 15 rainy days, must Miriam and Adam eat the same number of sweets in total? Explain your answer.

[MAT, 2020Q6]

For applicants in {CS, Math & CS, and CS & Philosophy} only.

The cancellation of the Wimbledon tournament has led to a world surplus of tennis balls, and Santa has decided to use them as stocking fillers. He comes down the chimney with n identical tennis balls, and he finds k named stockings waiting for him.

Let $g(n,k)$ be the number of ways that Santa can put the n balls into the k stockings; for example, $g(2,2)=3$, because with two balls and two children, Miriam and Adam, he can give both balls to Miriam, or both to Adam, or he can give them one ball each.

(i) What is the value of $g(1,k)$ for $k \geq 1$?

(ii) What is the value of $g(n,1)$?

(iii) If there are $n \geq 2$ balls and $k \geq 2$ children, then Santa can either give the first ball to the first child, then distribute the remaining balls among all k children, or he can give the first child none, and distribute all the balls among the remaining children. Use this observation to formulate an equation relating the value of $g(n,k)$ to other values taken by g.

(iv) What is the value of $g(7,5)$?

(v) After the first house, Rudolf reminds Santa that he ought to give at least one ball to each child. Let $h(n,k)$ be the number of ways of distributing the balls according to this restriction. What is the value of $h(7,5)$?

[MAT, 2020Q7]

For applicants in {CS and CS & Philosophy} only.

Quantiles is game for a single player. It is played with an inexhaustible supply of tiles, each bearing one of the symbols A, B or C. In each move, the player lays down a row of tiles containing exactly one A and one B, but varying numbers of C's. The rules are as follows:

- The player may play the basic rows $CACBCC$ and $CCACBCCC$.
- If the player has already played rows of the form $rAsBt$ and $xAyBz$ (they may be the same row), where each of r, s, t, x, y, z represents a sequence of C's, then he or she may add the row $rxAsyBtz$, in which copies of the sequences of C's from the previous rows are concatenated with an intervening A and B: this is called a *join move*. The original rows remain, and may be used again in subsequent join moves.
- No other rows may be played.

The player attempts to play one row after another so as to finish with a specified goal row.

(i) Give two examples of rows, other than basic rows, that may be played in the game.

(ii) Give two examples of rows, each containing exactly one A and one B, that may never be played, and explain why.

(iii) Let C^n denote an unbroken sequence of n tiles each labelled with C. Can the goal row $C^{64}AC^{48}BC^{112}$ be achieved? Justify your answer.

(iv) Can the goal row $C^{128}AC^{48}BC^{176}$ be achieved? Justify your answer.

(v) The goal row $C^{31}AC^{16}BC^{47}$ is achievable; show that it can be reached with 7 join moves.

(vi) In any game, we call a row *useless* if it repeats an earlier row or it is not used in a subsequent join move. What is the maximum number of join moves in a game that ends with $C^{31}AC^{16}BC^{47}$ and contains no useless rows?

MAT 2021

MAT 2021
On-line Exam

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Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

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1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given five possible answers, just one of which is correct. Indicate for each part A-J which answer (A), (B), (C), (D), or (E) you think is correct with a tick (√) in the corresponding column in the table below

Please show any rough working in the space provided between the parts.

	(A)	(B)	(C)	(D)	(E)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

[MAT, 2021Q1(A)]

A regular dodecagon is a 12-sided polygon with all sides the same length and all internal angles equal. If I construct a regular dodecagon by connecting 12 equally-spaced points on a circle of radius 1, then the area of this polygon is

(A) $6+3\sqrt{3}$

(B) $2\sqrt{2}$

(C) $3\sqrt{2}$

(D) $3\sqrt{3}$

(E) 3

[MAT, 2021Q1(B)]

The positive number a satisfies

$$\int_0^a \left(\sqrt{x}+x^2\right) \mathrm{d}x = 5$$

if

(A) $a=\left(\sqrt{21}-1\right)^{\frac{1}{3}}$

(B) $a=\sqrt{3}$

(C) $a=3^{\frac{2}{3}}$

(D) $a=\left(\sqrt{6}-1\right)^{\frac{2}{3}}$

(E) $a=5^{\frac{2}{3}}$

[MAT, 2021Q1(C)]

Tangents to $y=\mathrm{e}^x$ are drawn at (p,e^p) and (q,e^q). These tangents cross the x-axis at a and b respectively. It follows that, for all p and q,

(A) $pa=qb$

(B) $p-a<q-b$

(C) $p-a=q-b$

(D) $p-a>q-b$

(E) $p+q=a+b$

[MAT, 2021Q1(D)]

The area of the region bounded by the curve $y = \mathrm{e}^x$, the curve $y = 1 - \mathrm{e}^x$, and the y-axis equals

[Note that $\ln x$ is alternative notation for $\log_e x$.]

(A) 0

(B) $1 - \ln 2$

(C) $\frac{1}{2} - \frac{1}{2}\ln 2$

(D) $\ln 2 - 1$

(E) $1 - \ln\frac{1}{2}$

[MAT, 2021Q1(E)]

Six vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ are each chosen to be either $\binom{1}{1}$ or $\binom{3}{2}$ with equal probability, with each choice made independently. The probability that the sum $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 + \mathbf{v}_6$ is equal to the vector $\binom{10}{8}$ is

(A) 0

(B) $\frac{3}{64}$

(C) $\frac{15}{64}$

(D) $\frac{1}{6}$

(E) $\frac{5}{16}$

[MAT, 2021Q1(F)]

The tangent to the curve $y = x^3 - 3x$ at the point $(a, a^3 - 3a)$ also passes through the point $(2, 0)$ for precisely.

(A) no value of a

(B) one value of a

(C) two values of a

(D) three values of a

(E) all values of a

[MAT, 2021Q1(G)]

The sum

$$\sin^2(1°) + \sin^2(2°) + \sin^3(3°) + \cdots + \sin^2(89°) + \sin^2(90°)$$

is equal to

(A) 44

(B) $44\frac{1}{2}$

(C) 45

(D) $45\frac{1}{2}$

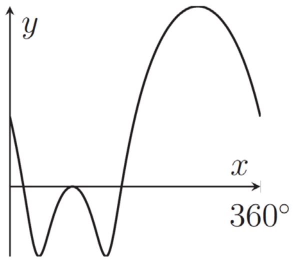
(E) 46

[MAT, 2021Q1(H)]

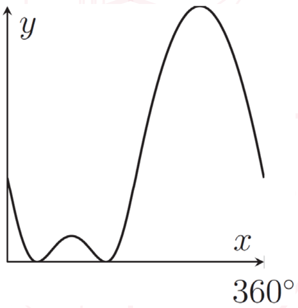
Which of the following graphs shows

$$y = \log_2(9 - 8\sin x - 6\cos^2 x)$$

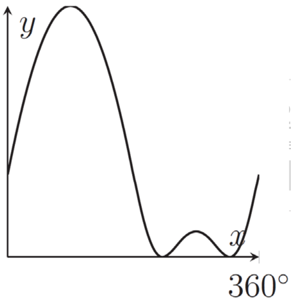
in the range $0 \le x \le 360°$?



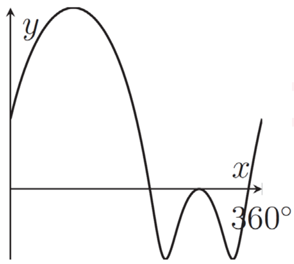
(A)



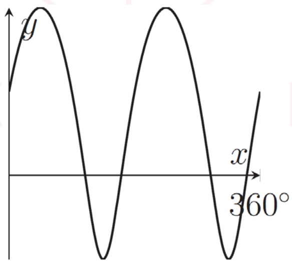
(B)



(C)



(D)



(E)

[MAT, 2021Q1(I)]

A sequence is defined by $a_0 = 2$ and then for $n \geq 1$, a_n is one more than the product of all previous terms (so $a_1 = 3$ and $a_2 = 7$, for example). It follows that for all $n \geq 1$,

(A) $a_n = 4a_{n-1} - 5$

(B) $a_n = a_{n-1}(a_{n-1} - 1) + 1$

(C) $a_n = 2a_{n-1}(a_{n-1} - 3) + 7$

(D) $a_n = \frac{3}{2}n^2 - \frac{1}{2}n + 2$

(E) None of the above

[MAT, 2021Q1(J)]

Four distinct real numbers a, b, c, and d are used to define four points

$$A = (a, b), \qquad B = (b, c), \qquad C = (c, d), \qquad D = (d, a).$$

The quadrilateral $ABCD$ has all four sides the same length

(A) if and only if $(a - b)^2 = (c - d)^2$

(B) if and only if $(a - c)^2 = (b - d)^2$

(C) if and only if $(a - d)^2 = (b - c)^2$

(D) if and only if $a - b + c - d = 0$

(E) for no values of a, b, c, d

[MAT, 2021Q2]

In this question you may use without proof the following fact:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots - \frac{x^n}{n} \cdots \qquad \text{for any } x \text{ with } |x| < 1.$$

[Note that $\ln x$ is alternative notation for $\log_e x$.]

(i) By choosing a particular value of x with $|x| < 1$, show that

$$\ln 2 = \frac{1}{2} + \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} + \frac{1}{4\times 2^4} + \frac{1}{5\times 2^5} + \cdots.$$

(ii) Use part (i) and the fact that

$$\frac{1}{n2^n} < \frac{1}{3\times 2^n} \qquad \text{for } n \geq 4$$

to find the integer k such that $\frac{k}{24} < \ln 2 < \frac{k+1}{24}$.

(iii) Show that

$$\ln\left(\frac{3}{2}\right) = \frac{1}{2} - \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} - \frac{1}{4\times 2^4} + \frac{1}{5\times 2^5} - \cdots$$

and deduce that

$$\ln 3 = 1 + \frac{1}{3\times 2^2} + \frac{1}{5\times 2^4} + \frac{1}{7\times 2^6} + \cdots.$$

(iv) Deduce that $\frac{13}{12} < \ln 3 < \frac{11}{10}$.

(v) Which is larger: 3^{17} or 4^{13}? Without calculating either number, justify your answer.

[MAT, 2021Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only.

The degree of a polynomial is the highest exponent that appears among its terms. For example, $2x^6 - 3x^2 + 1$ is a polynomial of degree 6.

(i) A polynomial $p(x)$ has a turning point at $(0, 0)$. Explain why $p(0) = 0$ and why $p'(0) = 0$, and explain why there is a polynomial $q(x)$ such that

$$p(x) = x^2 q(x). \qquad (*)$$

(ii) A polynomial $r(x)$ has a turning point at $(a, 0)$ for some real number a. Write down an expression for $r(x)$ that is of a similar form to the expression $(*)$ above. Justify your answer in terms of a transformation of a graph.

(iii) You are now given that $f(x)$ is a polynomial of degree 4, and that it has two turning points at $(a, 0)$ and at $(-a, 0)$ for some positive number a.

(a) Write down the most general possible expression for $f(x)$. Justify your answer.

(b) Describe a symmetry of the graph of $f(x)$, and prove algebraically that $f(x)$ does have this symmetry.

(c) Write down the x-coordinate of the third turning point of $f(x)$.

(iv) Is there a polynomial of degree 4 which has turning points at $(0, 0)$, at $(1, 3)$, and at $(2, 0)$? Justify your answer.

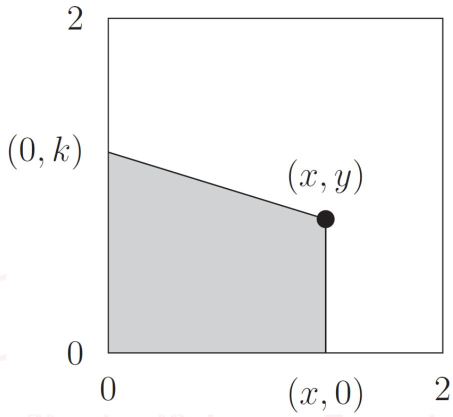
(v) Is there a polynomial of degree 4 which has turning points at $(1, 6)$, at $(2, 3)$, and at $(4, 6)$? Justify your answer.

[MAT, 2021Q4]

For applicants in {Math, Math & Statistics and Math &Philosophy} only.

Charlie is trying to cut a cake. The cake is a square with side length 2, and its corners are at $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$. Charlie's first cut is a straight line segment from the point (x, y) to $(x, 0)$, where $0 \le x \le 2$ and $0 \le y \le 2$.

Charlie plans to make a second straight cut from the point (x, y) to a point $(0, k)$ somewhere on the left-hand edge of the cake. This will make a slice of cake which is bounded to the left of the first cut and bounded below the second cut.



(i) Find the area of the slice of cake in terms of x, y, and k. Check your expression by verifying that if $x = 1$ and $y = 1$, then choosing $k = 1$ gives a slice of cake with area 1.

(ii) Find another point (x, y) on the cake such that choosing $k = 1$ gives a slice of cake with area 1.

(iii) Show that it is only possible to choose a value of k that gives a slice of cake with area 1 if both $xy \le 2$ and $x(2 + y) \ge 2$.

(iv) Sketch the region R of the cake for which both inequalities in part (iii) hold, indicating any relevant points on the edges of the cake.

(v) Charlie may instead plan to make the second straight cut from (x, y) to a point $(m, 2)$ on the top edge of the cake in order to make a slice bounded to the left of the two cuts. Find two necessary and sufficient inequalities for x and y which must both hold in order for this to give a slice of area 1 for some value of m. Sketch the region of the cake for which both inequalities hold.

[MAT, 2021Q5]

A *triangular triple* is a triple of positive integers (a, b, c) such that we can construct a triangle with sides of length a, b and c. This means that the sum of any two of the numbers is strictly greater than the third; so if $a \leq b \leq c$, then it is equivalent to requiring $a + b > c$. For example, $(3, 3, 3)$ and $(4, 5, 3)$ are triangular triples, but $(1, 3, 2)$ and $(3, 3, 6)$ are not. For any positive integer P, we define $f(P)$ to be the number of triangular triples such that the perimeter $a + b + c$ is equal to P. Triples with the same numbers, but in a different order, are counted as being distinct. So $f(12) = 10$, because there are 10 triangular triples with perimeter 12, shown below:

$(3, 4, 5)$	$(3, 5, 4)$	$(4, 3, 5)$	$(4, 5, 3)$	$(5, 3, 4)$	$(5, 4, 3)$
$(2, 5, 5)$	$(5, 2, 5)$	$(5, 5, 2)$			
$(4, 4, 4)$					

(i) Write down the values of $f(3)$, $f(4)$, $f(5)$ and $f(6)$.

(ii) If (a, b, c) is a triangular triple, show that $(a + 1, b + 1, c + 1)$ is also a triangular triple.

(iii) If (x, y, z) is a triangular triple, with $x + y + z$ equal to an even number greater than or equal to 6, show that each of x, y, z is at least 2 and that $(x - 1, y - 1, z - 1)$ is also a triangular triple.

(iv) Using the previous two parts, prove that for any positive integer $k \geq 3$,

$$f(2k - 3) = f(2k).$$

(v) We will now consider the case where $P \geq 6$ is even, and we will write $P = 2S$.

(a) Show that in this case (a, b, c) is a triangular triple with $a + b + c = P$ if and only if each of a, b, c is strictly smaller than S.

(b) For any a such that $2 \leq a \leq S - 1$, show that the number of possible values of b such that $(a, b, P - a - b)$ is a triangular triple is $a - 1$. Hence find an expression for $f(P)$ for any even $P \geq 6$.

(vi) Find $f(21)$.

[MAT, 2021Q6]

For applicants in {CS, Math & CS, and CS & Philosophy} only.

Distinct numbers are arranged in an $m \times n$ rectangular table with m rows and n columns so that in each row the numbers are in increasing order (left to right), and in each column the numbers are in increasing order (top to bottom). Such a table is called a *sorted* table and each location of the table containing a number is called a *cell*. Two examples of sorted tables with 3 rows and 4 columns (and thus $3 \times 4 = 12$ cells) are shown below.

3	12	33	64
15	26	37	78
19	40	51	92

5	22	53	68
18	36	67	78
19	45	81	92

We index the cells of the table with a pair of integers (i, j), with the top-left corner being $(1, 1)$ and the bottom-right corner being (m, n). Observe that the smallest entry in a sorted table can only occur in cell $(1, 1)$; however, note that the second smallest entry can appear either in cell $(1, 2)$, as in the first example above, or in cell $(2, 1)$ as in the second example above.

(i) (a) Assuming that $m, n \geq 3$, where in an $m \times n$ sorted table can the third-smallest entry appear?

(b) For any $k \geq 4$ satisfying $m, n \geq k$, where in an $m \times n$ sorted table can the kth smallest entry appear? Justify your answer.

(ii) Given an $m \times n$ sorted table, consider the problem of determining whether a particular number y appears in the table. Outline a procedure that inspects at most $m + n - 1$ cells in the table, and that correctly determines whether or not y appears in the table. Briefly justify why your procedure terminates correctly in no more than $m + n - 1$ steps.

[**Hint:** As the first step, consider inspecting the top-right cell.]

(iii) Consider an $m \times n$ table, say A, which might not be sorted; an example is shown below. Obtain table B from A by re-arranging the entries in each row so that they are in sorted order. Then obtain table C from B by re-arranging the entries in each column so that they are in sorted order. Fill in tables B and C here:

A:

33	92	46	24
25	26	37	8
49	40	81	22

→

B:

→

C:

(iv) Show that for *any* $m \times n$ table A, performing the two operations from part (iii) results in a sorted table C.

[MAT, 2021Q7]

For applicants in {CS and CS & Philosophy} only.

Throughout this question, all functions will be Boolean functions of Boolean input variables. A Boolean variable can be either 0 or 1. A Boolean function may have one or more Boolean input variables, and the output of a Boolean function is also either 0 or 1. Three *elementary* Boolean functions are defined as follows:

- The function $\min(x_1, \dots, x_k)$ can take any number of inputs. It outputs the value 1 exactly when each of its inputs is 1, that is the output of the function is the *minimum* value among its inputs.
- The function $\max(x_1, \dots, x_k)$ can take any number of inputs. It outputs the value 1 exactly when at least one of its inputs is 1, that is the output of the function is the *maximum* value among its inputs.
- The function flip takes a single input and is defined as $\text{flip}(x_1) = 1 - x_1$.

First we will consider Boolean functions obtained by combining the three elementary Boolean functions. One such function is shown below:

$$f(x_1, x_2, x_3) = \min\big(\max(x_1, x_2, x_3), \text{flip}(\min(x_1, x_2, x_3))\big).$$

(i) Describe in words when the function f outputs 1 and when it outputs 0.

(ii) The function $\text{majority}(x_1, \dots, x_k)$ takes k inputs and outputs 1 exactly when strictly greater than $\frac{k}{2}$ of its inputs are 1. Explain how you could combine elementary Boolean functions to obtain the following functions:

(a) $\text{majority}(x_1, x_2)$

(b) $\text{majority}(x_1, x_2, x_3)$

Now we will consider Boolean functions that can be obtained by combining only majority functions.

(iii) There are exactly 16 distinct Boolean functions of two input variables. Some of these can be represented using only majority functions that take 3 inputs; the use of 0 or 1 as fixed inputs to majority is permitted. For example, $\text{majority}(x_1, x_2, 1)$ represents the function $\max(x_1, x_2)$.

Find any four other Boolean functions of two variables that can be represented by combining one or more majority functions of 3 inputs. Write your answers in terms of majority functions.

(iv) Give an example of a Boolean function g of two input variables that cannot be represented by combining majority functions (of any number of inputs). You should write your answer by explicitly specifying $g(0,0)$, $g(0,1)$, $g(1,0)$ and $g(1,1)$. Justify your answer.

In the last part, you may express Boolean functions by combining any of the elementary Boolean functions or the majority function.

(v) Consider four input variables x_1, x_2, x_3, x_4. Let $z_1 = \min(x_1, x_2)$, $z_2 = \min(x_2, x_3)$, $z_3 = \min(x_3, x_4)$, $z_4 = \min(x_4, x_1)$. It is sometimes possible to represent a function $s(x_1, x_2, x_3, x_4)$ using a function $t(z_1, z_2, z_3, z_4)$. For example, $\min(x_1, x_2, x_3, x_4) = \min(z_1, z_2, z_3, z_4)$, as both functions output 1 if and only if all four x_i are 1.

Can you represent the following functions of inputs x_1, x_2, x_3, x_4 as some Boolean function of inputs z_1, z_2, z_3, z_4? Justify your answers.

(a) majority(x_1, x_2, x_3, x_4).

(b) The function parity(x_1, x_2, x_3, x_4) which outputs 1 exactly when an odd number of its inputs are 1.

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MAT 2022

MAT 2022
On-line Exam

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[MAT, 2022Q1(A)]

How many real solutions x are there to the equation $x|x| + 1 = 3|x|$?

[Note that $|x|$ is equal to x if $x \geq 0$, and equal to $-x$ otherwise.]

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

[MAT, 2022Q1(B)]

One hundred circles all share the same centre, and they are named C_1, C_2, C_3, and so on up to C_{100}. For each whole number n between 1 and 99 inclusive, a tangent to circle C_n crosses circle C_{n+1} at two points that are separated by a distance of 2. Given that C_1 has radius 1, it follows that the radius of C_{100} is

(A) 1

(B) 2

(C) $\sqrt{10}$

(D) 10

(E) 100

[MAT, 2022Q1(C)]

The equation $x^2 - 4kx + y^2 - 4y + 8 = k^3 - k$ is the equation of a circle

(A) for all real values of k.

(B) if and only if either $-4 < k < -1$ or $k > 1$.

(C) if and only if $k > 1$.

(D) if and only if $k < -1$.

(E) if and only if either $-1 < k < 0$ or $k > 1$.

[MAT, 2022Q1(D)]

A sequence has $a_0 = 3$, and then for $n \geq 1$ the sequence satisfies $a_n = 8(a_{n-1})^4$. The value of a_{10} is

(A) $\frac{2^{(2^{20})}}{3}$

(B) $\frac{6^{(2^{20})}}{3}$

(C) $\frac{3^{(2^{20})}}{2}$

(D) $\frac{18^{(2^{20})}}{2}$

(E) $\frac{6^{(2^{20})}}{2}$

[MAT, 2022Q1(E)]

If the expression $\left(x + 1 + \frac{1}{x}\right)^4$ is fully expanded term-by-term and like terms are collected together, there is one term which is independent of x. The value of this term is

(A) 10

(B) 14

(C) 19

(D) 51

(E) 81

[MAT, 2022Q1(F)]

Given that

$$\sin(5\theta) = 5\sin\theta - 20(\sin\theta)^3 + 16(\sin\theta)^5$$

for all real θ, it follows that the value of $\sin(72°)$ is

(A) $\sqrt{\frac{5+\sqrt{5}}{8}}$

(B) 0

(C) $-\sqrt{\frac{5+\sqrt{5}}{8}}$

(D) $\sqrt{\frac{5-\sqrt{5}}{8}}$

(E) $-\sqrt{\frac{5-\sqrt{5}}{8}}$

[MAT, 2022Q1(G)]

For all real n, it is the case that $n^4 + 1 = \left(n^2 + \sqrt{2}n + 1\right)\left(n^2 - \sqrt{2}n + 1\right)$. From this we may deduce that $n^4 + 4$ is

(A) never a prime number for any positive whole number n.

(B) a prime number for exactly one positive whole number n.

(C) a prime number for exactly two positive whole numbers n.

(D) a prime number for exactly three positive whole numbers n.

(E) a prime number for exactly four positive whole numbers n.

[MAT, 2022Q1(H)]

How many real solutions x are there to the following equation?

$$\log_2(2x^3 + 7x^2 + 2x + 3) = 3\log_2(x + 1) + 1$$

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

[MAT, 2022Q1(I)]

Alice and Bob each toss five fair coins (each coin lands on either heads or tails, with equal probability and with each outcome independent of each other). Alice wins if strictly more of her coins land on heads than Bob's coins do, and we call the probability of this event p_1. The game is a draw if the same number of coins land on heads for each of Alice and Bob, and we call the probability of this event p_2. Which of the following is correct?

(A) $p_1 = \frac{193}{512}$ and $p_2 = \frac{63}{256}$.

(B) $p_1 = \frac{201}{512}$ and $p_2 = \frac{55}{256}$.

(C) $p_1 = \frac{243}{512}$ and $p_2 = \frac{13}{256}$.

(D) $p_1 = \frac{247}{512}$ and $p_2 = \frac{9}{256}$.

(E) $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{3}$.

[MAT, 2022Q1(J)]

The real numbers m and c are such that the equation

$$x^2 + (mx + c)^2 = 1$$

has a repeated root x, and also the equation

$$(x - 3)^2 + (mx + c - 1)^2 = 1$$

has a repeated root x (which is not necessarily the same value of x as the root of the first equation). How many possibilities are there for the line $y = mx + c$?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

[MAT, 2022Q2]

(i) Suppose x, y, and z are whole numbers such that $x^2 - 19y^2 = z$. Show that for any such x, y and z, it is true that

$$(x^2 + Ny^2)^2 - 19(2xy)^2 = z^2$$

where N is a particular whole number which you should determine.

(ii) Find z if $x = 13$ and $y = 3$. Hence find a pair of whole numbers (x, y) with $x^2 - 19y^2 = 4$ and with $x > 2$.

(iii) Hence find a pair of positive whole numbers (x, y) with $x^2 - 19y^2 = 1$ and with $x > 1$.

Is your solution the only such pair of positive whole numbers (x, y)? Justify your answer.

(iv) Prove that there are no whole number solutions (x, y) to $x^2 - 25y^2 = 1$ with $x > 1$.

(v) Find a pair of positive whole numbers (x, y) with $x^2 - 17y^2 = 1$ and with $x > 1$.

[MAT, 2022Q3]

For applicants in {Math, Math & Statistics, Math & Philosophy and Math & CS} only.

(i) Sketch $y = (x^2 - 1)^n$ for $n = 2$ and for $n = 3$ on the same axes, labelling any points that lie on both curves, or that lie on either the x-axis or the y-axis.

(ii) Without calculating the integral explicitly, explain why there is no positive value of a such that $\int_0^a (x^2 - 1)^n \, dx = 0$ if n is even.

If $n > 0$ is odd we will write $n = 2m - 1$ and define $a_m > 0$ to be the positive real number that satisfies

$$\int_0^{a_m} (x^2 - 1)^{2m-1} \, dx = 0,$$

if such a number exists.

(iii) Explain why such a number a_m exists for each whole number $m \geq 1$.

(iv) Find a_1.

(v) Prove that $\sqrt{2} < a_2 < \sqrt{3}$.

(vi) Without calculating further integrals, find the approximate value of a_m when m is a very large positive whole number. You may use without proof the fact that

$$\int_0^{\sqrt{2}} (x^2 - 1)^{2m-1} \, dx < 0$$

for any sufficiently large whole number m.

[MAT, 2022Q4]

For applicants in {Math, Math & Statistics and Math &Philosophy} only.

(i) Sketch the graph of $y = \sqrt{x} - \frac{x}{4}$ for $x \geq 0$, and find the coordinates of the turning point.

(ii) Describe in words how the graph of $y = \sqrt{4x+1} - x - 1$ for $x \geq -\frac{1}{4}$ is related to the graph that you sketched in part (i). Write down the coordinates of the turning point of this new graph.

Point A is at $(-1,0)$ and point B is at $(1,0)$. Curve C is defined to be all points P that satisfy the equation $|AP| \times |BP| = 1$, that is, the distance from P to A, multiplied by the distance from P to B, is 1.

(iii) Find all points that lie on both the x-axis and also on the curve C.

(iv) Find an equation in the form $y = f(x)$ for the part of the curve C in the region where $x > 0$ and $y > 0$. You should explicitly determine the function $f(x)$.

(v) Use part (ii) to determine the coordinates of any turning points of the curve C in the region where $x > 0$ and $y > 0$.

(vi) Sketch the curve C, including any parts of the curve with $x < 0$ or $y < 0$ or both.

[MAT, 2022Q5]

Alice is participating in a TV game show where n *distinct* items are placed behind n closed doors. The game proceeds as follows. Alice picks a door i which is opened and the item behind it is revealed. Then the door is shut again and the host *secretly* swaps the item behind door i with the item behind one of the neighbouring doors, $i-1$ or $i+1$. If Alice picks door 1, the host has to then swap the item with the one behind door 2; similarly, if Alice picks door n, the host has to swap the item with the one behind door $n-1$. Alice then gets to pick any door again, and the process repeats for a certain fixed number of rounds. At the end of the game, Alice wins all the items that were revealed to her.

As a concrete example, suppose $n=3$, and if the original items behind the three doors were (a_1, a_2, a_3), then if first Alice picks door 2, the arrangement after the host has swapped items could be either (a_2, a_1, a_3) or (a_1, a_3, a_2). So if Alice was allowed to pick twice, had she chosen door 2 followed by door 1, in the former case she would only get the item a_2, whereas in the latter she would get items a_2 and a_1. Alice's aim is to find a sequence of door choices that *guarantee* her winning a large number of items, no matter how the swaps were performed.

(i) For $n=13$, give an *increasing* sequence of length 7 of *distinct* doors that Alice can pick that guarantees she wins 7 items.

(ii) For any n of the form $2k+1$, give a strategy to pick an *increasing* sequence of $k+1$ *distinct* doors that Alice can use to guarantee that she wins $k+1$ items. Briefly justify your answer.

(iii) For $n=13$, give a sequence of length 10 of doors that Alice can pick that guarantees she wins 10 items.

(iv) For any n of the form $3k+1$, give a strategy to pick a sequence of $2k+2$ doors that Alice can use to guarantee that she wins $2k+2$ items. Briefly justify your answer.

(v) (a) For $n=3$, give a sequence of length 3 of doors that Alice can pick that guarantees she wins all 3 items.

(b) For $n=5$, give a sequence of length 5 of doors that Alice can pick that guarantees she wins all 5 items.

(vi) For $n=13$, give a sequence of length 11 of doors that Alice can pick that guarantees she wins 11 items.

(vii) For any n of the form $4k+1$, give a strategy to pick a sequence of $3k+2$ doors that Alice can use to guarantee that she wins $3k+2$ items. Briefly justify your answer.

(viii) For $n=6$, is there a sequence of *any length* of doors that Alice can choose that will *guarantee* that she wins all 6 items? Justify your answer.

[MAT, 2022Q6]

For applicants in {CS, Math & CS, and CS & Philosophy} only.

This question is about *influencer networks*. An influencer network consists of n *influencers* denoted by circles, and arrows between them. Throughout this question, each influencer holds one of two opinions, represented by either a △ or a □ in the circle. We say that an influencer A follows influencer B if there is an arrow from B to A; this indicates that B has ability to influence A.

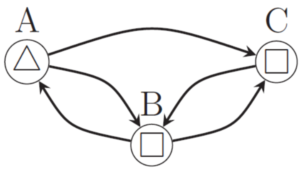


Figure 1

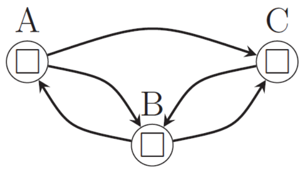
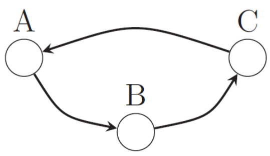


Figure 2

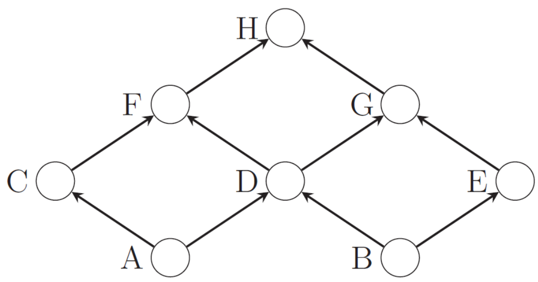
The example in Figure 1 above shows a network with A and B following each other, B and C following each other, and C also following A. In this example, initially B and C have opinion □, while A has opinion △. An influencer will change their mind according to the *strict* majority rule, that is, they change their opinion if *strictly* more than half of the influencers they are following have an opinion different from theirs. Opinions in an influence network change in rounds. In each round, each influencer will look at the influencers they are following and *simultaneously* change their opinion at the end of the round according to the strict majority rule. In the above network, after one round, A changes their opinion because the only influencer they are following, B, has a differing opinion, and the network becomes as shown in Figure 2 above.

An influencer network with an initial set of opinions is *stable* if no influencer changes their opinion, and a network (with initial opinions) is *eventually stable* if after a finite number of rounds it becomes stable. The network in the above example is eventually stable as it becomes stable after one round.

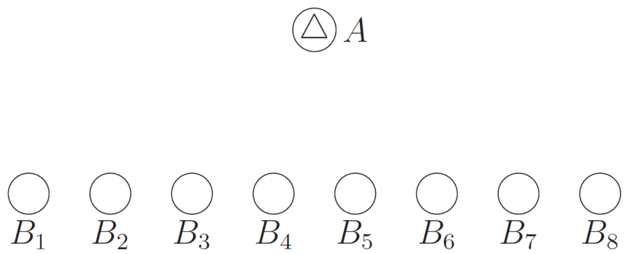
(i) A network of three influencers (without opinions) is shown below. Is this influencer network eventually stable regardless of the initial opinions of the influencers A, B and C? Justify your answer.



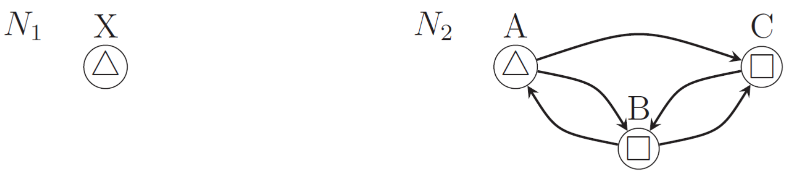
(ii) Another network of influencers (without opinions) is shown below. Is this influencer network eventually stable regardless of the initial opinions of the influencers? Justify your answer.



(iii) A *partial* network of influencers (without opinions for $B_1, \ldots, B_8$) is shown below. You can add *at most six* additional influencers, assign any opinion of your choice to the new influencers, and add any arrows to the network to describe follower relationships. Design a network that is eventually stable regardless of initial opinions, and has the property that when it becomes *stable* A has opinion □ if and only if each of $B_1, B_2, \ldots, B_8$ had opinion □ at the start. Justify your answer.



(iv) You are given two influencer networks, N_1 and N_2, with disjoint sets of influencers shown below. Both are *eventually stable*. Suppose one of the influencers from network N_2 follows the influencer X from the network N_1. Is the resulting network guaranteed to be eventually stable? Justify your answer.



(v) (a) Given a network with n influencers, where the arrows are fixed, but you are allowed to assign *opinions* (△ or □) to each influencer, how many possible assignments of opinions is possible?

(b) Given an influencer network and an initial assignment of opinions, explain how you would determine whether the influencer network is eventually stable. Justify your answer.

[MAT, 2022Q7]

For applicants in {CS and CS & Philosophy} only.

A data operator is receiving tokens one by one through an *input* channel. The data operator will receive a total of n tokens, where $n \geq 6$, and these are numbered as $1, 2, \ldots, n$. The data operator is required to pass these tokens to the output channel; however they must do so using a *valid* sequence. A *valid* sequence is one where all the odd tokens appear first, followed by the even tokens, or the other way round; furthermore all the odd tokens must appear in either increasing or decreasing order, and likewise all the even tokens must appear in increasing or decreasing order. All *valid* sequences for $n = 6$ are listed below.

1 3 5 2 4 6	1 3 5 6 4 2	5 3 1 2 4 6	5 3 1 6 4 2
2 4 6 1 3 5	2 4 6 5 3 1	6 4 2 1 3 5	6 4 2 5 3 1

The data operator has a storage unit that can hold a sequence, and can perform the following operations as they receive the tokens one by one.

- **pass** : Input token goes straight to the output channel.
- **pop** : Instead of using a token from the input, the token from the *right* end of the storage unit is removed (provided one exists) and sent to the output channel.
- **pushL** : Input token is pushed in at the left end of the storage unit.
- **pushR** : Input token is pushed in at the right end of the storage unit.

As an illustrative example, when $n = 6$, the storage and output channel are shown for the following sequence of operations, which results in the *valid* output sequence 1 3 5 6 4 2.

Input Token	Operation	Storage	Output
1	**pass**	[ ]	1
2	**pushR**	[2]	1
3	**pass**	[2]	1 3
4	**pushR**	[2 4]	1 3
5	**pass**	[2 4]	1 3 5
6	**pass**	[2 4]	1 3 5 6
	pop	[2]	1 3 5 6 4
	pop	[ ]	1 3 5 6 4 2

(i) For $n = 6$, which valid sequences can the data operator achieve?

(ii) For $n \geq 6$ and even, how many valid sequences are there? Justify your answer.

(iii) For $n \geq 6$ and even, how many valid sequences can be achieved by the data operator? Briefly justify your answer.

(iv) In the remainder of the question, $n \geq 9$ is a multiple of 3. A 3-*valid output sequence* is one where among the tokens $1, 2, \ldots, n$, all tokens of the form $3k$ appear together in increasing or decreasing order, likewise all tokens of the form $3k + 1$ appear together in increasing or decreasing order, and the same is the case for all tokens of the form $3k + 2$. As examples, the two sequences on the left below are 3-valid, whereas the two on the right are not – the first because it mixes groups and the second because although the groups are separate, the tokens of the form $3k + 2$ are in neither increasing nor decreasing order.

3-valid	**not 3-valid**
1 4 7 9 6 3 8 5 2	1 3 5 6 4 2 7 8 9
2 5 8 1 4 7 3 6 9	2 8 5 9 6 3 1 4 7

For $n \geq 9$ and multiple of 3, how many 3-valid sequences of length n are there? Justify your answer.

(v) For $n \geq 9$ and multiple of 3, given the input sequence of tokens $1, 2, \ldots, n$, how many 3-valid sequences can be achieved by the operator using a single storage unit and the operations **pass**, **pop**, **pushL** and **pushR**? Justify your answer.

MAT 2023

MAT 2023
On-line Exam

Scan the QR code or click on the link to take an on-line exam.
Full solutions can be accessed after submission.

TIME ALLOWED: 150 MINUTES

You must use a pen throughout the test.

Calculators are not permitted.

Please write your answers and rough work in the Candidate Answer Booklet.

During the test, please write your **Candidate Number** at the top of each page of the answer booklet as indicated.

The test contains 6 questions of which you should attempt 5.

If you are applying to Oxford for the degree course:

- Mathematics / Mathematics & Statistics, *or* Mathematics & Philosophy, you should attempt Questions **1, 2, 3, 4, 5**.
- Computer Science, *or* Mathematics & Computer Science, *or* Computer Science & Philosophy, you should attempt Questions **1, 2, 3, 5, 6**.

If you are not an Oxford applicant, you should attempt Questions **1, 2, 3, 4, 5**.

Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple-choice question with ten parts. Marks are given solely for correct answers, but any rough working should be shown in the pages in the answer booklet. Answer Question 1 on the grid in the answer booklet. Each part is worth 4 marks.

Answers to Questions 2–6 should be written in the space provided in the answer booklet, continuing onto the blank pages at the end of the answer booklet if necessary. Each of Questions 2–6 is worth 15 marks.

[MAT, 2023Q1(A)]

In this question we write $\alpha = \log_{10} 2$, $\beta = \log_{10} 3$ and $\gamma = \log_{10} 7$. Each of the following numbers is close to an integer. Which is the closest to an integer?

(A) 2β

(B) $5\alpha + \beta$

(C) $\alpha + 2\gamma$

(D) $2\alpha + 5\beta$

(E) $2\alpha + \beta + \gamma$

[Hint: $2^2 \times 3^5 = 9.72 \times 10^2$]

[MAT, 2023Q1(B)]

Exactly one of these five numbers is a square number. Which one?

(A) 99,999,999

(B) 123,333,333

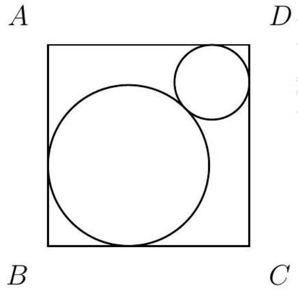
(C) 649,485,225

(D) 713,291,035

(E) 987,654,000

[MAT, 2023Q1(C)]

Two circles are inside a square $ABCD$ of side-length 1. One of the circles is tangent to sides AB and to BC. The other circle is tangent to sides CD and DA. The circles are tangent to each other. The area of the larger circle is 10 times the area of the smaller circle. The diagram is not to scale.



The sum of the radii of the circles is:

(A) $\frac{1+\sqrt{10}}{8}$

(B) $2 - \sqrt{2}$

(C) $\frac{\sqrt{2}}{2}$

(D) $\sqrt{\frac{2}{5}}$

(E) $\sqrt{10} - 1$

[MAT, 2023Q1(D)]

How many distinct real solutions x are there to the equation

$$(((x^2-1)^2-2)^2-3)^2 = 4\ ?$$

(A) 5

(B) 6

(C) 7

(D) 8

(E) 9

[MAT, 2023Q1(E)]

The first few positive whole numbers that are not powers of 3 are 2, 4, 5, 6, 7, 8, 10. What is the sum of all the positive whole numbers that are less than 3^{10} and are also not powers of 3?

(A) $\frac{(3^{10}-1)^2}{2}$

(B) $\frac{(3^{11}-1)^2}{2}$

(C) $\frac{3(3^{10}-1)^2}{2}$

(D) $(3^{10}-1)^2$

(E) $(3^{11}-1)^2$

[MAT, 2023Q1(F)]

The coefficient of x^{12} in

$$(1-2x)^5(1+4x^2)^5(1+2x)^5$$

is

(A) $-2^{13}\times 5$

(B) $-2^{12}\times 5$

(C) $2^{12}\times 5$

(D) $2^{13}\times 5$

(E) 0

[MAT, 2023Q1(G)]

The real numbers a, b, and c are non-zero. Each of the following quadratic equations has a repeated real root (not necessarily the same value).

$$ax^2 + bx + c = 0,\ bx^2 + cx + a = 0.$$

How many distinct real roots does the equation $cx^2 + ax + b = 0$ have?

(A) 0

(B) 1

(C) 2

(D) 3

(E) It depends on a.

[MAT, 2023Q1(H)]

Which of these triangles has the largest area?

(A) an isosceles triangle with side lengths 10, 10, 1.

(B) an isosceles triangle with side lengths 10, 10, 5.

(C) an isosceles triangle with side lengths 10, 10, 10.

(D) an isosceles triangle with side lengths 10, 10, 15.

(E) an isosceles triangle with side lengths 10, 10, 19.

[MAT, 2023Q1(I)]

The polynomial $p(x)$ has degree 3 and has $p(0) = 0, p(1) = 1, p(2) = 2$. The polynomial has a repeated root at $x = M$ with $M > 0$. The value of M is

(A) $\frac{7}{6}$

(B) $\frac{6}{5}$

(C) $\frac{5}{4}$

(D) $\frac{4}{3}$

(E) $\frac{3}{2}$

[MAT, 2023Q1(J)]

Let $\lfloor x \rfloor$ denote the largest whole number that is less than or equal to x. For example, $\lfloor -\pi \rfloor = -4$. A function $f(x)$ is defined as follows; if $0 < x < 2$ then

$$f(x) = \left(\frac{3}{4}\right)^{\lfloor \log_2(x) \rfloor}$$

and $f(x) = 0$ otherwise. Note that, for example, $f\left(\frac{1}{2}\right) = \frac{4}{3}$. The value of $\int_0^2 f(x)\mathrm{d}x$ is

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

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[MAT, 2023Q2]

For ALL APPLICANTS.

For n a positive whole number, and for $x \neq 0$, let $p_n(x) = x^n + x^{-n}$.

(i) Sketch the graph of $y = p_1(x)$. Label any turning points on your sketch. [3]

(ii) Show that $p_2(x) = p_1(x)^2 - 2$. [1]

(iii) Find an expression for $p_3(x)$ in terms of $p_1(x)$. [1]

(iv) Find all real solutions x to the equation [5]

$$x^4 + x^3 - 10x^2 + x + 1 = 0.$$

(v) Find all real solutions x to the equation [5]

$$x^7 + 2x^6 - 5x^5 - 7x^4 + 7x^3 + 5x^2 - 2x - 1 = 0.$$

[MAT, 2023Q3]

For ALL APPLICANTS.

Note that the arguments of all trigonometric functions in this question are given in terms of degrees. You are not expected to differentiate such a function. The notation $\cos^n x$ means $(\cos x)^n$ throughout.

(i) Without differentiating, write down the maximum value of $\cos(2x + 30°)$. [1]

(ii) Again without differentiating, find the maximum value of [4]

$$\cos(2x + 30°)(1 - cos(2x + 30°)).$$

(iii) Hence write down the maximum value of [4]

$$\cos^5(2x + 30°)(1 - cos(2x + 30°))^5.$$

(iv) Find the maximum value of [6]

$$\left(1 - \cos^2 (3x - 60°)\right)^4 \left(3 - \cos (150° - 3x)\right)^8.$$

[MAT, 2023Q4]

For **Oxford applicants** in Mathematics / Mathematics & Statistics / Mathematics & Philosophy, OR those not applying to Oxford, ONLY.

Point A is on the parabola $y = \frac{1}{2}x^2$ at $\left(a, \frac{1}{2}a^2\right)$ with $a > 0$. The line L is normal to the parabola at A, and point B lies on L such that the distance $|AB|$ is a fixed positive number d, with B above and to the left of A.

(i) Find the coordinates of B in terms of a and d. [6]

(ii) Show that in order for B to lie on the parabola, we must have [4]

$$a^2 d = 2(1 + a^2)^{\frac{3}{2}}. \qquad (*)$$

(iii) Let $t = a^2$ and express the equality $(*)$ in the form $d^{\frac{2}{3}} = f(t)$ for some function f which you should determine explicitly. [2]

(iv) Find the minimum value of $f(t)$. Hence show that the equality $(*)$ holds for some real value of a if and only if d is greater than or equal to some value, which you should identify. [3]

[MAT, 2023Q5]

For ALL APPLICANTS.

Define the sequence, F_n, as follows: $F_1 = 1$, $F_2 = 1$, and for $n \geq 3$,

$$F_n = F_{n-1} + F_{n-2}. \qquad (*)$$

(i) What are the values F_3, F_4, F_5? [3]

(ii) Using the equation $(*)$ repeatedly, in terms of n, how many additions do you need to calculate F_n? [1]

We now consider sequences of 0's and 1's of length n, that do not have two consecutive 1's. So, for $n = 5$, for example, (0,1,0,0,1) and (1,0,1,0,1) would be valid sequences, but (0,1,1,0,0) would not. Let S_n denote the number of valid sequences of length n.

(iii) What are S_1 and S_2? [1]

(iv) For $n \geq 3$, by considering the first element of the sequence of 0's and 1's, show that S_n satisfies the same equation $(*)$. Hence conclude that $S_n = F_{n+2}$ for all n. [3]

(v) For $n \geq 2$, by considering valid sequences of length $2n - 3$ and focusing on the element in the $(n-1)$th position, show that, [2]

$$F_{2n-1} = F_n^2 + F_{n-1}^2. \qquad (\mathrm{O})$$

(vi) For $n \geq 2$, show that, [3]

$$F_{2n} = F_n^2 + 2F_nF_{n-1}. \qquad (\mathrm{E})$$

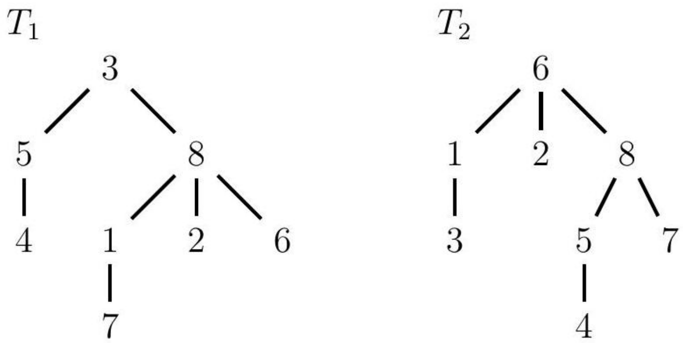
(vii) Let $k \geq 3$ be an integer. By using the equations (O) and (E) repeatedly, how many arithmetic operations do you need to calculate F_{2^k}? You should only count additions and multiplications needed to calculate values using the equations (O) and (E). [2]

[MAT, 2023Q6]

For **Oxford applicants** in Computer Science / Mathematics & Computer Science / Computer Science & Philosophy ONLY.

In an *octatree*, *all* the digits 1 to 8 are arranged in a diagram like *trees* T_1 and T_2 shown below. There is a single digit at the *root*, drawn at the top (so the root is 3 in T_1), and every other digit has another digit as its *parent*, so that by moving up the tree from parent to parent, each non-root digit has a unique path to the root. The order in which the children of any parent are drawn does not matter, so for simplicity we show them in increasing order from left to right.

A *leaf* is a digit that is not the parent of any other digit: in tree T_1, the leaves are 2, 4, 6 and 7.



The *code* for an octatree is a sequence of seven digits obtained as follows. We use T_1 as an example.

- Remove the numerically smallest leaf and write down its parent. In T_1, we remove 2 and write down its parent **8**.
- In the tree that remains, remove the smallest leaf and write down its parent. In T_1, after having removed 2, we remove 4 and write down its parent **5**.
- Continue in this way until only the root remains. In T_1, we would have deleted the digits 2, 4, 5, 6, 7, 1, 8 in that order and obtained the code **8538183**.

(i) Find the code for the octatree T_2. [1]

(ii) Draw the octatree that has the code **8888888**. [1]

(iii) Draw the octatree that has the code **3165472**. [2]

(iv) What are the leaves of the octatree that has the code **1618388**? Justify your answer. [3]

(v) Find all the digits in the octatree that has the code **1618388** that have 1 as their parent. [2]

(vi) Reconstruct the whole tree that has the code **1618388**. [2]

(vii) Briefly describe a procedure that given a sequence of seven digits from 1 to 8 constructs an octatree with that sequence as its code. [2]

(viii) Is the number of distinct octatrees greater than or smaller than 2,000,000? Justify your answer. (You may use the fact that $2^{10} = 1024$.) [2]

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